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W.Weller, V.N.Prigodin*, Yu.A.Firsov*

LOCALIZATION IN A RANDOM SYSTEM OF N COUPLED CHAINS.

I. Microscopic Model, General Scheme

*A.F.Ioffe Physico-Technical Institute, Leningrad, USSR

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1. INTRODUCTION

Recently there has been much interest in the problem of localization of electrons in random systems, first treated by Anderson [1] for a strongly disordered three-dimensional system. Of special interest are one-dimensional (1D), quasi-1D systems and thin wires (and 2D systems), where large effects arise already from small disorder because of strong quantum mechanical coherence in multiple scattering. However, at present time only the assertion by Mott and Twose [2] about the localization of all eigen states in a 1D system for even arbitrarily small random potential seems to be strictly proven. Berezinskii [3] first attacked the 1D problem by usual field theoretic methods based on the important concept of diagrams ordered along the chain It is the aim of this and the following papers to investigate the effects of the transverse motion for N weakly coupled chains. We show localization in an infinitely long thin wire for arbitrarily small random potential, describing the wire by a simple microscopic model which takes into account the quantization of the transverse motion.

Already in 1977 Thouless [4] gave arguments based on a simple scaling method that in a disordered infinitely long thin wire all states are localized independently of the strength of the randomness. Anderson et al. [5] and Azbel [6] developed a scaling theory for the 1D case and for the case of N transverse channels.

Our model for the N coupled chains (Sec. 2) represents microscopically such an N channel system as discussed in the scalling theories [5,6]. We need no additional hypothesis like that about the independence of stochastical quantities belonging to neighbouring pieces of the wire used in the scaling theories. A microscopic model with N states at each site of a d-dimensional random lattice was recently proposed and investigated for $N \to \infty$ by Wegner et al. [7]. Our work differs in two respects from Wegner's: (i) Our derivation of the model leads to a selection rule for the scattering between states (see text below (6)) not present in Wegner's formal model, (ii) Wegner investigates infinite systems for $d > 1$ in the limit $N \to \infty$, whereas we consider a wire with finite N.

As a criterium for localization we calculate in Sec. 3,4 by means of diagram technique the average of a special density-density correlation function for the infinitely long system (compare Economou and Cohen [8]), instead of calculating the conductance. Thus we do not use the generalization [5,11] of the Landauer conductance formula [9] to the N channel case. Our calculated density-density correlation function is the conditional probability density that the electron for infinitely long time remains in a given cross section $z=0$ (z coordinate along the wire). In other words, we average the normalization constant of the conditional probability $P(z)\Delta z$ (that means the probability that the electron initially at $z=0$ is for $t\to\infty$ between z and $z+\Delta z$).). Therefore, in essence we average the inverse localization length $1/L_{loc}$, a selfaveraging quantity [5,11-14].

The introduction of N different states essentially complicates the diagram analysis (already for $N=2$), and it was necessary (i) to simplify Berezinskii's technique by avoiding the ordering of the lines at any z (ii) to develop an effective method (Sec.5) for the treatment of the N^2 variables m_{ij} (m_{ij} number of pairs with lines i, j). This problem of N^2 variables exists in principle also in the scaling theories, because there the rank of the scattering matrix is of that order. The main idea is to consider deviations of the m_{ij} from a main variable. In this way it was possible to reduce the problem to the solution of two ordinary differential equations with the main variable as the independent one. As in the case of the 1D problem [3] we made calculations for weak random potential, that means in Born approximation. For the considered model the diagram analysis is then developed and evaluated without further approximations.

In Sec.6 we consider the resulting two ordinary differential equations in an approximated form and show that the density density correlation function exhibits the singularity necessary for localization (compare [8]). In the following paper we consider these differential equations in their exact form.

2. THE MODEL

The Hamiltonian without impurities is given by

$$H^{(0)} = \int dz \left\{ \sum_{\nu} \tilde{a}^{+}_{\nu}(z)\, \epsilon_{\parallel}\, (-ih\frac{d}{dz})\tilde{a}_{\nu}(z) + \sum_{\substack{\nu\nu' \\ (\nu\neq\nu')}} t_{\nu\nu'}\tilde{a}^{+}_{\nu}(z)\tilde{a}_{\nu'}(z) \right\}, \qquad (1)$$

ν, ν' number of chains. We diagonalize with respect to the motion perpendicular to z (U unitary):

$$\vec{a}_\nu(z) = \sum_j U_{\nu j} a_j(z), \qquad (2)$$

after the transformation the energy eigenvalues of the transverse motion are $\delta\epsilon'_j$. These states j may be related to the channels in refs. [5,6]. The unperturbed retarded Green function is then (for weakly coupled chains $|\delta\epsilon_j| \ll \epsilon_F$)

$$G^{(0)}_{jj}(z, \epsilon_F + is') = -\frac{i}{hv_F} e^{i(k_F + \delta\epsilon_j + is)|z|}, \qquad (3)$$

where $\delta\epsilon_j = \delta\epsilon'_j / \hbar v_F$; $s = s'/hv_F$. The Hamiltonian of the interaction with the impurities is

$$H^{imp} = \int dz \sum_{j,j'} V_\nu(z) (U^+)_{j\nu} a^+_j(z) U_{\nu j'} a_{j'}(z). \qquad (4)$$

Following [3] we consider the possible vertices for diagrams ordered along the z-axis, generalized to the case of N states. The correlator of the stochastic impurity potential V(z) is

$$\langle V_\nu(z_1) V_{\nu'}(z_2)\rangle = \delta_{\nu\nu'} W(|z_1 - z_2|). \qquad (5)$$

