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INTRINSIC SYMMETRY OF THE SCALING LAWS AND GENERALIZED RELATIONS FOR CRITICAL INDICES

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Many of the fundamental physical laws can be expressed as a consequence of some symmetry principle. Here we want to show that the standard scaling laws for critical indices are not an exception to the rule.

Denote an arbitrary system with Hamiltonian Γ, temperature $\theta = kT$, number of particles N as Γ/θ. Consider a set of order parameters A, B, C,... – Hermitian operators of the additive type normalized per particle; we shall call below "order parameters" both operators and their equilibrium averages $\langle A\rangle_{\Gamma/\theta}$, $\langle B\rangle_{\Gamma/\theta}$, ... Introduce also generalized susceptibility $\chi_{AB}(\Gamma/\theta) = [\partial\langle A\rangle_{\Gamma - xNB/\theta}/\partial x]_{x=0}$.

Consider a conventional ferromagnet with a fixed Hamiltonian H, critical temperature θ_C, one-component order parameter S. For a nonzero magnetic field $h > 0$ the Hamiltonian of the system will be $H - hNS$. Introduce also the correctly normalized "temperature order parameter" $L = N^{-1}(\langle H\rangle_{H/\theta_C} - H)$ and dimensionless temperature deviation $t = (\theta - \theta_C)/\theta_C$, $\theta \equiv \theta_C(1-t)$.

Consider a standard set of order parameters (averages) and susceptibilities: S (magnetization), L (singular part of the energy), χ_{SS} (magnetic susceptibility), χ_{LL} (specific heat), $\chi_{SL} \equiv \chi_{LS}$ (derivatives $\partial\langle S\rangle/\partial\theta$, $\partial\langle L\rangle/\partial h$) taken on the critical isobar in the ordered phase $\theta \equiv \theta_C(1-t) < \theta_C$, $h \equiv 0$, and on the critical isotherm $\theta \equiv \theta_C$, $h > 0$. Assuming the proper power asymptotics, we write:

$$S(t) \backsim t^{\beta}, \quad L(t) \backsim t^{1-\alpha'}, \quad \chi_{SS}(t) \backsim t^{-\gamma'}, \quad \chi_{LL}(t) \backsim t^{-\alpha'},$$

$$\chi_{SL}(t) \backsim t^{\beta-1}; \quad S(h) \backsim h^{1/\delta}, \quad L(h) \backsim h^{\zeta}, \quad \chi_{SS}(h) \backsim h^{1/\delta - 1},$$

$$\chi_{LL}(h) \backsim h^{-\epsilon}, \quad \chi_{SL}(h) \backsim h^{\zeta - 1}$$

(we use here the primed notation for indices below θ_C), here and below $X \backsim Y$ means $X/Y = \text{const.}$ $(\neq 0, \infty)$ as $t \to 0, h \to 0$. As is well known, there are 4 scaling relations for 6 indices in these asymptotic forms (see, e.g., ref. [1]):

$$\gamma' = \beta(\delta - 1),$$

$$\alpha' + 2\beta + \gamma' = 2,$$

$$1 - \alpha' = \beta\delta\zeta,$$

$$\alpha' = \beta\delta\epsilon.$$

Though the symmetry we want to show can be demonstrated directly in relations (1), we shall consider the problem from a more general point of view and obtain first heuristic generalized relations for critical indices for the case of arbitrary order parameters and variations of the Hamiltonian taking the system away from the critical point, which manifestly possess a symmetrical form and include (1) as a particular case.

As the starting point can be considered the observation [2] that scaling laws (1) can be derived from the following physical hypothesis: in the ordered phase $\theta \equiv \theta_C(1-t) < \theta_C$, $h \equiv 0$ there exists the temperature-dependent "inner field" $h(t)$, associated with spontaneous ordering, which provides the same effect as the corresponding external field and gives a finite contribution into all order parameters and susceptibilities, so that every quantity of a set $X = \{S, L, \chi_{SS}, \chi_{LL}, \chi_{SL}\}$ in the ordered phase, $X(t)$, is of the same order of magnitude as $X(h)$ (critical isotherm) for $h = h(t)$:

$$X(t) \backsim X(h = h(t)). \tag{2}$$

Choosing $h(t) \backsim t^{\sigma}$ and substituting the power asymptotics for $S, \dots, \chi_{SL}$ into (2), one finds 5 relations for 7 indices α', β, γ', δ, ζ, ϵ, σ. Eliminating σ one obtains just 4 scaling equalities (1) [2]. It is also easily seen that $\sigma = \gamma' + \beta$, i.e., $h(t) \backsim t^{\gamma' + \beta}$.

