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CNOIDAL WAVES IN THE MODEL φ^4 WITH A CURRENT-CURRENT TYPE SELF-ACTION

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In recent years the study of nonlinear effects has become still more and more actual. The understanding of many phenomena of the nonlinear origin is of fundamental importance for elementary particle physics, nonlinear optics, solid-state physics, that of plasma.

The nonlinearity is naturally considered to be weak if changes it causes in the wave amplitude are slow as compared to fast harmonic oscillations. Otherwise the strong nonlinearity is meant.

When nonlinear dynamic equations are studied in the case of weak nonlinearity, there arise difficulties due to the presence of secular terms. These difficulties are eliminated by using either the Bogolubov-Krylov general method [1,2] or the method of many-time successive approximations which is in fact a modification of the Bogolubov-Krylov method.

This paper is a sequel to ref. [3] dealing with the study of the model with nonlinear interaction of currents. This representation first, is useful in view of a simple interpretation of nonlinear interactions on the basis of exact solutions; second, it allows one to include higher-order dispersion effects and to consider the nonlinear properties without restricting them to be small.

Analytic solutions obtained here for the given model are expressed in terms of elliptic functions, a particular case of which is soliton solutions [3].

Following ref. [3] we consider the model with the Lagrangian

$$L = \phi_{,\mu}\ \phi^{*}_{,\mu} - m^{2}(\phi\phi^{*}) + g_{1}(\phi\phi^{*})^{2} - g_{2}J_{\mu}J^{\mu}$$
$$J_{\mu} = \frac{i}{2}\phi^{*}\overleftrightarrow{\phi}_{,\mu}, \tag{1}$$

where

$$\phi_{,\mu} \equiv \frac{\partial\phi}{\partial x_{\mu}}, \quad \phi_{,\mu} = g_{\mu\nu}\frac{\partial^{2}\phi}{\partial x_{\mu}\partial x_{\nu}}, \quad \mu = 0,1 .$$

Here the metric from [3] is used with the signature

$$g_{\mu\nu} = \mathrm{diag}\{1, -1\}$$

$$a_{\mu}b^{\mu} = g_{\mu\nu}a_{\mu}b_{\nu} = a_{0}b_{0} - a_{1}b_{1} .$$

In what follows we shall consider the Lagrangian (1) on the class of harmonic functions with the use of the Lorentz-invariant representation

$$\phi = \rho(\theta) \cdot e^{ip_\mu x^\mu} . \tag{2}$$

Here $x_\mu=(t,x)$, $u_\mu=(\gamma, v\gamma)$, $p_\mu=(\omega\gamma, \omega v\gamma)$ are Lorentz coordinates squares of which are invariants with the corresponding normalization

$$u_\mu u^\mu = 1, \quad p_\mu p^\mu = \omega^2;$$

the contractions are: 1) $\theta = \epsilon_{\mu\nu} u_\mu x_\nu = \gamma(x - vt)$, which defines the shift; 2) $p_\mu x^\mu = \omega\gamma(t - vx)$, the invariant phase, where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ is the fully antisymmetric tensor, $\epsilon_{01} = 1$, $\gamma = 1/\sqrt{1-v^2}$.

If one uses (2), the Lagrangian (1) takes the form

$$L_{\text{эфф}} = -(\rho_\theta)^2 - V(\rho),$$

$$V(\rho) = -\tilde{g}[\rho^2 - (\frac{\tilde{m}^2}{2\tilde{g}})]^2 - \frac{\tilde{m}^4}{4\tilde{g}}, \quad \tilde{g} = g_1 + g_2 p_\mu^2 , \tag{3}$$

$$\tilde{m}^2 = m^2 - p_\mu^2 .$$

In deriving (3) we have used the formula

$$|\phi_{,\mu}|^2 = \frac{1}{2}(\phi_{,\mu} g_{\mu\nu} \phi^*_{,\nu} + \phi^*_{,\mu} g_{\mu\nu} \phi_{,\nu}) = -u_\nu^2 (\rho_\theta)^2 + p_\mu^2 \rho^2$$

at $u_\nu^2 = 1$, $|\phi_{,\mu}|^2 = -(\rho_\theta)^2 + p_\mu^2 \rho^2$.