We consider in more detail only the vertex shown in Fig.1:

$$\gamma = -\frac{1}{(hv_F)^2} \int_{-\infty}^{\infty} d\zeta W(|\zeta|) e^{i(\delta\epsilon_j - \delta\epsilon_k)} e^{i(\delta\epsilon_j - \delta\epsilon_k + \delta\epsilon_\ell - \delta\epsilon_m)z} \times \qquad (6)$$

$$\times \sum_\nu (U^+)_{j\nu} U_{\nu k} (U^+)_{\ell\nu} U_{\nu m}.$$

We now simplify the model: (i) We take into account only terms for which the indices j, k, ℓ, m of the U coincide pairwise, because in such cases we obtain phase-independent quantities like $|U_{\nu j}|^2$. (ii) We neglect vertices (oscillating vertices), where in the exponent the factor in front of z does not vanish. The dependence of the diagram blocks L (or X) on z following from Eqs. (17,54) shows, that the mean distance between two vertices and thus the characteristic integration domain for z is the localization length $L_{loc} \approx N\ell_2$ (ℓ_2 is mean free path for backward scattering, see (11)). Thus the necessary condition is

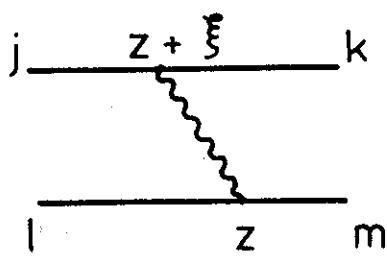


Fig.1. Vertex. ⟶ retarded line, ⟹ advanced line. The analytical expression is independent direction of the arrows.

$$\min_{j \neq k} |\delta\epsilon_j - \delta\epsilon_k| N\ell_2 \gg 1. \tag{7}$$

For simple transverse lattices we get

$$\frac{8\pi^2 t_\perp \ell_2}{h v_F} = 4\pi^2 \frac{t_\perp}{\epsilon_F} k_F \ell_2 \gg 1, \qquad \text{for} \quad d-1=2, \tag{7a}$$

$$\frac{4\pi^2 t_\perp \ell_2}{h v_F N} = 2\pi^2 \frac{t_\perp}{\epsilon_F} \frac{k_F \ell_2}{N} \gg 1, \qquad \text{for} \quad d-1=1. \tag{7b}$$

(iii) The exponent containing ζ can be neglected because of the small range of W. Because the sum over ν contains a large number of terms, we substitute this sum by its average with respect to j, ℓ:

$$\frac{1}{N} \sum_{\substack{j\ell \\ \nu}} |U_{\nu j}|^2 |U_{\nu \ell}|^2 = \frac{1}{N}. \tag{8}$$

To this vertex corresponds the mean free path

$$\frac{1}{\ell_1} = \frac{2}{h v_F} \int_0^\infty W(\zeta)\, d\zeta. \tag{9}$$

Thus finally

$$\gamma = -\frac{1}{\ell_1} \qquad \text{for} \quad \begin{matrix} j=k \\ \ell=m \end{matrix} \qquad \text{for} \quad \begin{matrix} j=m, \\ \ell=k. \end{matrix} \tag{10}$$

For the vertices with backward scattering we need the mean free path

$$\frac{1}{\ell_2} = \frac{2}{(h v_F)^2} \int_0^\infty W(\zeta) \cos(2k_F \zeta)\, d\zeta. \tag{11}$$

All vertices and the corresponding analytic expressions are shown in Fig.2.

3. DIAGRAM ANALYSIS, SUMMATION OF THE DIAGRAMS FOR THE LEFT PART

Our aim is the calculation of the conditional probability that the electron for an infinitely long time remains at a given "cross section". Instead of using a state strictly localized at a point z it is convenient to use normalizable states $\psi_\pm(z)$ centered at $z=0$ with a width Δz; for example $\psi_\pm(z) = (\frac{\Delta k}{\pi})^{1/2} \frac{\sin(\Delta k z)}{\Delta k z} e^{\pm i k_F z}$, $\Delta k \ll k_F$, $\Delta k \Delta z = \pi$. The conditional probability $P\Delta z$, that the electron initially in the state $\psi_\pm(z)$ remains for $t \to \infty$ in these states is then (compare /8/)

$$P\Delta z = \lim_{s'\to 0} \frac{s'}{\pi} \int_{-\infty}^{\infty} d\epsilon \int \psi^*(\zeta_1') d\zeta_1' \int \psi(\zeta_1) d\zeta_1 \int \psi^*(\zeta_2) d\zeta_2 \int \psi(\zeta_2') d\zeta_2' \times \tag{12}$$

$$\times \frac{1}{N} \sum_{uj} \langle G_{ij}(\zeta_1, \zeta_1'; \epsilon + is') G_{ji}(\zeta_2', \zeta_2; \epsilon - is') \rangle .$$

For narrow $\psi(z)$ ($\Delta z \ll L_{loc} \approx N\ell_2$) we can put $\zeta_1 = \zeta_1' = \zeta_2 = \zeta_2' = 0$ in the Green functions; only the neighbourhood of ϵ_F contributes to the ϵ-integral. Finally we obtain