If one accepts the view point that the "inner field" is a real physical object [2] (see also remarks 1 and 2 at the end of the paper), one can then suppose that the inner field would give also finite contribution into some other order parameters A, B, ..., except S, L, and corresponding susceptibilities. Moreover, one can expect that the inner field will appear every time when the system H/θ_C is taken away from the critical point by means of some "ordering" variation of the Hamiltonian $\delta H \equiv V$, $H/\theta_C \to H + V/\theta_C$, which gives rise to spontaneous magnetization. The lowering of the temperature $\theta_C \to \theta_C(1-t)$ corresponds to variation $V_t = tH/(1-t) \approx tH$, nonzero external field corresponds to $V_h = -hNS$. We shall consider below a class of only such ordering variations $\{V\}$ including V_t and V_h.

Let us now assume that energy V is characterized by a small parameter $\xi > 0$, $V \equiv V(\xi) \to 0$ as $\xi \to +0$, and for a set of order parameters A, B and S and corresponding susceptibilities for the variations under consideration the power asymptotic forms hold true. [Common scaling arguments (see, e.g., ref. [3]) lead to the preference of the power laws for arbitrary order parameters constructed from the same suboperators as S, L (see remark 3 at the end of the paper)]. By analogy with a standard situation we write:

$$\chi_{XY}(V(\xi)) \propto \xi^{-\gamma_V^{XY}}, \quad Y(V(\xi)) \propto \xi^{\beta_V^Y}, \tag{3a}$$

and in a special case of the field variation:

$$\chi_{XY}(h) \propto h^{-\epsilon^{XY}}, \quad Y(h) \propto h^{\zeta^Y}, \tag{3b}$$

where X, Y independently run over {A, B, S} ; Y, χ_{XY} in (3a) are taken for the system $H + V(\xi)/\theta_C$ (in particular, $Y(V(\xi)) \equiv \equiv \langle Y \rangle_{H+V(\xi)/\theta_C}$, it is assumed that $\langle Y \rangle_{H/\theta_C} = 0$ for all Y = A, B, S). Here $\gamma_V^{XY}, \beta_V^Y, \epsilon^{XY}, \zeta^Y$ are constants (generalized critical indices), the subindex V labels the type of the variation $V(\xi)$. We have introduced a separate notation for the field variation $V_h = -hNS$ in view of its particular role; note that $\epsilon^{AS} = 1 - \zeta^A$ and $\zeta^S = 1/\delta$, $\epsilon^{SS} = (\delta-1)/\delta$.

In accordance with the above discussion we shall futher proceed from the heuristic assumption generalizing (2) and being of the analogous physical sense (existence and finite contribution of the inner field $h(V(\xi))$ into order parameters and suseptibilities in the system $H + V(\xi)/\theta_C$).

$$\chi_{XY}(V(\xi)) \propto \chi_{XY}(h = h(V(\xi)), \tag{4a}$$

$$Y(V(\xi)) \propto Y(h = h(V(\xi)), \tag{4b}$$

where X, Y = {A, B, S}. Assuming $h(V(\xi)) \propto \xi^{\sigma_V}$ and substituting asymptonic forms (3) into (4), one obtains by equating indices in both sides:

$$\gamma_V^{XY} = \epsilon^{XY}\sigma_V, \quad \beta_V^Y = \zeta^Y\sigma_V, \tag{5}$$

where X, Y = {A, B, S}.

A notable feature here is the factorization of the dependences on X, Y and V. For the class of variations considered ϵ^{XY}, ζ^Y are independent of V, while σ_V is independent of X, Y. So, the ratio of any two indices from a set $\{\gamma_V^{XY}, \beta_V^Z\}$ appears to be independent of the variation V, for instance:

$$\frac{\gamma_V^{AA}}{\gamma_V^{BB}} = \frac{\epsilon^{AA}}{\epsilon^{BB}} = \mathrm{inv}(V), \tag{6}$$

$$\frac{\gamma_V^{AA}}{\beta_V^B} = \frac{\epsilon^{AA}}{\zeta^B} = \mathrm{inv}(V), \tag{7}$$

$$\frac{\gamma_V^{AS}}{\beta_V^B} = \frac{\epsilon^{AS}}{\zeta^B} = \mathrm{inv}(V), \tag{8}$$

$$\frac{\beta_V^A}{\beta_V^B} = \frac{\zeta^A}{\zeta^B} = \mathrm{inv}(V), \tag{9}$$

and so on. Equalities (6)-(9) all follow from the generalized relation

$$\frac{\gamma_V^{XY}}{\beta_V^Z} = \frac{\epsilon^{XY}}{\zeta^Z} = \mathrm{inv}(V), \tag{10}$$

where X, Y, Z independently run over {A, B, S}. This makes it possible to formulate the following heuristic principle - the Symmetry Principle for critical indices.

The Principle. Given a set of critical indices for order parameters and susceptibilities for a set of ordering variations of the Hamiltonian $\{V(\xi)\}$ and field variation $V_h \equiv -hNS$ (see (3)), Then the ratio of any two indices for two fixed quantities does not change when passing from one variation $V(\xi)$ to another or to V_h.