The equation of motion corresponding to (3) reads

$$\rho_{\theta\theta} - \frac{1}{2} \frac{\partial V}{\partial \rho} = 0. \tag{4}$$

It is evident that the equation obtained from the Lagrangian (1)

$$(\Box + m^2)\phi = 2g_1 |\phi|^2 \phi + g_2 \cdot \frac{\delta(J_\mu J^\mu)}{\delta\phi^*} \tag{4a}$$

transforms into (4) if (2) is substituted into (4a). Therefore these two equations are adequate. Multiplying (4) by ρ_θ, taking into account that $\frac{1}{2}(\rho_\theta)^2 = \rho_\theta \cdot \rho_{\theta\theta}$ and $\frac{\partial V}{\partial \rho} \rho_\theta = V_\theta$ we obtain finally equation (4) in the form

$$(\rho_\theta)^2_\theta - V_\theta = 0. \tag{5}$$

Equation (5) is easily integrated. The general solution to (5) can be written as

$$\theta = \int \frac{d\rho}{\sqrt{C - U(\rho)}}, \tag{6}$$

where $U(\rho) = -V(\rho)$, C is the integration constant. The solution (6) can be interpreted as a certain "classical" motion in the field of the nonlinear effective potential $U(\rho)$.

The picture of trajectories on the phase plane $(\pi = \rho_\theta, \rho)$ is shown in Fig.1: various cases of the finite motion are drawn versus the position of the constant c.

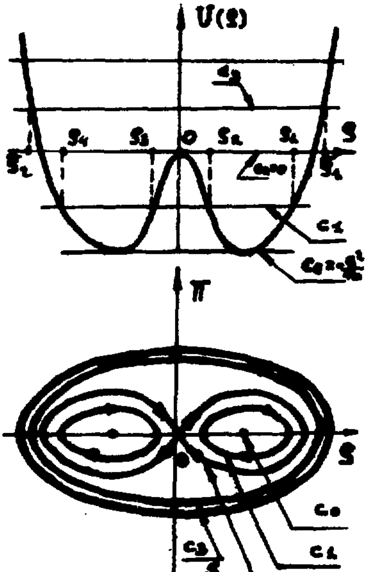

Fig.1. ρ_1, ρ_2, ρ_3, ρ_4 are rotation points of the finite motion; C_0, C_1, C_2, C_3 are planes of the cross section; C=const is the limiting cross section, in this case the trajectory degenerates into a point; in case $C_2 = 0$ the cross section has a trajectory, the separatrix S (its rotation point ($\pi = 0$, $\rho = 0$) is the bifurcation point).

a) Consider the case $C_2 = 0$ (see Fig.1). Expression (6) takes the form

$$\theta = \int \frac{d\rho}{\rho\sqrt{a - b\rho^2}},$$

where

$$a = m^2 - p^2,$$

$$b = g_1 + g_2 p_\mu^2 .$$

Upon integrating we obtain

$$\theta = \frac{1}{2(m^2 - p_\mu^2)^{1/2}} \cdot \ln\frac{\sqrt{a} - \sqrt{a - b\rho^2}}{\sqrt{a} + \sqrt{a - b\rho^2}},$$

or, reversing this equation for the function $\rho = \rho(\theta)$, we arrive at the well-known soliton solution [3] with natural boundary conditions $\{\Gamma: \rho|_{\theta\to\infty} = 0\}$

$$\rho(\theta) = \left(\frac{m^2 - p_\mu^2}{g_1 + g_2 p_\mu^2}\right)^{1/2} \mathrm{sech} \cdot \{(m^2 - p_\mu^2)^{1/2} \cdot \theta\}. \tag{7}$$

b) Consider now $C=C_1$, where $C_2 \geq C_1 \geq C_0$ (see Fig. 1). We shall assume that the polynomial in (6)

$$P_4 = C - U = -b\rho^4 + a\rho^2 + C \tag{8}$$

has the roots $\rho_1, \rho_2, \rho_3, \rho_4$ all real. Due to symmetry of the polynomial $P_4(\rho)=P_4(-\rho)$ the roots $\rho_1=-\rho_4, \rho_2=-\rho_3$ and are given by the formulae

$$\rho_1^2 = \frac{a+\sqrt{a^2+4bC}}{2b}, \quad \rho_2^2 = \frac{a-\sqrt{a^2+4bC}}{2b}. \tag{9}$$

Therefore (6) can be written as

$$\sqrt{b}\cdot\theta = \int^{\rho(\theta)} \frac{d\rho}{\sqrt{(\rho_1-\rho)(\rho-\rho_2)(\rho-\rho_3)(\rho-\rho_4)}} = g\cdot \mathrm{sn}^{-1}(\sin\phi, k), \tag{10}$$

where

$$g = 2/(\rho_1+\rho_2).$$

Inverting this integral we express the function $\rho=\rho(\theta)$ in terms of the Jacobi elliptic functions /4/.

