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ISOSCALAR FACTORS FOR ANTIUNITARY SHUBNIKOV GROUPS

1. Cubic Point Groups

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1. Introduction

The quantum theory of angular momentum, or Wigner-Racah algebra, is widely applied in the atomic and nuclear spectroscopy, as well as in the solid state spectroscopy. The basic elements of this powerful mathematical apparatus are the Clebsch-Gordan (or Wigner) and Racah coefficients (in the symmetrized form known as 3j-, 6j-, 9j -symbols, etc.), also the isoscalar factors and the coefficients of fractional parentage (see, e.g., [1-6] and the references given there). In the last 10-15 years the magnetic or Shubnikov groups [7-10] as well as other types of colour groups [10-12] are often used in the investigation of paramagnetic and magnetically ordered crystals. The operator of time reversal, which is an element of the so-called "grey" Shubnikov groups or it is contained in the black-white Shubnikov groups only in a combination with a space symmetry operator, is an antiunitary element. The theory of the so-called Wigner corepresentations (not the theory of the ordinary representations) is valid for the antiunitary groups (groups of unitary and antiunitary elements) [1,8,13]. The main theorems of the representation theory and the algebra of Wigner-Racah cannot be applied automatically to the systems with antiunitary symmetry, because there is no homomorphism between the antiunitary groups and their corepresentations. The theory of corepresentations is developed in close analogy with the theory of the ordinary representations in a number of papers (see, e.g., [8,14-18]). In particular the orthogonality relations and Clebsch-Gordan coefficients (CGC) for the corepresentations, the projection operators and two forms of the generalization of Wigner-Eckart theorem are given in [15,16]. Using the generalized Schur lemma [14], the generalization of the well-known Racah lemma [23], which gives the relation between the CGC for the corepresentations of a group and its antiunitary subgroups, is discussed in [17,18]. The main difference, for the corepresentation case, is that the corresponding isoscalar factors should be real. On the bases of the generalized Racah lemma an effective method for the calculation of the CGC for the corepresentations of antiunitary Shubnikov groups is proposed in [17,18], and the CGC for the single and double valued corepresentations of the 90 antiunitary Shubnikov point groups are calculated and tabulated in [19-22], for even (in respect to space inversion), as well as for odd basic functions [24]. One of the advantages of this method and the tables is that the CGC are correlated (by Racah lemma) for the different point groups as well as with the Wigner coefficients.

In this paper, the isoscalar factors for the corepresentations of antiunitary group-subgroup chains are discussed on the bases of [17,19], and also the complete tables of the isoscalar factors for the cubic magnetic point groups are given.

2. Isoscalar factors for the corepresentations

The isoscalar factors (IF) relate the CGC for the representations of a group and its subgroups. The IF appear most naturally in the lemma of Racah [23] (see also [26,25]). The IF for the crystallographic groups are applied for the calculations of the CGC for point groups [26,27] and in the crystal field theory [28,29].

As has been shown in [17,18], Racah lemma is valid for the corepresentations of the antiunitary groups, but the IF should be real (it follows from the generalization of Schur lemma). The generalized lemma of Racah can be written in a matrix form as follows:

$$\Big(\bigoplus_{\substack{\beta_1\in\alpha_1\\ \beta_2\in\alpha_2}} U^{\beta_1\beta_2}\Big) = \overline{U}^{\alpha_1\alpha_2}\,\big(X^{\alpha_1\alpha_2}\big)^{-1}, \tag{1}$$

where

$$\overline{U}^{\alpha_1\alpha_2} \equiv \big(S^{\alpha_1}\otimes S^{\alpha_2}\big)^{-1}\,U^{\alpha_1\alpha_2}\Big(\bigoplus_{\alpha r_\alpha} S^{\alpha r_\alpha}\Big), \tag{2}$$

$$\big(X^{\alpha_1\alpha_2}\big)^{-1} = \widetilde{X}^{\alpha_1\alpha_2}. \tag{3}$$