Let us just show that the Principle entirely contains scaling equalities (1). Indeed, relations (6)-(9) follow from the Principle, but considering the case of standard variations V_t, V_h and putting in (8) A = B = S, one gets $\gamma' = \beta(\delta-1)$, putting in (7) A = L, B = S, one obtains $\alpha' = \beta\delta\epsilon$, choosing in (9) A = L, B = S, one has $1 - \alpha' = \beta\delta\zeta$, and putting in (8) A = B = L, one finds $1 - \alpha' = \zeta(2 - \alpha' - \beta)$ and hence, in view of the above relations, $2 = \alpha' + 2\beta + \gamma'$. Here we have taken into account that in the standard notation $\gamma_{SS} = \gamma'$, $\gamma_{SL} = 1 - \beta$, $\beta_L = 1 - \alpha'$,

So, the conventional scaling laws can be expressed "by words" as a consequence of a simple symmetry principle.

However, the heuristic equalities for indices (6)-(10) are probably of a more general significance. These relations seem to be quite universal (through they don't claim, of course, to be undoubtedly correct in all cases without exception). As one sees, of a considerable interest is the study of notraditional order parameters and "artificial" variations of the Hamiltonian in different specific systems.

Let us formulate some concluding remarks.

1) The concept of the inner field is qualitatively (but not quantitatively) similar to the proper "molecular field" appearing in the simplest approximating methods. The switching on of an external field $h > 0$ means appearance of the ordering force constant over the whole sample. The existence of spontaneous magnetization for $\theta < \theta_C$ can also be interpreted as appearing

of a "nonfluctuating mode" in effective forces acting on separate magnetic moments, i.e., one can speak about an "inner field" (see also ref. [2]). In such an interpretation there is no significant difference between assumption (2) leading to the proper scaling laws and assumptions (4) leading to generalized relations (6)-(10) (if the variation $V(\xi)$ produces spontaneous ordering and inner field, and if order parameters A, B respond to the magnetic field). Note that the assumption itself of existence and finite contribution of the inner field determines its value $h(t) \backsim t^{\gamma'+\beta}$.

2) One can, in principle, accept a more cautious point of view and consider the correspondence $t \rightleftarrows h(t) \backsim t^{\gamma'+\beta}$ to be a consequence of some "indirect" and "unknown in all details" reasons. Nevertheless, if one assumes that these "unknown reasons" are physical and the correspondence $t \rightleftarrows h(t)$ is not a kind of a happy chance, even then there are some grounds for suppositions (4) leading to (6)-(10). However, I prefer the direct physical interpretation of $h(t)$.

3) Order parameters A, B should be constructed from the same suboperators as S, L. For instance, in the case of the Ising model one can consider a "cluster" of some neighbour spins $\lambda_{ij\ldots k}(\sigma_i\sigma_j \ldots \sigma_k), \ldots,$ $\sigma_j = \pm 1, \ldots,$ and symmetrize it in all spins, one than obtains order parameter $A \backsim \Sigma\lambda_{ij\ldots k}(\sigma_i\sigma_j\ldots\sigma_k)$. The standard order parameters $S = N^{-1}\Sigma\sigma_i$, $L \backsim N^{-1}\Sigma J_{ij}\sigma_i\sigma_j$ are simplest examples of this type.

4) Consider for illustration some "aritifical" (and in a sense, trivial) example of order parameters $A = S^2$, $B = L, S$ and variations V_t, V_h. The critical indices for $A = S^2$ can be easily obtained. For instance, $A(t) = \langle S^2 \rangle_t \equiv \langle S \rangle_t^2 \backsim t^{2\beta}$ and $\beta_t^A = 2\beta$; $\chi_{AA}(t) = 4\langle S \rangle_t^2 \chi_{SS}(t)$ $t^{2\beta-\gamma'}$ and $\gamma_t^{AA} = \gamma' - 2\beta$ etc. One can easily verify that here relations (6)-(10) hold and are equivalent (as it should be) to the standard relations (1). The same is valid also for a more general set of $A = S^m$, $B = L^n$, S and V_t, V_h.

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Внутренняя симметрия скейлинговых равенств и обобщенные соотношения для критических индексов

Показано, что скейлинговые законы для критических индексов могут быть выражены как следствие простого принципа симметрии. Представлены эвристические обобщенные соотношения для критических индексов в случае произвольных параметров порядка и вариаций гамильтониана, выводящих систему из критической точки, которые явно имеют симметричную форму и включают стандартные скейлинговые равенства как частный случай.

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Intrinsic Symmetry of the Scaling Laws and Generalized Relations for Critical Indices

It is shown that the standard scaling laws can be expressed as a consequence of a simple symmetry principle. Heuristic relations for critical indices generalizing scaling laws to the case of arbitrary order parameters are represented, which manifestly have a symmetric form and include the scaling laws as a particular case.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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