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ON GAUGE EQUIVALENCE OF LANDAU-LIFSHITZ AND NONLINEAR SCHRÖDINGER EQUATIONS

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The present investigation is a continuation of a series of works devoted to the study of nonlinear Schrödinger equation (NLSE) for field variables which are elements of some finite dimensional vector space [1,2] (isospace). When the Hamiltonian of the system is invariant under some symmetry group acting in the isospace, the system becomes an integrable one allowing the formulation of corresponding linear problem for it.

It has been already stressed in a previous work [3] that in spite of the seeming triviality of the generalizations of U(1) and U(2) symmetric models to the general group GL(n,C) and even the simple replacement of a compact isogroup U(2) by a noncompact U(1,1), it yields significant nontrivial results connected with much richer set of physically admissible boundary conditions.

Moreover in the case of nontrivial boundary conditions even a global rotation in the isospace generalises a whole class of new solutions. Local transformations on the other hand give rise to gauge equivalence of different systems and in particular equivalence of NLSE and Landau-Lifshitz equation (LLE). Note that the gauge equivalence of vector NLSE and LLE for different compact groups G was established in ref. [4] for $G=SU(2)$, in [5] for $G=SU(3)$ and in principle for $SU(N)$. Finally the equivalence in the case of general matrix $m\times n$ NLSE with compact group $G=U(n+m)$ is analysed in ref. [6]. We, therefore, limit ourselves only with the cases which differ from these considered previously. The most natural in our case is the language of group theory and bundle spaces. The linear problem may be represented as

$$\psi_x = U\psi, \qquad \psi_t = V\psi. \tag{1}$$

Then the compatibility or the trivial flatness condition

$$\partial_t U - \partial_x V + [U,V] = 0, \tag{2}$$

where

$$U = i\begin{pmatrix} -\frac{\lambda}{m} I_m & R \\ Q & \frac{\lambda}{n} I_n \end{pmatrix} \tag{3}$$

$$V = \begin{pmatrix} \frac{1}{m}\,\frac{m+n}{mn}\, i\lambda^2 I_m - iRQ, & -\frac{m+n}{mn}\, i\lambda R + R_x \\ -\frac{m+n}{mn}\, i\lambda Q - Q_x, & -\frac{1}{n}\,\frac{m+n}{mn}\, i\lambda^2 I_n + iQR \end{pmatrix} \qquad (4)$$

gives the system of matrix nonlinear Schrödinger equations (MNLSE)

$$\begin{aligned} -iR_t + R_{xx} + 2(RQ)R = 0, \\ iQ_t + Q_{xx} + 2Q(RQ) = 0. \end{aligned} \qquad (5)$$

Here R and Q are, respectively, the $n\times m$ and $n\times m$ matrices. It is convenient to introduce the following notations

$$U = A_0 + \lambda A_1, \quad A_0 = \begin{pmatrix} 0 & iR \\ iQ & 0 \end{pmatrix}, \quad A_1 = -i\Sigma, \quad \Sigma = \begin{pmatrix} \frac{1}{m} I_m & 0 \\ 0 & -\frac{1}{n} I_n \end{pmatrix}$$

$$V = B_0 + \lambda B_1 + \lambda^2 B_2, \quad B_0 = \begin{pmatrix} -iRQ & R_x \\ -Q_x & iQR \end{pmatrix}, \quad B_1 = -\frac{m+n}{mn} A_0, \quad B_2 = -\frac{m+n}{mn} A_1.$$

It is then not difficult to check that $\mathrm{Tr}\Sigma = 0$ and

$$\Sigma^2 = \frac{1}{mn} I + \frac{n-m}{mn}\Sigma. \qquad (6)$$

Problem (1) is defined up to some gauge transformations $\psi' \to g\psi$ (when $g \in G$ is a global group, U and V are subject to similarity transformation: $(U,V) \longrightarrow g(U,V)g^{-1}$). Let us introduce the space of Jost matrix solutions Ψ, the gauge group G acting on it transitively. Moreover Ψ is the linear representation of G and its principal (left) homogeneous space. In general $G = GL(mn,c)$. Group G acts as well on the manifold $M = (U,V)$ which is also its homogeneous space. Let us consider the isotropic group H_Σ keeping the point $\Sigma \in M$ fixed. One may easily check that $H = GL(m,C) \otimes GL(n,C)$ with field variables R and Q changing as

$$R' = g_1 R g_2^{-1}, \quad Q' = g_2 Q g_1^{-1}, \quad g_1 \in GL(m,C) = G_1, \quad g_2 \in GL(n,C) = G_2.$$