$$P = \lim_{s'\to 0} 2s'\hbar v_F K , \tag{13}$$

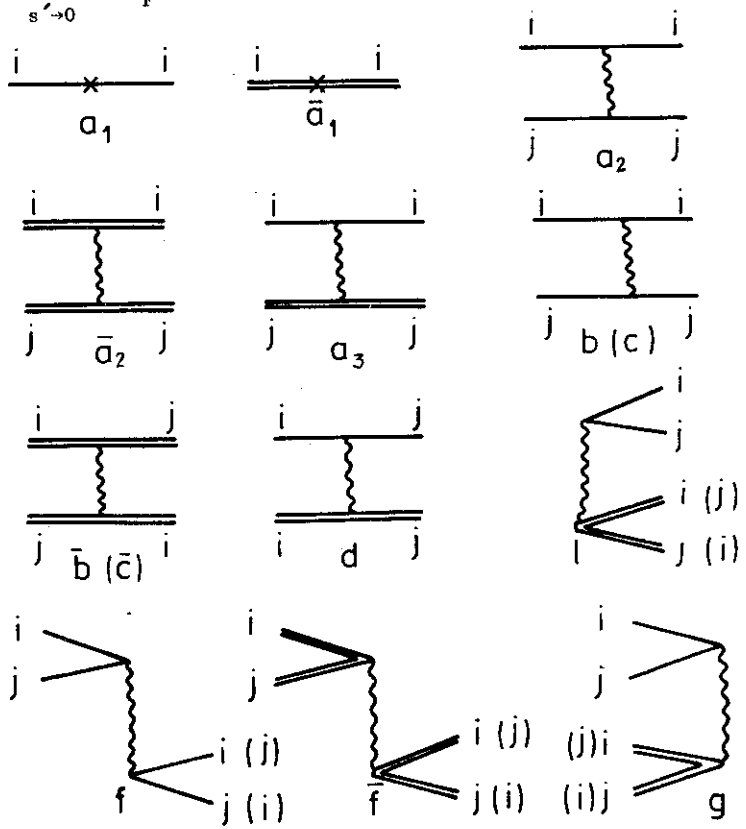


Fig.2. The vertices of the model: $\gamma^{a_1} = \gamma^{\bar{a}_1} - (\frac{1}{2\ell_2} + \frac{1}{2\ell_1})$ (mass operator, imaginary parts are ommitted because their contributions from retarded and advanced lines cancel); $\gamma^{a_2} = \gamma^{\bar{a}_2} = -\gamma^{a_3} = -1/N\ell_1$; $\gamma^b = \gamma^{\bar{b}} = \gamma^c = \gamma^{\bar{c}} = -1/N\ell_1$ (b intrapair scattering, c interpair scattering); $\gamma^d = 1/N\ell_1$; $\gamma^e = e^{4sz}/N\ell_2$; $\gamma^f = \gamma^{\bar{f}} = -1/N\ell_2$; $\gamma^g = e^{-4sz}/N\ell_2$.

$$K = \frac{1}{N} \sum_{ij} \langle G_{ij}(0,0; \epsilon_F + is') G_{ji}(0,0; \epsilon_F - is') \rangle . \qquad (14)$$

We now develop the diagram analysis for K, and introduce the following definitions.

Pair of lines: Any pair i,j has two lines, one from left to right (first index), second from right to left (second index). m_{ij} is the number of pairs i,j in a diagram at a coordinate z; the vertices provide for any z

$$M = \sum_{ij} m_{ij} = \sum_{ij} \bar{m}_{ij} . \qquad (15)$$

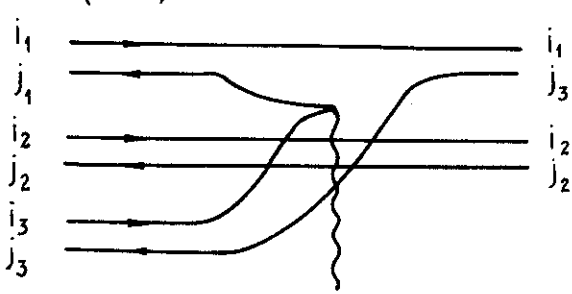
Left block $L\binom{z}{m_{ij}}$: Sum of all diagrams from $z = -\infty$ to z

$$L\binom{z}{0} = 1 \qquad (16)$$

by definition. To avoid the serious complications which arise for $N \gg 1$ if we follow Berezinskii's definitions[3] based on ordering of the lines for any z, we change the definitions of the blocks. Any diagram begins with vertex 2e; the vertices $2a_1, \ldots, 2d$ are adjoined in the same way as in[3]. If there appear further vertices 2e no ordering of the pairs is introduced. Additional closed loops (besides the basic retarded one advanced one corresponding to the considered correlator (14)) are avoided by the following rules for adjoining the vertices 2f, $2\bar{f}$, 2g: (i) The two lines of a pair are never coupled into one end of the impurity line. (ii) If one line of one pair is coupled to the line in opposite direction of a second pair, the two remaining lines represent a new pair. All possible couplings should be taken. These rules are illustrated in Fig.3.

We now transform

$$L\begin{pmatrix} z \\ \{m_{k\ell}\} \\ \{\bar{m}_{\bar{k}\bar{\ell}}\} \end{pmatrix} = e^{4sMz}\, \tilde{L}\begin{pmatrix} \{m_{k\ell}\} \\ \{\bar{m}_{\bar{k}\bar{\ell}}\} \end{pmatrix} \qquad (17)$$



Inspection of special diagrams shows that for an infinitely long system

Fig.3. Coupling of lines by backward scattering vertices.

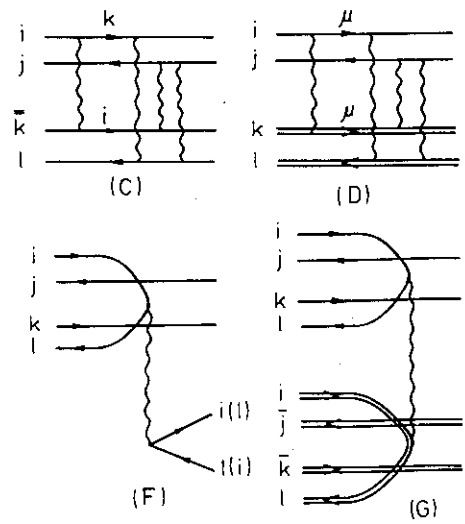


Fig.4. Diagrams for the derivation of the operators $\hat{C},\ldots,\hat{G}$:

The diagrams for the operators $\hat{A}$ (vertices, Fig.2, $a_1,\ldots,a_3$), $\hat{B}$ (vertices b, b), and $\hat{E}$ (vertex e) coincide with the corresponding ones in Fig.2. C) The four terms in (24) (two coincide) correspond to the four possible connections. F) In addition to the coupling of the lines i,ℓ also coupling of j,k should be taken; the two resulting terms coincide. G) Only one of the 8 possible couplings is shown.

