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ON NONLINEAR EFFECTS IN MAGNETIC CHAINS

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Occurring of solitary waves in a variety of one-dimensional models of condensed matter physics was discussed by now in several papers[1-4]. Further investigations concerning statistical properties of such excitations have been carried out in refs.[5-7] using the ideas of pioneering work of Krumhansl and Schrieffer [1]. Some of these theoretical results were confirmed in neutron experiments [8].

In what follows our concern will be of soliton excitations in ferromagnetic chains in the classical limit with special attention paid to the effects of phonon anharmonism and nonlinearity in exchange integrals

1. Let us consider for that the Hamiltonian of the interacting phonon and spin systems

$$H = H_L + H_S , \tag{1.1}$$

where

$$H_L = T + U , \tag{1.2a}$$

$$H_s = -\frac{1}{4}\sum_{j\delta} J_{jj+\delta}[S_j^+ S_{j+\delta}^- + S_j^- S_{j+\delta}^+] - \frac{1}{2}\sum_{j\delta} \bar{J}_{jj+\delta} S_j^z S_{j+\delta}^z - \mu h \sum_j S^z , \tag{1.2b}$$

T and U being the kinetic and the potential energies of the lattice oscillations and $J_{jj+\delta} \equiv J((x_{j+\delta} - x_j))$ are the exchange integrals with the property $J_{ij} = J_{ji}$, h is the external magnetic field, and μ is the magnetic susceptibility, j labels the atoms occupying position x_j, δ runs over nearest neighbours, $S_j^{\pm}$ are given by the relation

$$S_j^{\pm} = S_j^x \pm i S_j^y .$$

The previous investigations [3,4] were limited only with the following approximations.

i) Phonon effects in the harmonic approximation:

$$U = \frac{mv_0^2}{2}\sum_j (x_{j+1} - x_j - a)^2 , \tag{1.3}$$

a being the lattice distance.

ii) The exchange integrals in the linear approximation

$$J(|x_{j+\delta}-x_j|) \approx J(0) - J_1 \cdot |x_{j+\delta} - x_j|, \qquad (1.4)$$

where

$$J_1 = -\frac{\partial J}{\partial x_j}\Big|_{x_j = x_{j+\delta}} < 0.$$

We will, however, reject these limitations away. At low temperature when only a few spin waves are excited we may neglect all the nonlinear terms in the Holstein-Primakov representation of the spin operators[9] reducing them to

$$S_j^+ = \sqrt{2s}\, a_j, \quad S_j^- = \sqrt{2s}\, a_j^+, \quad S_j^z = s - a_j^+ a_j.$$

Hamiltonian (1.1) is consequently expressed through Bose operators a_j^+ and a_j. Introducing the coherent state

$$|\Phi\rangle = \exp\{i\Sigma(\phi_j a_j^+ + \phi_j^* a_j)\}|0\rangle,$$

where "vacuum" $|0\rangle$ is the completely magnetized state, i.e., $S_j^+|0\rangle = 0$, and assuming a weak nonlinearity, i.e.:

$$\langle\Phi|a_j^+ a_j a_{j+\delta}^+ a_{j+\delta}|\Phi\rangle = \int \langle\Phi|a_j^+ a_j|\Phi'\rangle\langle\Phi'|a_{j+\delta}^+ a_{j+\delta}|\Phi\rangle d^2\Phi' \approx$$

$$\approx \langle\Phi|a_j^+ a_j|\Phi\rangle\langle\Phi|a_{j+\delta}^+ a_{j+\delta}|\Phi\rangle$$

from eq. (1.2) one gets its classical equivalent

$$\langle\Phi|H|\Phi\rangle = \mathcal{H} = \mathcal{H}_0 - \frac{1}{2}\sum_{j\delta} J_{jj+\delta}[s(\phi_j^*\phi_{j+\delta} + \phi_{j+\delta}^*\phi_j) - \qquad (1.5)$$

$$- \rho\{s(|\phi_j|^2 + |\phi_{j+\delta}|^2) - |\phi_j|^2|\phi_{j+\delta}|^2\}] - \mu h \sum_j |\phi_j|^2,$$

where $\rho = \bar{J}/J$ and $\mathcal{H}_0 = -sJ(0)N/2 + s^2 \sum_{j\delta} J_{j+\delta} + sh\mu N, \ \rho > 0$.