Thus, for instance, for the interval

$$\rho_2 \leq \rho \leq \rho_1$$

we obtain the following solution

$$\rho(\theta) = \rho_1 \frac{[1-k\cdot \mathrm{sn}^2(\frac{\sqrt{b}}{g}\theta, k)]}{[1+k\cdot \mathrm{sn}^2(\frac{\sqrt{b}}{g}\theta, k)]}, \tag{11}$$

where $k^2=(\frac{\rho_1-\rho_2}{\rho_1+\rho_2})^2$ is the modulus of the elliptic function. If one uses the Gauss transformation formulae for elliptic functions /4/, the solution can be finally written as

$$\rho(\theta) = \rho_1 \cdot \mathrm{dn}\,(\frac{\sqrt{b}}{g}(1+k)\cdot\theta,\ k_1). \tag{12}$$

Here the function modulus is defined by $k_1 = \frac{2\sqrt{k}}{1+k} = \frac{\sqrt{\rho_1^2-\rho_2^2}}{\rho_1}$

$0 \leq k_1 \leq 1$. From periodicity of the function (12) $\mathrm{dn}(u+2K(k_1),k_1) = \mathrm{dn}(u, k_1)$ it follows that the period $T_u = 2K(k_1)$ is defined by the complete elliptic integral of the first kind $K(k_1) = F(\pi/2, k_1)$. Then the period T_θ for variable θ is given by the condition

$$\frac{\sqrt{b}}{g}(1+k)\cdot T_\theta = T_u \tag{13}$$

or

$$T_\theta = \frac{g}{(1+k)\sqrt{b}} T_u = \left(\frac{2}{a+\sqrt{a^2+4bC}}\right)^{1/2} \cdot T_u ,$$

where

$$T_u = 2 \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k_1^2 \sin^2\phi}} .$$

Obviously, for the constant $C=C_1$, $-\frac{a^2}{4b}<C_1\leq 0$ there is one more interval of the finite motion, namely, for

$\rho_4 \geq \rho \geq \rho_3$.

In this case the solution is represented by the function

$$\rho(\theta) = \rho_4 \cdot \mathrm{dn}\left(\frac{\sqrt{b}}{g}(1+k)\,\theta,\ k_1\right) . \qquad (14)$$

Solutions (12), (14) can be interpreted as classical motion in the field of the nonlinear effective potential $U(\rho)$ (fig.2a). It is to be noted that solutions (12), (14) in the limiting case at $k_1 = \frac{2\sqrt{k}}{1+k} = 1$ or $k=1$ pass into the soliton solution (7). Really, with the connection between hyperbolic functions $\mathrm{dn}(u,1) = \mathrm{sech}\,u$ we have

$$\rho(\theta) = \pm\rho_1 \cdot \mathrm{dn}\left(\frac{2\sqrt{b}}{g}\theta,\ 1\right) = \pm\sqrt{\frac{a}{b}}\,\mathrm{sech}\,(\sqrt{a}\cdot\theta),$$

that is, equivalent to the transition of the cross section from the position $C=C_1$ to $C=C_2=0$ (Fig.2b). Therefore, on the trajectory $S(C_2=0)$, because from (13) the period $T_u=\infty$ at $k_1=1$, functions of the delta amplitude (12), (14) degenerate into solitons (7).