The unitary matrices $U^{\alpha_1\alpha_2}$ in (2) and $U^{\beta_1\beta_2}$ in (1) reduce the Kronecker products of corepresentations $D^{\alpha_1}\times D^{\alpha_2}$ of the antiunitary groups A and $D^{\beta_1}\times D^{\beta_2}$ of its antiunitary subgroup $B\subset A$, i.e., their matrix elements are the corresponding CGC for the corepresentations of A and $B\subset A$:

$$U^{\alpha_1\alpha_2}_{a_1a_2,\,\alpha r_\alpha a} \equiv [\alpha_1 a_1, \alpha_2 a_2 \,|\, \alpha r_\alpha a], \tag{4}$$

$$U^{\beta_1\beta_2}_{b_1b_2,\,\beta r_\beta b} \equiv [\beta_1 b_1, \beta_2 b_2 \,|\, \beta r_\beta b]. \tag{5}$$

The indices a and b numerate the basic functions $|\alpha a\rangle$ and $|\beta b\rangle$ of the corepresentations D^α and D^β, and the reduction multiplicity indices $r_\alpha = 1, \dots, (\alpha_1\alpha_2|\alpha)$ and $r_\beta = 1, \dots, (\beta_1\beta_2|\beta)$ numerate the repeated equivalent corepresentations $D^{\alpha r_\alpha}$ and $D^{\beta r_\beta}$ in $D^{\alpha_1}\times D^{\alpha_2}$ and $D^{\beta_1}\times D^{\beta_2}$. However, all $U^{\lambda_1\lambda_2}$, $(\lambda_i = \alpha_i; \beta_i)$, are chosen in such a way that the matrices of the equivalent corepresentations identically coincide for all r_λ and $\lambda = \alpha; \beta$:

$$D^{\lambda r_\lambda}(g) = D^\lambda(g). \tag{6}$$

From (6) it follows that $S^{\alpha r_\alpha} = S^\alpha$, where the unitary matrices

$$S^\alpha = \| S^\alpha_{a,\beta s_\beta b} \| = \| (\alpha a \,|\, \alpha\beta s_\beta b) \| \tag{7}$$

are the so-called subduction matrices. They transform the basic functions $|\alpha, a\rangle$ of D^{α} into $|\alpha, \beta s_{\beta} b\rangle$ in such a way that the transformed function is also a basic function for the corepresentations D^{β} of $B \subset A$. The corepresentation D^{α} in the new base is equivalent to $\bar{D}^{\alpha}$, and its subduction to $B \subset A$, has a block diagonal form

$$\bar{D}^{\alpha} \downarrow B = [(S^{\alpha})^{-1} D^{\alpha} S^{\alpha(*)}] \downarrow B = \bigoplus_{\beta s_{\beta}} D^{\beta s_{\beta}}. \tag{8}$$

The subduction multiplicity index $s_{\beta} = 1, \ldots, (\alpha/\beta)$ numerates the equivalent corepresentations D^{β}. The matrices S^{α} are such, that

$$D^{\beta s_{\beta}}(g') = D^{\beta}(g') \tag{9}$$

for all s_{β}, β and $g' \in B \subset A$. The matrix $\bar{U}^{\alpha_1 \alpha_2}$, contains CGC for the group A in the new base $|\alpha_i, \beta_i s_{\beta_i} b_i\rangle$, i.e., a_i are replaced by $\beta_i s_{\beta_i} b_i$:

$$\bar{U}^{\alpha_1\alpha_2}_{\beta_1 s_{\beta_1} b_1, \beta_2 s_{\beta_2} b_2, \alpha\tau_{\alpha}\beta s_{\beta} b} = [\alpha_1\beta_1 s_{\beta_1} b_1, \alpha_2\beta_2 s_{\beta_2} b_2 | \alpha\tau_{\alpha}\beta s_{\beta} b]. \tag{10}$$

The direct sum of the l.h.s. of eq. (1) consists of $U^{\beta_1\beta_2}$ for all $D^{\beta_1} \times D^{\beta_2}$, which are contained in $D^{\alpha_1} \times D^{\alpha_2}$, and $U^{\beta_1 s_{\beta_1} \beta_2 s_{\beta_2}} = U^{\beta_1\beta_2}$, as it follows from eq. (9).