We take the kinetic energy of the lattice systems as

$$T = \frac{m}{2}\sum_j \dot{x}_j^2$$

and in potential energy (1.3) add the next anharmonic term:

$$U = \frac{mv_0^2}{2}\sum_j (x_{j+1} - x_j - a)^2 + \frac{U^{III}}{3!}\sum_j (x_{j+1} - x_j - a)^3. \qquad (1.6)$$

The exchange integral, however, would be taken so far in the linear approximation (1.4) to distinguish effects of these different nonlinearities. We assume x_j to be monotonic function which allows one to write $|x_{j\pm 1} - x_j| = \pm(x_{j\pm 1} - x_j)$. We will show below the validity of this assumption. Considering further the longwave limit we have the expansion

$$x_{j\pm 1} = x \pm x_\xi a + \frac{1}{2}x_{\xi\xi}a^2 \pm \frac{1}{3!}x_{\xi\xi\xi}a^3 + \frac{1}{4!}x_{\xi\xi\xi\xi}a^4.$$

In the expansion of $\phi_{j\pm 1}$ we retain, however, the terms up to $\phi_{\xi\xi}$ assuming ϕ to be of the same order as $x_{\xi\xi}$. The field equations derived from the classical Hamiltonian (1.5) under above assumption take the form

$$i\dot{\phi} = -A\phi_{\xi\xi} - \tilde{\mu}\phi + g x_\xi \phi - \lambda|\phi|^2\phi \qquad (1.7a)$$

and

$$m\ddot{x} = C x_{\xi\xi} + D(x_\xi^2)_\xi + E x_{\xi\xi\xi\xi} + g(|\phi|^2)_\xi, \qquad (1.7b)$$

where

$$C = a^2(mv_0^2 - U^{III}a), \quad D = \frac{a^3}{2}U^{III}, \quad E = mv_0^2 a^4/12, \quad \lambda = 2J(0)\rho,$$

$$g = -2J_1 s(\rho - 1), \quad A = J(0)s, \quad \tilde{\mu} = 2s(J(0) - J_1)(1-\rho) - h\mu.$$

In (1.7) the higher order term $\frac{2}{3}sJ_1\{(x\phi)_{\xi\xi\xi} - \rho\phi x_{\xi\xi\xi}\}$ is neglected.

Differentiating Eq. (1.7b) once by ξ and passing to the function $\eta = x_\xi$ one gets the set of equations

$$i\dot{\phi} = -A\phi_{\xi\xi} - \tilde{\mu}\phi + g\eta\phi - \lambda|\phi|^2\phi, \qquad (1.8a)$$

$$\ddot{\eta} = (C/m)\eta_{\xi\xi} + (D/m)(\eta^2)_{\xi\xi} + (E/m)\partial_\xi^4\eta + (g/m)(|\phi|^2)_{\xi\xi} \qquad (1.8b)$$

obtained earlier with $\lambda=0$ in ref.[9] for coupled ion-sound and Langmuir plasma waves.

These equations describe solitons moving with near sound velocities $(v^2 \to c^2 = A/m)$ and were analyzed in detailes in reviews[10]. Since our Eqs. differ from plasma ones by the last term in (1.8a) we discuss here extra soliton solutions occurring in our system.

Using the variables

$$|\phi| \to \frac{C}{\sqrt{AB^2/m}}|\phi|, \quad \eta \to \frac{C}{ABm}\eta, \quad \xi \to \frac{A}{\sqrt{C/m}}\xi, \quad t \to \frac{A}{C/m}t,$$

where $A, C>0$ and introducing the ansatz

$$\phi = \exp[i(\frac{v}{2}x - (\omega - \tilde{\mu})t)]\psi(\kappa\zeta), \quad \zeta = x - vt - x_0,$$

$$\eta = \eta(\kappa\zeta),$$

we get

$$\Omega\psi + \psi_{\zeta\zeta} - \psi(\eta + \delta|\psi|^2) = 0, \qquad (1.9)$$

$$(v^2 - 1)\eta - a\eta^2 - \beta\eta_{\zeta\zeta} - |\psi| = 0$$

with

$$a = \frac{D}{Agm}, \quad \beta = \frac{E}{A^2m} > 0, \quad \delta = -\frac{\lambda C^2}{g^2}, \quad \Omega = \omega - \frac{v^2}{4}, \quad C > 0.$$