$\tilde{L}$ does not depent on z. A differential equation for $\tilde{L}$ is obtained (as in [8]) going in the construction of $\tilde{L}$ from z to $z+dz$ by connecting the lines at z by vertices:

$$4s\hat{M}\Lambda = (\hat{A} + \hat{B} + \hat{C} + \hat{D} + \hat{E} + \hat{F} + \hat{G})\,\Lambda\,. \tag{18}$$

In order to give all formulae in a simple and short form operator formalism is used. The operators $\alpha^+(\beta^+)$ create retarded (advanced) pairs, $\alpha^-(\beta^-)$ annihilate pairs:

$$\begin{aligned} \alpha^+_{ij}\,|m_{ij}) &= |m_{ij}+1)\,, \\ \alpha^-_{ij}\,|m_{ij}) &= m_{ij}\,|m_{ij}-1)\,, \end{aligned} \tag{19}$$

and analogously for the advanced pairs (commutation between $\alpha^\pm,\alpha^\pm$ and $\beta^\pm,\beta^\pm$ and $\alpha^\pm,\beta^\pm$ for different indices). We introduce the generating functional

$$\Lambda = \sum_{\substack{\{m_{ij}\}=0 \\ \{\bar{m}_{ij}\}=0}}^{\infty} \tilde{L}\begin{pmatrix}\{m_{ij}\}\\ \{\bar{m}_{ij}\}\end{pmatrix}\begin{pmatrix} m_{ij} \\ \bar{m}_{ij}\end{pmatrix}\,, \tag{20}$$

and the operator

$$\hat{M} = \sum_{ij} \alpha^+_{ij}\,\alpha^-_{ij}\,. \tag{21}$$

The operators $\hat{A},\ldots,\hat{G}$ result from connecting the lines by the vertices (Fig.2a ,...,2g). The vertices $2a_1,\ldots,2a_3$ contribute to $\hat{A}$, the vertices 2b, 2b, to B:

$$\hat{A} = -\left\{\frac{1}{\ell_2} + (1 - \frac{1}{N})\frac{1}{\ell_1}\right\} \sum_{ij} a^+_{ij}\, a^-_{ij} \qquad \text{- advanced,} \qquad (22)$$

$$\hat{B} = -\frac{1}{N\ell_1} \sum_{ij} (1 - \delta_{ji})\, a^+_{ji}\, a^-_{ij} \qquad \text{- advanced.} \qquad (23)$$

The operators $\hat{C},\ldots,\hat{G}$ follow from the vertices 2c,...,2g as is shown in Fig.4.

$$\hat{C} = -\frac{1}{2N\ell_1} \sum_{\substack{ij \\ k\ell}} \Big\{(1-\delta_{ik})a^+_{kj}a^+_{i\ell} + 2(1-\delta_{i\ell})a^+_{\ell j}a^+_{ki} + \qquad (24)$$

$$+ (1-\delta_{j\ell})\, a^+_{i\ell}a^+_{kj}\Big\} a^-_{ij}\, a^-_{k\ell} \qquad \text{- advanced,} \qquad (25)$$

$$\hat{D} = \frac{1}{N\ell_1} \sum_{ij} \sum_{r} \Big\{\delta_{ik}(1-\delta_{ir})a^+_{rj}\beta^+_{r\ell} + \delta_{i\ell}(1-\delta_{ir})a^+_{rj}\beta^+_{kr} +$$

$$+ \delta_{jk}(1-\delta_{jr})a^+_{ir}\beta^+_{r\ell} + \delta_{j\ell}(1-\delta_{jr})a^+_{ir}\beta^+_{kr}\Big\} a^-_{ij}\beta^-_{k\ell}\,, \qquad (26)$$

$$\hat{E} = \frac{e^{4sz}}{N\ell_2} \sum_{ij} \Big[\delta_{ij}\, a^+_{ij}\, \beta^+_{ii} + (1-\delta_{ij})a^+_{ij}(\beta^+_{ij} + \beta^+_{ji})\Big]\,, \qquad (27)$$

$$\hat{F} = -\frac{1}{N\ell_2} \sum_{ij} \Big[(1-\delta_{i\ell})a^+_{kj}(a^+_{i\ell} + a^+_{\ell i}) + \delta_{i\ell}a^+_{kj}a^+_{ii}\Big] a^-_{ij}\, a^-_{k\ell} \quad \text{- advanced,} \qquad (28)$$

$$\hat{G} = \frac{e^{-4sz}}{N\ell_2} \sum_{ij} a^+_{kj}\, a^-_{ij}\, a^-_{k\ell} \Big[(1-\delta_{i\ell}) \sum_{\bar{k}\bar{j}} \beta^+_{\bar{k}\bar{j}} \Big\{\beta^-_{\bar{i}\bar{j}}\beta^-_{\bar{k}\ell} + \beta^-_{\bar{\ell}\bar{j}}\beta^-_{\bar{k}\bar{i}}\Big\} +$$

$$+ \delta_{i\ell} \sum_{\substack{k\ell \\ \bar{k}\bar{j}}} \beta^+_{kj}\beta^-_{\bar{i}\bar{j}}\beta^-_{\bar{k}i}\Big]\,.$$