Note that H_Σ at the same time is the symmetry group of the Hamiltonian of the system

$$\mathcal{H} = \mathrm{Tr}\,(R_x Q_x - RQRQ) = \mathrm{Tr}\,(R'_x Q'_x - R'Q'R'Q') \qquad (7)$$

and equations (5) are covariant under H_Σ.

Let G be now the local gauge transformation. We consider its linear representation Ψ and denote the Jost solution at the point $\lambda = \lambda_0$ by $g(\lambda_0; x,t) = \psi(\lambda_0; x,t)$:

$$g_x = U_0(\lambda_0) g, \qquad g_t = V_0(\lambda_0) g.$$

Introducing the matrix

$$S(\lambda_0; x,t) = g^{-1}(\lambda_0)\, \Sigma\, g(\lambda_0)$$

one may easily verify that under the transformation

$$\Psi = g(\lambda_0)\Psi_1$$

we get

$$\psi_{1x} = U_1(\lambda,\lambda_0)\psi_1,$$

$$\psi_{1t} = V_1(\lambda,\lambda_0)\psi_1,$$

where

$$U_1(\lambda,\lambda_0) = g^{-1}Ug - g^{-1}g_x = -i(\lambda-\lambda_0)\, S(\lambda_0; x,t), \qquad (8)$$

$$V_1(\lambda,\lambda_0) = g^{-1}Vg - g^{-1}g_t = i\,\frac{m+n}{mn}(\lambda^2-\lambda_0^2)S - \frac{2mn}{m+n}(\lambda-\lambda_0)SS_x + \frac{n-m}{m+n}(\lambda-\lambda_0)S_x$$

and

$$S^2 = \frac{1}{mn} I_n + \frac{n-m}{mn} S, \qquad N = m+n. \qquad (9)$$

Using (2) we may deduce the following equation for matrix S

$$S_t = \frac{mn}{m+n}\,\frac{1}{i}[S, S_{xx}] - 2\lambda_0 \frac{m+n}{mn} S_x \qquad (10)$$

which is a generalization of the known SU(2) LLE obtained in the case $S \in SU(2)$, $\lambda_0 = 0$, $m = n = 1$, when (9) becomes $S^2 = I$.

It has been mentioned in ref.[5] that by means of gauge transformation corresponding to Galilean transformation $(x',t') = (x - vt, t)$ one may easily eliminate the term proportional to λ_0 in Eq.(10). We note, however, that this may be done only in the case of trivial boundary conditions (for both compact and noncompact group G). For nontrivial boundary conditions (and noncompact groups G) the presence of nonzero vacuum (condensate) states breaks the Galilean invariance and hence the choice of λ_0 will completely define the solution of the gauge equivalent LL model.It is the second nontrivial conclusion emerging from noncompact groups. Note that the representation Ψ of group G may only be constructed with matrix Jost solutions corresponding to continuous spectrum $\mathrm{Im}\lambda = 0$. One can, therefore, come to the point $\lambda = 0$ through Galilean transformation of coordinates only in the case of trivial boundary conditions of NLSE both with compact as well as noncompact internal group symmetry. The continuous spectrum in the case of nontrivial boundary conditions (most interesting when the isogroup is noncompact) possesses gap which results to some limitations in the choice of λ_0. In particular cases of U(0,1) and U(1,1), for example, $|\lambda_0| > \mu$, where μ is defined by boundary conditions $(|\phi(\infty)|^2 = \mu,\ \phi = Q = R^*)$.