Consider another limiting case $k_1=0$. It is equivalent to the lowest position of the plane $C=C_0=-\frac{a^2}{4b}$ (Fig.1). In this case, since $\mathrm{dn}(u,0)=1$, the solutions (12), (14) degenerate into constants ρ_1, ρ_4 : $\rho = \left\{ \begin{matrix} \rho_1 \\ \rho_4 \end{matrix} \right\}$. Allowing for the phase-dependence of the solution (2) we have

$$\phi = \left\{ \begin{matrix} \rho_1 \cdot \mathrm{dn}\left(\frac{2\sqrt{b}}{g}\theta,\ k_1\right) \\ \rho_4 \cdot \mathrm{dn}\left(\frac{2\sqrt{b}}{g}\theta,\ k_1\right) \end{matrix} \right\} e^{ip_\mu x^\mu} . \qquad (15)$$

We see that essential nonlinear waves of the form (15) in the limiting case $k_1=0$ turn into the vacuum condensate (plane waves) $\phi = \left\{ \begin{matrix} \rho_1 \\ \rho_4 \end{matrix} \right\} \Big|_{C=-a^2/4b} \cdot e^{ip_\mu x^\mu}$ with the period of linear oscillations given by the formula

$$T_\Lambda = \frac{2\pi}{\omega\gamma} \tag{16}$$

c) Consider next all possible positions of planes

$C = C_3 > 0$

(Figs.1,2b). In this variant the polynomial (8) has two real roots

$$\tilde{\rho}_{1,2} = \pm\left(\frac{a+\sqrt{a^2+4bC}}{2b}\right)^{1/2} \tag{17}$$

and two complex conjugate roots $z, \bar{z}$, where

$$z = \left(\frac{a-\sqrt{a^2+4bC}}{2b}\right)^{1/2} \quad \text{at} \quad (a^2+4bC)^{1/2} > a.$$

The solution (6) can be written as

$$\sqrt{b}\cdot\theta = \int^{\rho(\theta)} \frac{d\rho}{\sqrt{(\tilde{\rho}_1-\rho)(\rho-\tilde{\rho}_2)[(\rho-b_1)^2+a_1^2]}} = g\,cn^{-1}(\cos\phi, k) \tag{18}$$

with the notation

$$b_1 = \frac{z+\bar{z}}{2},\quad a_1^2 = -\frac{(z-\bar{z})^2}{4},\quad A^2 = (\tilde{\rho}_1-b_1)^2+a_1^2,$$
$$B^2 = (\tilde{\rho}_2-b_1)^2+a_1^2,\quad g = 1/\sqrt{AB}. \tag{19}$$

Inverting the integral (18) we obtain the solution $\rho = \rho(\theta)$

$$\rho(\theta) = \frac{A\tilde{\rho}_2[1+cn(\frac{\sqrt{b}}{g}\theta,k)] + B\tilde{\rho}_1[1-cn(\frac{\sqrt{b}}{g}\theta,k)]}{A[1+cn(\frac{\sqrt{b}}{g}\theta,k)] + B[1-cn(\frac{\sqrt{b}}{g}\theta,k)]}, \tag{20}$$

With the notation (19) the solution (20) for $\tilde{\rho}_2 \le \rho \le \tilde{\rho}_1$ takes the form

$$\rho(\theta) = \tilde{\rho}_2\cdot cn(\frac{\sqrt{b}}{g}\theta, k). \tag{21}$$

Frequency properties of the solution (21) are determined by the formulae

$$\nu = \frac{1}{T_\theta},\quad T_\theta = \frac{g}{\sqrt{b}}\cdot T_u = \frac{1}{(a^2+4bC)^{1/4}}\cdot T_u, \tag{22}$$