4. SUMMATION OF THE DIAGRAMS FOR THE CORRELATION FUNCTION

The procedure of construction of L from $z = -\infty$ to $z = 0$ extends in the same way to $z > 0$. However for $z > 0$ we have to take into account that the retarded closed loop and the advanced one start and end at $z = 0$. Out of the four possibilities (Fig.5, 5a) we consider explicitly only the one shown in Fig.5. The extension of the considered procedure to $z > 0$ defines the block X. This block depends not only on the pairs $m'_{k\ell}$ at $\bar{m}'_{\bar{k}\bar{\ell}}$ at a coordinate $z > 0$ but also on the pairs $m_{k\ell}$, $\bar{m}_{\bar{k}\bar{\ell}}$ excisting at $z = 0$ (including the additional retarded pair $_{i,j}$ and advanced pair j,i). For $z' \to +0$ of course

$$X\left(\begin{array}{lll} z=0 & & z'\to+0 \\ \{m_{k\ell}\}\ ; & & \{m'_{k\ell}\} \\ \{\bar{m}_{\bar{k}\bar{\ell}}\} & & \{\bar{m}'_{\bar{k}\bar{\ell}}\} \end{array}\right) = \delta_{\{m_{k\ell}\},\{m'_{k\ell}\}}\,\delta_{\{\bar{m}_{\bar{k}\bar{\ell}}\},\{\bar{m}'_{\bar{k}\bar{\ell}}\}} \qquad (29)$$

The correlator is then given by

$$K = \frac{1}{N(hv_F)^2}\sum_{ij}\sum_{\substack{\{m_{k\ell}\}=0 \\ \{\bar{m}_{\bar{k}\bar{\ell}}\}=0}}^{\infty} L\left(\begin{array}{l} z=0 \\ \{m_{k\ell}\} \\ \{\bar{m}_{\bar{k}\bar{\ell}}\} \end{array}\right) \times$$

$$\times \sum_{rq}\int_0^\infty \frac{dz'}{N\ell_2}\, e^{-4sz'}\left[X\left(\begin{array}{lll} z=0 & & z' \\ \{m_{k\ell}+\delta_{k\ell}\}; & m'_{rq}=1 & \\ ij & & \\ \{\bar{m}_{\bar{k}\bar{\ell}}+\delta_{\bar{k}\bar{\ell}}\} & \bar{m}'_{rq}=1 & \\ ji & & \end{array}\right) + (1-\delta_{rq})X\left(\begin{array}{l} z' \\ ; m'_{rq}=1 \\ \bar{m}'_{rq}=1 \end{array}\right)\right] + \qquad (30)$$

+ further terms corresponding to Fig.5a.
(The notation $m'_{rq}=1$, $\bar{m}'_{rq}=1$ means that all other variables are zero; $\delta_{k\ell}=\delta_{ki}\delta_{\ell j}$). In analogy with (17) we transform

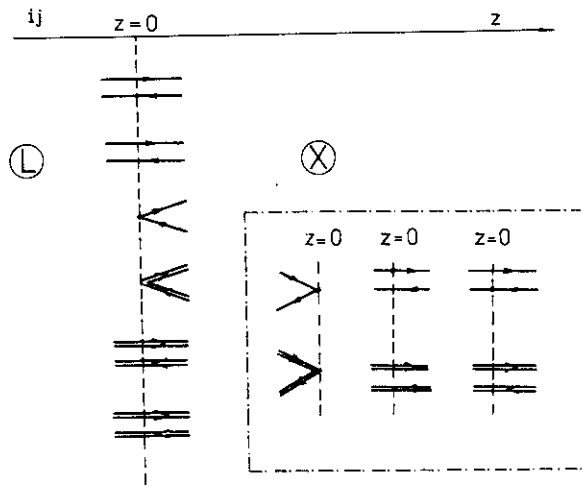


Fig.5. Pairs of lines at $z=0$. Each point • corresponds to a pair of operators $a\,(z=0)$, $a^+(z=0)$ from one Green function in (14). The insert 5a shows the 3 further possibilities.

$$X\begin{pmatrix} z' \\ ; \{m'_{rq}\} \\ \{\bar{m}'_{rq}\} \end{pmatrix} = e^{4sM'z'}\,\tilde{X}\begin{pmatrix} z' \\ ; \{m'_{rq}\} \\ \{\bar{m}'_{rq}\} \end{pmatrix}. \qquad (31)$$

Inspectation of special diagrams shows that for $\tilde{X}$ dependence on z remains. Because the construction of $\tilde{X}$ is the same as that for L, the r.h.s. of the differential equations are the same:

$$\{4s\hat{M}' + \frac{d}{dz'}\}\,\Xi(\;;z') = \{\hat{A}+\hat{B}+\ldots+\hat{G}\}\,\Xi(\;;z'\infty. \qquad (32)$$

The generating functional Ξ (with respect to the right variables) is related to $\tilde{X}$ in analogy with (20). As initial condition (analogous to (16)) we use

$$\tilde{X}\begin{pmatrix} z' \\ ; \{m'_{rq}=0\} \\ \{\bar{m}'_{rq}=0\} \end{pmatrix} = 0\,. \qquad (33)$$

Eq. (30) for the correlator K can be written in the form

$$K = \frac{1}{N(\hbar v_F)^2}\sum_{rq}\left\{F\begin{pmatrix} m'_{rq}=1 \\ \bar{m}'_{rq}=1 \end{pmatrix} + \left(1-\delta_{rq}\right)F\begin{pmatrix} m'_{rq}=1 \\ \bar{m}'_{rq}=1 \end{pmatrix}\right\} \qquad (34)$$