Let us consider now quite general case with the transformation groups and the groups of internal symmetry being noncompact. For this we concentrate on the equation (10) with $\lambda_0 \neq 0$:

$$S_t = \frac{1}{2i}[S, S_{xx}] + 4\lambda_0 S_x, \tag{11}$$

where

$$S = \sum_a S^a \tau_a, \qquad i\tau_a \in su(p,q)$$

with

$$\tau_a = (\lambda_1, \ldots, \lambda_\ell, -i\lambda_{\ell+1}, -i\lambda_{\ell+2}, \ldots, -i\lambda_{\ell+k}$$

$$i\lambda_a \in su(N), \quad N = p + q.$$

Then

$$su(p,q) = \tilde{k} \oplus \tilde{\ell}, \qquad \dim \tilde{k} = p^2 + q^2 - 1 = k$$

$$\dim \tilde{\ell} = 2pq = \ell.$$

The generators τ_a have the algebraic property

$$\tau_a \tau_\beta = g_{a\beta} + d_{a\beta\gamma}\tau^\gamma + i f_{a\beta\gamma}\tau^\gamma, \qquad \mathrm{Tr}\,\tau_a = 0. \tag{12}$$

From the above relations one gets

$$\mathrm{Tr}\, S = 0. \quad \Gamma^\circ S^+ \Gamma^\circ = S. \qquad \Gamma^\circ = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}. \tag{13}$$

We impose the σ model condition (9) on S which in the component form reads as

$$S^a g_{a\beta} S^\beta = \frac{1}{mn} \tag{14}$$

$$S^a S^\beta d_{a\beta\gamma} = \frac{n-m}{mn} S_\gamma.$$

We have

$$g_{a\beta} = (\underbrace{1, 1, \ldots, 1}_{\ell \text{ times}}, \underbrace{-1, \ldots, +1}_{k \text{ times}})$$

and denote further

$$r = p - m, \qquad n = q + r \ (r \geq 0).$$

Consider some subgroup G of group SU(p,q) elements $\bar{g}$ of which reduce matrix S through similarity transformations to a diagonal one*:

$$S = g \Sigma_g g^{-1}, \quad (\Gamma^\circ g^+ \Gamma^\circ) g = 1. \tag{15}$$

It is immediate that Σ_g obeys equations (13) and (14) and hence diagonal and real and furthermore coinciding with Σ given above.

Note that one may find some subgroup G_0 in the group $\bar{G}$ the elements of which keep matrix S unchanged (isotropy group), i.e., we get for

$$\tilde{g} = g g_0, \quad \tilde{S} = \tilde{g}\Sigma\tilde{g}^{-1} = g\Sigma g^{-1} = S,$$

if $[\Sigma, g_0] = 0$. This naturally implies

$$G_0 = S(U(p-r) \oplus U(r,q)), \quad (r \geq 0)$$

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*We have shown before in a constructing way that such transformations really exist.

and coincides with isotropy group H_Σ of the linear problem in the space Ψ. Thus we have Ψ isomorphic to G/H_Σ or in our case isomorphic to $SU(p,q)/S(U(p-r)\ \ U(r,q))$.

Suppose further $\tilde{G} = H_\Sigma\ \ L$, then the corresponding Lie algebra should be given by $A\ell\tilde{G} = h\ \ \tilde{\ell}$, with h and $\tilde{\ell}$ having the form

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \qquad \tilde{\ell} = \begin{pmatrix} 0 & \bar{\ell} \\ -\ell & 0 \end{pmatrix},$$

h_1 and h_2 matrices have dimensions $m\times n$, $n\times n$ respectively, and ℓ is a $n\times m$ matrix with

$$\bar{\ell} = \ell^{+}\gamma_0, \qquad \gamma_0 = \begin{pmatrix} I_r & 0 \\ 0 & -I_q \end{pmatrix}.$$

Introducing further the left chiral current $A^L_\mu = -g^{-1}\partial_\mu g$ we may express S_μ through g, Σ, and $A^L_\mu\ (\mu = 0,1)$ as

$$S_\mu = g[\Sigma, A^L_\mu]\, g^{-1}.$$

Let A^L_1 takes values only in the complement algebra $\tilde{\ell}$:

$$A^L_1 = A_0 = \begin{pmatrix} 0 & \bar{\phi} \\ -\phi & 0 \end{pmatrix}. \qquad (16)$$

Denoting then $A^L_0 = B_0$ from the compatibility conditions $g_{xt} = g_{tx}$ or $A_{0t} - B_{0x} + [A_0, B_0] = 0$ one gets

$$B_0 = -i\,\frac{m+n}{2mn}\begin{pmatrix} \bar{\phi}\phi - \mu I_m, & \bar{\phi}_x \\ \phi_x, & -(\phi\bar{\phi} - \mu I_n) \end{pmatrix}, \qquad (17)$$

μ being the integration constant and ϕ, $\bar{\phi}$ being, respectively, the $n\times m$ and $m\times n$ matrices.