Soliton-like solutions to (1.10) may be of two forms

$$\eta = \frac{a}{\cosh^2\kappa\zeta}, \quad |\psi| = b\frac{\tanh\kappa\zeta}{\cosh\kappa\zeta}, \qquad (1.10a)$$

$$\eta = \frac{a}{\cosh^2\kappa\zeta}, \quad |\psi| = \frac{b}{\cosh\kappa\zeta}. \qquad (1.10b)$$

Substituting the first one into (1.9) we find (if $\delta|\phi|^2 \ll \eta$),

$$\Omega = -\kappa^2, \quad a = -6\kappa^2, \quad b = a\sqrt{a+\beta},$$

$$\kappa^2 = \frac{1}{2\gamma^2}(3a + \beta)^{-1}, \quad (a+\beta) > 0, \quad \gamma^2 = (1 - v^2)^{-1}, \qquad (1.11)$$

i.e., the solution known from [10]. The condition of its validity $\delta|\phi|^2 \ll \eta$ is therefore equivalent to $a \gg \delta b^2$ or

$$\frac{\gamma^2}{3}\frac{3a+\beta}{a+\beta} \gg |\delta|$$

and is satisfied when $\gamma^2 \gg |\delta|$. It occurs always at $\rho = 0$ (i.e., for x-y system). For more details about solution (1.11) see ref.[10]

The second solution appears only when $\delta \neq 0$ and gives

$$a = -2\kappa^2 - \delta b^2, \quad \Omega = -\kappa^2,$$

$$= 6\kappa^2(\frac{\beta}{a}), \quad 1 - v^2 = -(4\beta\kappa^2 + \frac{b^2}{a}). \qquad (1.12)$$

There are again two possibilities

(i) $\delta > 0$, i.e., the usual "compact" case and

(ii) $\delta < 0$, i.e., "noncompact" case.

Let us discuss first case (i). From (1.12) it follows that $(\beta/a) < 0$ or since $\beta > 0$ we have $a < 0$. From (1.12) one can also obtain

$$b^2 = \frac{2\kappa^2}{\delta}(3\frac{\beta}{|a|} - 1), \quad \text{i.e.,} \quad |a| < 3\beta \quad \text{or} \quad U^{III} < mv_0^2 a\frac{J_1|\rho - 1|}{J(0)},$$

then

$$1 - v^2 + 4\beta\kappa^2 = \frac{1}{\delta}(1 - \frac{1}{3}\frac{|a|}{\beta}) \equiv \epsilon > 0.$$

The right-hand side of this relation depends only on parameters of the system considered: a, β, δ, hence $4\beta\kappa^2 = \epsilon + v^2 - 1 > 0$ and $v^2 > 1 - \epsilon$.

In the region $v^2 \to 1$ the soliton amplitude is fixed and defined by system parameters

$$\kappa = \frac{\epsilon}{4\beta}, \quad a = -\frac{3}{2}\frac{\epsilon}{|a|}.$$

Moreover solitons (1.12) unlike (1.11) may be supersound as well as subsound $v \gtrless 1$ and even sound like $v = 1$. Their inverse widths κ grow with v, and if $\epsilon < 1$ they have minimal velocity $v_{min} = \sqrt{1 - \epsilon}$. All these features differ them substantially from that studied earlier (see ref.[10] and references cited there in).

Analogous picture takes place in the second case ($\delta < 0$) only with the difference that here a may be both negative and positive since in this case

$$b^2 = \frac{2\kappa^2}{|\delta|}(1 + 3\frac{\beta}{a}),$$

$|a| > 3\beta$ if $a < 0$. Whence

$$4\beta\kappa^2 = v^2 - 1 = \frac{1}{|\delta|}(1 + \frac{1}{3}\cdot\frac{a}{\beta}) \quad \text{or}$$

$$v^2 > 1 + \frac{1}{|\delta|}(1 + \frac{1}{3}\cdot\frac{a}{\beta}),$$

i.e., at $a > 0$ solitons may be only supersound. To proceed further let us verify the above assumptions. Note that since $\kappa_\xi = \eta = -a/\cosh^2\kappa\zeta$ we have for both types of solutions