+ further terms corresponding to Fig.5a,

$$F\begin{pmatrix} \{m'_{rq}\} \\ \{\bar{m}'_{rq}\} \end{pmatrix} = \sum_{ij}\sum_{\substack{\{m_{k\ell}\} \\ \{\bar{m}_{k\ell}\}}}\tilde{L}\begin{pmatrix} \{m_{k\ell}\} \\ \{\bar{m}_{k\ell}\} \end{pmatrix}\int_0^\infty \frac{dz'}{N\ell_2}\,\tilde{X}\begin{pmatrix} 0 & z' \\ \{m_{k\ell}+\delta_{k\ell}\}_{ij} & \{m'_{rq}\} \\ \{\bar{m}_{k\ell}+\delta_{k\ell}\}_{ji} & \{\bar{m}'_{rq}\} \end{pmatrix} \qquad (35)$$

A similar method of attack was used in [3] (see there Eq. (26)).

We transform (32) into an equation for F by (i) applying to (32) the operations acting on $\tilde{X}$ in (35), (ii) using (33) at the lower integration limit, (iii) using that $\tilde{X}\to 0$ for $z'\to\infty$ $z'\to\infty$ and $m'_{rq}\neq 0$, $\bar{m}'_{rq}\neq 0$ (what can be shown by inspection of special diagrams), (iv) introducing for F a generating functional Φ in analogy with (20):

$$4s\hat{M}'\Phi - \frac{1}{N\ell_2}\sum_{ij}\alpha^+_{ij}\beta^+_{ij}\,\Lambda = \{\hat{A}+\hat{B}+\ldots+\hat{G}\}\,\Phi\,. \qquad (36)$$

In order to obtain more symmetrical differential equations (especially with respect to the terms $\hat{E}, \hat{G}$) we transform

$$\tilde{\tilde{L}}\left(\begin{matrix}\{m_{rq}\}\\ \{\bar{m}_{rq}\}\end{matrix}\right) = N^{2M}\prod_{rq}(m_{rq}!\,\bar{m}_{rq}!)\,\tilde{L}\left(\begin{matrix}\{m_{rq}\}\\ \{\bar{m}_{rq}\}\end{matrix}\right) \tag{37}$$

and analogously $F \to \tilde{F}$. This transformation can be performed by changing in all equations written in operator form from $a^{\pm}, \beta^{\pm}$ to $\tilde{a}^{\pm}$, $\tilde{\beta}^{\pm}$

$$\tilde{a}^{+}_{ij}\,\widetilde{(m_{ij})} = (m_{ij}+1)\widetilde{(m_{ij}+1)}\,,$$
$$\tilde{a}^{-}_{ij}\,\widetilde{(m_{ij})} = \widetilde{(m_{ij}-1)}\,, \tag{38}$$

$\tilde{a}^{-}(0) = 0$ (and analogously for the advanced pairs), and by adding to E the factor N^2 and to G the factor $1/N^2$.

5. TREATMENT OF THE N^2 VARIABLES m_{ij}

The generating functionals Λ, Φ are generalizations of the corresponding functionals in [3] to the case of many variables. To see this more explicitly it is convenient to use a special representation of the operators and states: $\tilde{a}^{+} = \zeta$; $\tilde{a}^{-} = \partial/\partial\zeta$ $(\tilde{m}) = \frac{1}{m!}(\zeta)^{m}$; analogously for $\tilde{\beta}$. In this representation the equations (18, 32) for the generating functionals take the form of partial differential equations with $2N^2$ independent varibles $\zeta_{ij}, \bar{\zeta}_{ij}$. One can show that these differential equations are of second order. Because we are only interested in the limit $s \to 0$ we can avoid the solution of these complicated differential equations in many variables by a much simpler method of attack. Instead of introducing the variables $\zeta_{ij}, \bar{\zeta}_{ij}$ we remain with the variables $m_{ij}, \bar{m}_{ij}$ or, more strictly, with certain linear combinations of them.

The variables $\vec{m} = \{\ ,\ ,\ m_{ij},\ \}$ (and $\vec{\bar{m}}$) satisfy the condition (15) which can be interpreted as a plane perpendicular to the vector

$$\vec{I} = \frac{1}{N^2}\{1, 1, \ldots, 1\}, \tag{39}$$

in the N^2-dimensional space of the m_{ij}. We introduce a complete system of orthogonal vectors $\vec{e}^{\,\rho}$ ($\rho = 1, \ldots, N^2 - 1$) on this plane. For any value of the main variable M we can write

$$\vec{m} = M\vec{I} + \sum \vec{e}^{\,\rho}\delta\mu^{\rho}, \tag{40}$$

analogously for $\vec{\bar{m}}$. $M\vec{I}$ is the vector from the origin to the centre of the plane (it describes the "homogeneous distribution" of the pairs among the indices), the second term descri-

bes the "general" deviation from the homogeneous distribution. As an example we consider the first of the four terms of $\hat{C}\Lambda$ in (18, 24):

$$C_1 = -\frac{1}{2N\ell_1} \sum_{\substack{ij \\ k\ell}} (1-\delta_{ik}) m_{kj} m_{i\ell} \left[(1-\delta_{j\ell}) \tilde{\tilde{L}} \begin{pmatrix} m_{ij}+1 \\ m_{k\ell}+1 \\ m_{kj}-1 \\ m_{i\ell}-1 \end{pmatrix} + \delta_{j\ell} \tilde{\tilde{L}} \right] - \quad (41)$$

– advanced.