The linear problem (Eq. (8) with $\lambda_0 \neq 0$) for (11) under the transformation $\psi = g\tilde{\psi}$ reduces to

$$\tilde{U} = g^{-1}Ug - g^{-1}g_x = i(\lambda - \lambda_0)\Sigma + A_0$$

$$\tilde{V} = g^{-1}Vg - g^{-1}g_t = 2i(\lambda^2 - \lambda_0^2)\Sigma + 2(\lambda - \lambda_0)\left(\frac{m+n}{mn}\right)^2 A_0 + B_0.$$

From Eq. (2) we get now the NLSE sought for

$$i\,\frac{m+n}{2mn}\phi_t + \phi_{xx} + 2(\phi\bar{\phi}\phi - \mu\phi) = 0.$$

Or denoting $t' = \frac{m+n}{2mn}t$ one comes to equation (5) in which $\phi = -iQ$, $\bar{\phi} = iR$. The internal group symmetry of NLSE, as we have observed before, coincides with H_Σ and in our case it is

$$S(U(p-r) \otimes U(r,q)) \quad \text{for } r \geq 0,$$

$$S(U(p,r) \otimes U(q-r)) \quad \text{for } r < 0.$$

Constraint (9) completely defines the isotropic group as well as the projectors on the space in which the field variables of NLSE take their values:

$$P_{m,n} = \frac{m}{m+n}(I - nS), \qquad P^2 = P.$$

Changing the constraint leads to certain new gauge equivalent reductions of LLE (see, e.g., ref. /7/).

Let us consider simple example showing possible reductions of LL system to the NLSE:

$SU(2) \to U(1)$

$SU(1,1) \to U(0,1)$

$SU(3) \to U(2)$, $U(1)$

$SU(2,1) \to U(1,1)$, $U(2,0)$, $U(1,0)$, $U(0,1)$

$SU(4) \to U(3)$ vector, $U(2)$ matrix

$SU(3,1) \to U(2,1)$, $U(3,0)$ vector; $U(1,1)$, $U(2,0)$ matrix

etc.

It is evident from the above diagram that the number of possible reductions in the noncompact cases is much greater compared to that in the corresponding compact groups. Note that the reductions of SU(2,1) into U(1,1) and U(2,0) are gauge equivalent to each other only for trivial boundary conditions.

Previous investigations on U(1,1) NLSE/1-3/ showed that in noncompact group models the soliton spectrum is considerably richer due to the existence of various possible boundary conditions. Now, we have demonstrated in a simple example of the classical Heisenberg model (with σ model constraint) that the spectrum of reductions (and hence solutions) in the case of noncompact gauge group is much richer. As it has been mentioned in ref./8/ this result is very important in the theory of extended supergravity with noncompact groups/9/.

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Кунду А., Маханьков В.Г., Пашаев О. E17-82-601

О калибровочной эквивалентности уравнения Ландау-Лифшица и нелинейного уравнения Шредингера

Установлена калибровочная эквивалентность обобщенной спиновой цепочки Гейзенберга в классическом и континуальном пределе нелинейному уравнению Шредингера /НУШ/. Особый упор сделан на случай некомпактных групп. Некоторые особенности модели с нетривиальными граничными условиями обсуждаются подробно.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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On Gauge Equivalence of Landau-Lifshitz and Nonlinear Schrödinger Equations

The gauge equivalence between a generalized Heisenberg spin chain in the classical and continuum limit and the nonlinear Schrödinger equation (NLSE), with the special attention to the noncompact groups, is established. It has been demonstrated that the noncompact groups allow a richer spectrum of possible reductions of the Heisenberg system to the NLSE. Some specialities of the model with nontrivial boundary conditions are discussed.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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