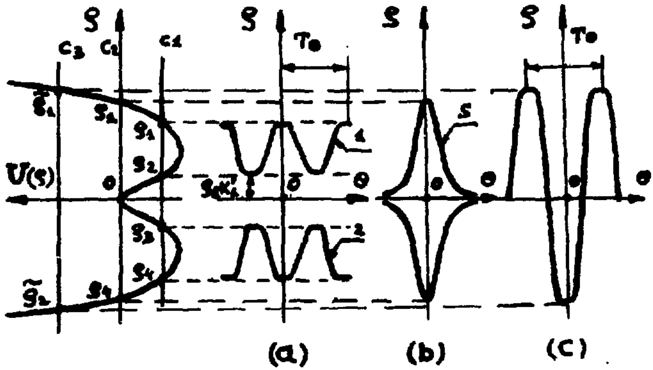

Fig.2. a) Plane C takes the position $C = C_1 < 0$, 1 is the solution in the form of nonlinear waves $\rho = \rho_1 \cdot dn(\frac{\sqrt{b}}{g}\theta(1+k), k_1)$ for $\rho_2 \le \rho \le \rho_1$, 2 is the analogous solution for the finite interval $\rho_4 \le \rho \le \rho_3$ in the form $\rho = \rho_4 dn(\frac{\sqrt{b}}{g}(1+k)\theta, k_1)$.
b) The case $C = C_2 = 0$, the solution on the trajectory, separatrix, in the form of soliton S with period $T_s = \infty$.
c) The case $C = C_3 > 0$, the solution is given by "cnoidal" waves $\rho = \tilde{\rho}_2 \cdot cn(\frac{\sqrt{b}}{g}\theta, k)$ for $\tilde{\rho}_2 \le \rho \le \tilde{\rho}_1$. Here $\rho_1 \cdot k_1' = \rho_1\sqrt{1-k_1^2} = \rho_2$ is an extra modulus of the elliptic function dn which coincides with the value of the second root ρ_2.

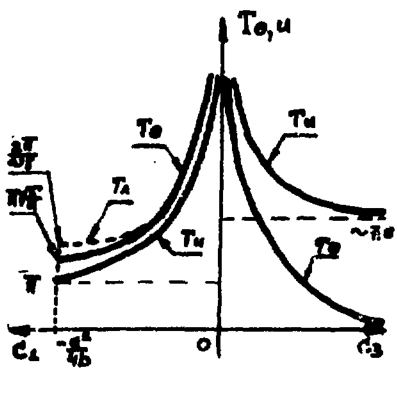

Fig.3. The periods T_θ and T_u of the nonlinear oscillations as functions of the parameter C. In the limit case $C = -a^2/4b$ (point formation of the condensate) the nonlinear waves tend to the corresponding linear waves with the period $T_\Lambda = 2\pi/\omega\gamma$.

where

$$T_u = 4 \cdot \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}},$$

$$k^2 = \frac{(\tilde{\rho}_1 - \tilde{\rho}_2)^2 - (A - B)^2}{4AB} = \frac{(a + \sqrt{a^2 + 4bC})}{2\sqrt{a^2 + 4bC}}.$$

In contrast to the functions (12), (14) the frequency properties (22) for the solution (21) in the interval $\tilde{\rho}_2 \le \rho \le \tilde{\rho}_1$ and for $0 \le C < \infty$ can change in a wide range of frequencies $0 \le \nu < \infty$ (see Fig.3). Indeed, $k^2 = \frac{(a/C^2 + \sqrt{a^2/C + 4b})}{2\sqrt{(a^2/C + 4b}}\Big|_{C \to \infty} = \frac{1}{2}$,

$$T_u = 4 \cdot \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} = 7.6$$

is bounded and

$$\nu = \frac{(a^2 + 4bC)^{1/4}}{T_u}\Big|_{C \to \infty} = \infty.$$

The complete picture of the solutions (12,14,21) versus the position of planes $C = C_1, C_2, C_3$ is shown in Fig.2. From this Figure it is seen how the solutions are transformed with the change of the position of planes for $\infty < C \le -a^2/4b$.

The frequency properties of the solutions (12,14,21) are drawn in Fig.3, where the periods of nonlinear waves for the corresponding positions of planes C are defined by the formulae:

for $C = C_1$, $\quad 0 \ge C_1 \ge -\frac{a^2}{4b}$

$$T_\theta = \left(\frac{2}{a + \sqrt{a^2 + 4bC_1}}\right)^{1/2} \cdot T_u, \quad T_u = 2 \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k_1^2 \sin^2\phi}}, \tag{23}$$

where

$$k_1^2 = \frac{2\sqrt{a^2 + 4bC_1}}{a + \sqrt{a^2 + 4bC_1}}, \quad 0 \le k_1^2 \le 1$$

for $C = C_3$, $\quad 0 \le C_3 < \infty$,

$$T_\theta = \frac{1}{(a^2 + 4bC_3)^{1/4}} \cdot T_u, \quad T_u = 4 \cdot \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}, \tag{24}$$

where

$$k^2 = \frac{a + \sqrt{a^2 + 4bC_3}}{2\sqrt{a^2 + 4bC_3}}, \quad \frac{1}{2} \le k^2 \le 1 .$$

Here constants a and b are given by parameters of the starting model (1) and equal $a = m^2 - p_\mu^2$, $b = g_1 + g_2 p_\mu^2$.