$\hat{C}$ like any other vertex leads to "special" deviations which add to the general deviation, and which shall be described by vectors $\vec{\eta}^{\,\sigma} \begin{pmatrix} ij \\ k\ell \end{pmatrix}$ (σ numerates the vertices). In (41) only variables showing such special deviations are written explicitly. Because the special deviations also lie on the plane (15),

$$\vec{\eta}^{\,\sigma} \begin{pmatrix} ij \\ k\ell \end{pmatrix} = \sum_{\rho} a^{\sigma\rho} \begin{pmatrix} ij \\ k\ell \end{pmatrix} \vec{e}^{\,\rho} \quad . \quad (42)$$

The $\delta\mu^{\rho}$, $\delta\bar{\mu}^{\rho}$ are arbitrary variables. According to (40) they are restricted to such values that all components of $\vec{m}$ are integers; however in the following we generalize the $\delta\mu^{\rho}$, $\delta\bar{\mu}^{\rho}$ (like M) to continuous variables (compare (47) below).

We expand the functions $\tilde{\tilde{L}}$, $\tilde{\tilde{F}}$ at $\vec{m} = M\vec{I}$ (homogeneous distribution) with respect to the general and special deviations into Taylor series. It is convenient to introduce derivatives with respect to the directions $\vec{e}^{\,\rho}$ and $\vec{\eta}^{\,\sigma}$:

$$\partial_{\rho} = \sum_{ij} e_{ij}^{\rho} \frac{\partial}{\partial m_{ij}}, \qquad \bar{\partial}_{\rho} = \sum_{ij} e_{ij}^{\rho} \frac{\partial}{\partial \bar{m}_{ij}},$$

$$\partial_{\sigma} = \sum_{ij} \eta_{ij}^{\sigma} \frac{\partial}{\partial m_{ij}}, \qquad \bar{\partial}_{\sigma} = \sum_{ij} \bar{\eta}_{ij}^{\sigma} \frac{\partial}{\partial \bar{m}_{ij}}. \quad (43)$$

The equations are symmetrical with respect to mirror reflections: $\vec{e}^{\,\rho'} \to -\vec{e}^{\,\rho'}$, $\vec{e}^{\,\rho} = \vec{e}^{\,\rho}$ ($\rho \neq \rho'$); this can also be seen by direct calculations. Therefore, all derivatives changing sign under such a transformation vanish. The Taylor series for $\tilde{\tilde{L}}$ is

$$\tilde{\tilde{L}} \begin{pmatrix} M\vec{I} + \sum_{\rho} \vec{e}^{\,\rho} \delta\mu^{\rho} + \vec{\eta}^{\,\sigma} \\ M\vec{I} + \sum_{\rho} \vec{e}^{\,\rho} \delta\bar{\mu}^{\rho} + \vec{\eta}^{\,\sigma} \end{pmatrix} = \tilde{\tilde{L}}(M) + \quad (44)$$

$$+ \frac{1}{2} \sum_{\rho} \left[(\delta\mu^{\rho} + a^{\sigma\rho}) \partial_{\rho} + (\delta\bar{\mu}^{\rho} + \bar{a}^{\sigma\rho}) \bar{\partial}_{\rho} \right]^2 \tilde{\tilde{L}}(M) + \text{higher order terms.}$$

The derivatives, e.g., $\partial_{\rho}^2 \tilde{\tilde{L}}(M)$, depend on M.

We substitute (44) into (18) and the analogous series for F into (36). Because the $\delta\mu^{\rho}$, $\delta\bar{\mu}^{\rho}$ are arbitrary variables, the Eqs. (18, 36) are equivalent to an infinite system of linear equations for L(F) and their derivatives to all orders. The system is of the type

$$\begin{aligned} &\tilde{L}(M) - M(\tilde{L}^{II} + \tilde{L}^{IV} + \ldots) = 0, \\ &\tilde{L}^{II}(M) - M(\tilde{L}^{IV} + \tilde{L}^{VI} + \ldots) = 0, \end{aligned} \tag{45}$$

where numerical coefficients of the order 1 (depending also on N) are omitted. The equation obtained in zeroth order in the $\delta\mu$ is not included in (45); it is the main equation, and it shall be treated below. Because the essential domain is that of large M (as in the one-dimensional case[3]) the system (45) can be simply solved, giving for the derivatives the order of magnitude

$$\tilde{L}^{(2n)} \approx \tilde{L} M^{-n}. \tag{46}$$

The vanishing of derivatives noninvariant with respect to the mentioned mirror reflection symmetry can also be proved in such a way.

As the further treatment shows (see (52,53)) the natural variables are

$$\begin{aligned} &p = N\ell_2 sM, \\ &\delta\mu'^{\rho} = \sqrt{NsM}\,\delta\mu^{\rho}, \end{aligned} \tag{47}$$

and the Taylor series rapidly converges for large as well as for small M*.

The zeroth order equation from (18) reads

$$4sM\tilde{L}(M) = A + B + C + D + E + F + G, \tag{49}$$

where A, B, ... correspond to $\hat{A}\Lambda$, $\hat{B}\Lambda$, ... As an example we write down C_1 (Eq. (41)):

$$-\frac{1}{2N\ell_1}\sum_{\substack{ij\\k\ell}}(1-\delta_{ik})\left(\frac{M}{N^2}\right)^2\left[\tilde{L}(M) + (1-\delta_{j\ell})\frac{1}{2}\sum_{\rho}(a^{C_1,\rho}\partial_{\rho})^2\tilde{L}(M) + \right. \tag{50}$$

$$\left. + \text{higher order terms}\right] - \text{advanced}.$$

$a^{C_1,\rho}$ corresponds to $\vec{\eta}^{C_1}$ (to the special deviations in (41)).