The IF are the matrix elements of the matrix $X^{\alpha_1\alpha_2}$, which are defined by the following equation:

$$(\alpha_1\beta_1 s_{\beta_1}, \alpha_2\beta_2 s_{\beta_2}; \beta\tau_{\beta} \| \alpha\tau_{\alpha}\beta s_{\beta})\, \delta_{\beta\beta'}\, \delta_{bb'} \equiv \sum_{b_1 b_2} [\beta_1 b_1, \beta_2 b_2 | \beta\tau_{\beta} b]^{*} [\alpha_1\beta_1 s_{\beta_1} b_1, \alpha_2\beta_2 s_{\beta_2} b_2 | \alpha\tau_{\alpha}\beta s_{\beta} b]. \tag{11}$$

Obviously, the matrix $X^{\alpha_1\alpha_2}$ is diagonal for the indices β and b, and it consists of submatrices $\chi^{\alpha_1\alpha_2\beta b} = \chi^{\alpha_1\alpha_2\beta}$. They do not depend on the indices b, i.e., on the explicit form of the D^{β} base. In the case of corepresentations the matrix $X^{\alpha_1\alpha_2}$ has only real matrix elements, i.e., $X^{\alpha_1\alpha_2}$ is an orthogonal matrix, or it is a direct sum of orthogonal submatrices

$$\chi^{\alpha_1\alpha_2\beta} = \chi^{\alpha_1\alpha_2\beta *} = \| (\alpha_1\beta_1 s_{\beta_1}, \alpha_2\beta_2 s_{\beta_2}; \beta\tau_{\beta} \| \alpha\tau_{\alpha}\beta s_{\beta}) \|. \tag{12}$$

The orthogonality relations for the IF for the corepresentations [17] follow from the orthogonality of the submatrices $\chi^{\alpha_1\alpha_2\beta}$ by rows and columns which are numerated correspondingly by the indices $\beta_1 s_{\beta_1} \beta_2 s_{\beta_2} \tau_{\beta}$ and $\alpha\tau_{\alpha} s_{\beta}$:

$$\sum_{\substack{\beta_1 s_{\beta_1} \\ \beta_2 s_{\beta_2} \\ \tau_{\beta}}} (\alpha_1\beta_1 s_{\beta_1}, \alpha_2\beta_2 s_{\beta_2}; \beta\tau_{\beta} \| \alpha\tau_{\alpha}\beta s_{\beta})(\alpha_1\beta_1 s_{\beta_1}, \alpha_2\beta_2 s_{\beta_2}; \beta\tau_{\beta} \| \alpha'\tau'_{\alpha'}\beta s'_{\beta}) = \delta_{\alpha\alpha'}\, \delta_{\tau_{\alpha}\tau'_{\alpha'}}\, \delta_{s_{\beta}s'_{\beta}}. \tag{13}$$

$$\sum_{\alpha\tau_{\alpha} s_{\beta}} (\alpha_1\beta_1 s_{\beta_1}, \alpha_2\beta_2 s_{\beta_2}; \beta\tau_{\beta} \| \alpha\tau_{\alpha}\beta s_{\beta})(\alpha_1\beta'_1 s'_{\beta'_1}, \alpha_2\beta'_2 s'_{\beta'_2}; \beta\tau'_{\beta} \| \alpha\tau_{\alpha}\beta s_{\beta}) = \delta_{\beta_1\beta'_1}\, \delta_{\beta_2\beta'_2}\, \delta_{s_{\beta_1}s'_{\beta_1}}\, \delta_{s_{\beta_2}s'_{\beta_2}}\, \delta_{\tau_{\beta}\tau'_{\beta}}. \tag{14}$$