As we may conclude from formulae (16), (23) and Fig.3, an important fact seems to be that in the limiting position $C = C_1 = -\frac{a^2}{4b}$ we obtain the limiting value of the period $T_\theta = \pi\sqrt{2/a}$, but in this case the solutions (12, 14) degenerate into constants ρ_1, ρ_4.

Formally, for such "constant" solutions periods T_θ may be any numbers. If we consider the general solutions (15), then in the limiting case $k_1 \to 0$, or, what is the same, $C_1 = -a^2/4b$, essentially nonlinear waves of the type (15) turn into the plane waves $\phi = \{{\rho_1 \atop \rho_4}\}\Big|_{C=-a^2/4b} \cdot e^{ip_\mu x^\mu}$ with the period of linear oscillations $T_\Lambda = \frac{2\pi}{\omega\gamma}$. Therefore it is to be assumed that the curve T_θ (Fig.3) reaches the period of linear oscillations $T_\Lambda = \frac{2\pi}{\omega\gamma}$ in the critical point $C_1 = -\frac{a^2}{4b}$ rather than the period $T = \pi\sqrt{2/a}$, as it follows from formulae (23).

Besides, the roots of polynomial P in the integral (10) become degenerate: at $C_1 = -\frac{a^2}{4b}$, $\rho_1 = \rho_2$, $\rho_3 = \rho_4$. Therefore in the integrand of (10) one may consider only the limiting transition as $C_1 \to -a^2/4b$ or $k_1 \to 0$. On the whole, the expression (10) is valid and gives correct results.

Full information on the existence of solutions (12), (14), (21) as functions of the parameter C, $\rho_c = \rho(c)$, is shown in Fig.4 for various possible values of C.

Boundaries of the regions of existence of solutions as functions of parameter C, $\rho_{1,2}^{(c)}$, $\rho_{3,4}^{(c)}$, $\tilde{\rho}_{1,2}^{(c)}$ are given by formulae (9), (17).

As a result of studies of the given work, solutions of a more general type are found to be nonlinear waves expressed in terms of elliptic functions.

The solutions constructed include solitons and condensate states (plane waves) as particular cases.

Upon studying the properties of these solutions a sufficiently complete picture is obtained on the evolution of the given system in spite of the absence of its complete integrability.

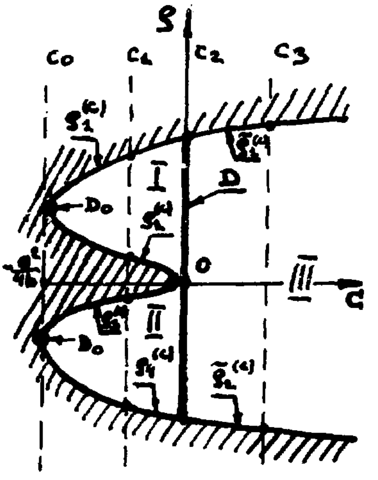

Fig.4. I,II are regions of nonlinear waves of the form $\rho = \{\rho_1, \rho_4\} \mathrm{dn}\,\theta$. III is the region of nonlinear waves of the form $\rho = \tilde{\rho}_2 \cdot \mathrm{cn}\,\theta$, D is the boundary which separates regions I and II from III and forms solitons, D_0 are boundary points where linear waves $\phi = \{\rho_1, \rho_4\} \Big|_{C=-a^2/4b} e^{ip_\mu x^\mu}$ are formed. The shaded region is a zone forbidden for given solutions.

REFERENCES

1. Bogolubov N.N., Mitropolski Yu.A. Asimptoticheskie metody v teorii nelineinyh kolebanii (in Russian). "Nauka", M., 1974.
2. Bogolubov N.N., Krylov N.M. Vvedenie v nelineinuyu mekhaníku (in Russian). "Nauka", M., 1974.
3. Grishin V.E., Fedyanin V.K. JINR, P17-81-659, Dubna, 1981.
4. Abramowitz M.A., Stegan I.A. Handbook of Mathematical Functions, National Bureau of Standards, Washington, 1964.

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