$$x = -\frac{a}{\kappa}\tanh\kappa\zeta,$$

i.e., kink-like profile for lattice displacements. It means that $x(\xi)$ is a monotonic function indeed. It is easily seen that in the velocity region $v \to 1$ the harmonic approximation $a \to 0$ is no longer valid, since

$$\eta \sim \frac{|\psi|^2}{1 - v^2} \to \infty.$$

To make assumption $|\phi| \le x_{\xi\xi}$ valid it is enough to have

$$\gamma \le \frac{1}{a + \beta}$$

in the first case and $\kappa \ge \epsilon$ or

$$v^2 - 1 \ge (4\beta\epsilon - 1)\epsilon$$

in the second one. And longwave limit means that $\kappa \ll \frac{1}{a}$, i.e.,

$$\frac{1}{\gamma^2} \ll 2\frac{(3a+\beta)}{a} \quad \text{or} \quad \epsilon \ll \frac{1}{a}.$$

2. Now we consider nonlinear effects due to the following expansion of exchange integral

$$J_{jj+\delta} \approx J(0) - J_1|x_{j+\delta} - x_j| + J_2|x_{j+\delta} - x_j|^2,$$

with potential U being harmonic (1.3). Then using the analogous procedure one gets the following field equations

$$\begin{aligned} i\dot\phi &= -A\phi_{\xi\xi} - \tilde\mu\phi + gx_\xi\phi + c_1(x_\xi)^2\phi - \lambda\phi|\phi|^2, \\ m\ddot x &= c_0^2 x_{\xi\xi} + g\partial_\xi(|\phi|^2) + c_1(x_\xi|\phi|^2)_\xi, \end{aligned} \tag{2.1}$$

where $c_0 = mv_0^2a^2$, $c_1 = 4sJ_1(\rho-1)$, other parameters are the same as in (1.7).

In the quasi-stationary limit ($|\ddot x| \ll \frac{c_0}{m}x_{\xi\xi}$) we may reduce this system to a single equation:

$$i\dot\phi = -A\phi_{\xi\xi} - \tilde\mu\phi - d\frac{|\phi|^2\phi}{1 + a|\phi|^2}, \tag{2.2}$$

where

$$d = 4[\frac{J_1^2s^2(1-\rho)^2}{c_0} + \frac{sJ(0)}{2}] \quad \text{and} \quad d = \frac{c_1}{c_0}$$

and higher nonlinear terms are neglected. Eq. (2.2) is the well-known Schrödinger equation with saturable nonlinearity. There is a great amount of works devoted to it (see ref.[10] and references cites there in). It possesses the following features, physically important from our point of view:

1) there are soliton-like solituions (SLS),
2) these solutions may be stable in $(D+1)$ space-time with $D = 1, 2, 3, \ldots$,
3) stable SLS may interact with each other creating in particular bound states (pulsons) and so one.

The detailed discussion of these and other peculiarities of Eq. (2.2) SLS may be found in refs.[10,11]. Probably the most interesting feature is the second one since it gives us the hope to obtain stable SLS in more than one-dimensional models of condensed matter physics.

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О нелинейных эффектах в магнитных цепочках

Исследованы эффекты фононного ангармонизма и нелинейности в обменных интегралах и их влияние на солитонные возбуждения в ферромагнитных цепочках в классическом и длинноволновом пределе. Показано, что первые приводят к системе связанных уравнений типа Буссинеска и Шредингера, допускающей два типа солитонных решений, вторые - к нелинейному уравнению Шредингера с насыщающейся нелинейностью.

Работа выполнена в Лаборатории вычислительной техники и автоматики ОИЯИ.

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On Nonlinear Effects in Magnetic Chains

The effects of phonon anharmonism and nonlinearity in exchange integrals on soliton excitations in ferromagnetic chains in the classical and longwave limit are studied. It has been first shown that the anharmonic effect leads to a system of coupled Boussinesq and nonlinear Schrödinger equations allowing two types of soliton solutions. The nonlinear effect, on the other hand, results in nonlinear Schrödinger equation with saturable nonlinearity admitting stable solitons in more than one-dimensional models.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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