Using the symmetry relations for the CGC accepted in [18]

$$[\lambda_2 l_2, \lambda_1 l_1 | \lambda\tau_{\lambda} l] = (-1)^{j(\lambda_1)+j(\lambda_2)-j(\lambda\tau_{\lambda})} [\lambda_1 l_1, \lambda_2 l_2 | \lambda\tau_{\lambda} l], \tag{15}$$

and eq. (11), the symmetry relations for the IF can be found

$$(\alpha_2\beta_2 s_{\beta_2}, \alpha_1\beta_1 s_{\beta_1}; \beta\tau_{\beta} \| \alpha\tau_{\alpha}\beta s_{\beta}) = (-1)^{j(\alpha_1)+j(\alpha_2)-j(\alpha\tau_{\alpha})}(-1)^{j(\beta_1)+j(\beta_2)-j(\beta\tau_{\beta})}(\alpha_1\beta_1 s_{\beta_1}, \alpha_2\beta_2 s_{\beta_2}; \beta\tau_{\beta} \| \alpha\tau_{\alpha}\beta s_{\beta}). \tag{16}$$

Here $j(\lambda)$ is the so-called "quasimoment" of the corepresentations D^{λ} (see also [19,24,26] for more details).

The IF can be calculated using the system of equations

$$\sum_{\tau_{\beta}} (\alpha_1\beta_1 s_{\beta_1}, \alpha_2\beta_2 s_{\beta_2}; \beta\tau_{\beta} \| \alpha\tau_{\alpha}\beta s_{\beta}) \cdot (\alpha_1\beta_1 s'_{\beta_1}, \alpha_2\beta_2 s'_{\beta_2}; \beta\tau_{\beta} \| \alpha'\tau'_{\alpha'}\beta s'_{\beta}) = \sum_{b_1 b_2} [\alpha_1\beta_1 s_{\beta_1} b_1, \alpha_2\beta_2 s_{\beta_2} b_2 | \alpha\tau_{\alpha}\beta s_{\beta} b] \cdot [\alpha_1\beta_1 s'_{\beta_1} b_1, \alpha_2\beta_2 s'_{\beta_2} b_2 | \alpha'\tau'_{\alpha'}\beta s'_{\beta} b]^{*}, \tag{17}$$

and the orthogonality relations (14),(13).

In a number of cases, an ambiguity exists in the solutions of (17), which can be used for a better correlation between the CGC for the corepresentations of the groups A and $B \subset A$.

We shall stress on a very useful property of IF. Let us consider the chain of antiunitary subgroups $A \supset B \supset C$. The IF matrix $X^{\alpha_1\alpha_2}_{C \subset A}$ for the transition from A to $C \subset A$ can be expressed by the corresponding IF matrices $\chi^{\alpha_1\alpha_2}_{B\subset A}$ and $\chi^{\beta_1\beta_2}_{C\subset B}$ for the transitions from A to B⊂A and from B to C⊂B. For their submatrices it can be written symbolically as

$$\chi^{\alpha_1\alpha_2\gamma}_{C\subset A} \sim \chi^{\beta_1\beta_2\gamma}_{C\subset B}\ \chi^{\alpha_1\alpha_2\beta}_{B\subset A} \quad (18)$$