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* The order of magnitude of the essential values of the $\delta\mu^{\rho}$ can be obtained from the statistical relation between the averages: $\langle(\delta\mu^{\rho})^2\rangle = \langle m_{ij}\rangle = \frac{M}{N^2}$. (48)

6. MAIN DIFFERENTIAL EQUATIONS

Evaluation of all terms in (49) leads to

$$4sM\tilde{L}(M) = \frac{1}{N\ell_2}(2-\frac{1}{N})M^2\{\tilde{L}(M-1)+\tilde{L}(M+1)-2\tilde{L}(M)\}, \qquad (51)$$

if we at first neglect all second (and higher order) derivatives with respect to ρ, which are present in all terms (see, e.g., (50)). Introducing we obtain (in the limit $s\to 0$)

$$4p\tilde{L}(p) = (2-\frac{1}{N})\frac{p^2}{(N\ell_2 s)^2}\{\tilde{L}(p-N\ell_2 s)+\tilde{L}(p+N\ell_2 s)-2\tilde{L}(p)\} =$$

$$= (2-\frac{1}{N})p^2\frac{d^2\tilde{L}(p)}{dp^2}. \qquad (52)$$

Inclusion of the second order derivatives leads to additional coefficients of the type $(1+a(N)/M)$ at any term in the curly brackets in (51). These terms do not violate the crucial effect that after introducing p the "adiabatic" parameter s no more appears explicitly. (However the form the second order differential equations changes). Therefore, these terms do not violate localization for finite N. All details of the calculations will be given in a following paper.

The analogous procedure for F gives

$$4p\tilde{F}(p) - \frac{p^2}{(N\ell_2 s)^2}\tilde{L}(p) = \left(2-\frac{1}{N}\right)p^2\frac{d^2\tilde{F}(p)}{dp^2}. \qquad (53)$$

Again, after the introduction of p the s disappeared (with the natural exclusion of the prefactor of the inhomogeneous term). Eq. (52) and the homogeneous part of (53) can be transformed to the standard Bessel's equation. The solutions, satisfying the necessary boundary conditions ($\tilde{L}(p)\to 1$, $\tilde{F}(p)\to 0$ for $p\to 0$ compare (16,33); $\tilde{L}(p)\to 0$, $F(p)\to 0$ for $p\to\infty$) are

$$\tilde{L}(p) = 2\sqrt{\alpha p}\,K_1(2\sqrt{\alpha p}), \qquad (54)$$

$$\tilde{F}(p) = \beta p \quad \text{for} \quad p\to 0; \quad \text{for} \qquad (55)$$

$$\alpha = \frac{4}{2-1/N}\approx 2; \qquad \beta = \frac{1}{12(N\ell_2 s)^2}. \qquad (56)$$

The procedure already applied to (18,36) gives for the correlator (34)

$$K = \frac{8}{(hv_F)^2}\tilde{F}(p)/p = N\ell_2 s = \frac{2}{3hv_F N\ell_2 s'}, \qquad (57)$$

($p = N\ell_2$ corresponds to $M=1$).

According to (13) the $1/s$-singularity in (57) manifests localization. The conditional probability density that the electron remains in the state at $z=0$ with width Δz is

$$P = \frac{2}{3N\ell_2}. \tag{58}$$

7. CONCLUSIONS

Comparing our main equations (51-53) which prove on a microscopic basis localization for the case of N weakly coupled chains (N finite) with the corresponding equations for the 1D case we note:

(i) The characteristic length is $N\ell_2$ instead of ℓ_2; (58) shows that the localization length $L_{loc} \approx N\ell_2$ in accordance with [4,5].

(ii) The prefactor of $p^2 d^2/dp^2$ is $2-1/N$ instead of 1 for the 1D case.

(iii) The situation is, however, different in the limit $N \to \infty$, which according to the conditions (7a,7b) can be performed in our model for $d=3$, but not for $d=2$. In this limit ($N \to \infty$), but $s \to 0$ and therefore M limited) the leading terms are that of order MN not containing interference terms (and not that of order M^2 as in (51-53)). Then from (49) follows

$$4sM\tilde{L}'(M) = \frac{M}{\ell_2}\{\tilde{L}'(M-1) + \tilde{L}'(M+1) - 2\tilde{L}'(M)\} \tag{59}$$

instead of (52). The quantity $\tilde{L}'(M)$ corresponds to $\tilde{\tilde{L}}_M(M)$ however in the transformation (37) the prefactor is N^M instead of N^{2M}. Introducing $p' = M\sqrt{\ell_2 s}$ gives

$$4\tilde{L}'(p') = \frac{d^2\tilde{L}'(p')}{dp'^2}. \tag{60}$$

Then for the correlator results

$$K \sim \sum_{M=0}^{\infty} \tilde{L}'^2(M) \sim \frac{1}{\sqrt{s}} \int dp' \tilde{L}'^2(p'), \tag{61}$$

where the $1/\sqrt{s}$-singularity manifests diffusions *. Concerning $d=2$ we have to include oscillating vertices, what means more scattering.

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* The singularity $1/\sqrt{s}$ corresponds to $1/\sqrt{t}$; we obtain this 1D law for the diffusion because we are considering only the behaviour along z (in (12) the states in the cross section are summed over).

(iv) We remark that we can get our result (57) in a very simple, but crude way: We consider the left part and the right part (identical to the $z=0$ part) satisfying (51) (without special deviations), and connect the lines at $z=0$ in any possible way, ignoring all the serious difficulties connected with the indexing of the lines (which forced us to go via (30) and Sec. 5). With the difference that p appears instead of p' this is then similar to the consideration in (61) for diffusion (where the mentioned difficulties do not arise). The s in p (instead of $\sqrt{s}$ in p') then gives the $1/s$ -singularity of the correlator. In this way it is very clearly seen that also in the case of N chains the interference terms change the singularity of the correlator (like in the 1D case).

Our general scheme allows for generalization of the model to include oscillating vertices, what will be investigated in a forthcoming paper.

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