The precise expression in matrix form can be found by the double use of eqs.(1) and (2), but is quite complicated and it is not convenient for practical application. For the corresponding matrix elements (i.e., the IF), the following simple relation can be derived:

$$(\alpha_1\beta_1 s_{\beta_1}\gamma_1 s_{\gamma_1}, \alpha_2\beta_2 s_{\beta_2}\gamma_2 s_{\gamma_2}; \gamma r_\gamma \| \alpha r_\alpha \beta s_\beta \gamma s_\gamma) =$$

$$= \sum_{r_\beta} (\beta_1\gamma_1 s_{\gamma_1}, \beta_2\gamma_2 s_{\gamma_2}; \gamma r_\gamma \| \beta r_\beta \gamma s_\gamma)(\alpha_1\beta_1 s_{\beta_1}, \alpha_2\beta_2 s_{\beta_2}; \beta r_\beta \| \alpha r_\alpha \beta s_\beta) . \quad (19)$$

Using this equation we can considerably decrease the volume of the calculations of the IF for a given set of subgroups (see the next section).

The IF are most often applied in the calculation of the CGC of a given group by the CGC of its subgroup or supergroup. With the help of IF we have calculated the CGC [18-23] for the corepresentations of all 90 antiunitary Shubnikov point groups, starting with the CGC for the corepresentations of their common supergroup $SO(3)\otimes\Theta$ (they coincide with the well-known Wigner coefficients). As another important application of the IF, we shall mention the relation between the reduced matrix elements of quantum mechanical operators, whose symmetry is described by the groups A and B⊂A (see [25,28-30]).

3. Isoscalar factors for cubic magnetic groups

It is known that 90 antiunitary magnetic point groups are subgroups of the full orthogonal group O(3) supplemented with the operator of time reversal Θ. The calculations of the IF for the corepresentations of these groups can be carreid

out using the chain of subgroups of the type A⊃B⊃C, where A=O(3)⊠Θ, B=O_h ⊠ Θ or D_{6h} ⊠ Θ. As it has been shown in [17-19], the CGC for O(3) ⊠ Θ coincide with the Wigner coefficients, i.e.,

$$[\alpha_1 a_1, \alpha_2 a_2 | \alpha r_\alpha a] \equiv [j_1 m_1, j_2 m_2 | j 1 m] = (j_1 m_1, j_2 m_2 | j m) . \quad (20)$$

In this paper we have calculated the IF for the 10 antiunitary cubic magnetic point groups, which are distributed into the following isomorphic sets:

O_h ⊠ Θ = O ⊠ Θ ⊠ C_i

O ⊠ Θ ≅ T_d ⊠ Θ ≅ O_h (O) ≅ $O_h(T_d)$

T_h ⊠ Θ = T ⊠ Θ ⊠ C_i

T ⊠ Θ ≅ T_h (T) (21)

$O_h(T_h)$ = O(T) ⊠ C_i

O(T) ≅ T_d(T)

The corepresentations $D^{\beta\pm}$ of the centrosymmetrical groups G ⊠ C_i are expressed by the corepresentations D^β of G(see [18-19]). The corepresentations of the isomorphic groups are equivalent and their matrices can be chosen in such a way that they coincide identically. Consequently their IF will also coincide and it is sufficient to calculate the IF for the chains:

$$SO(3)\otimes\Theta \supset O\otimes\Theta \lesssim {T\otimes O \atop O(T)} . \quad (22)$$

We have calculated all IF for single- and double-valued corepresentations, which are contained in the matrices:

$\chi^{j_1 j_2}_{(O\otimes O\subset SO(3)\otimes O)}$; j_1, j_2=1/2,1, 3/2,2,5/2,3 - Table 1,

$\chi^{\beta_1\beta_2}_{(T\otimes O\subset O\otimes O)}$ - Table 2,

$\chi^{\beta_1\beta_2}_{(O(T)\subset O\otimes O)}$ - Table 3.

Using eq. (19) and Table 1÷3, the matrices

$\chi^{j_1 j_2}(T \boxtimes O \subset SO(3) \boxtimes \Theta)$

$\chi^{j_1 j_2}_{\ldots}(O(T) \subset SO(3) \boxtimes \Theta)$

can be calculated easily, but because of the volume, such tables are not publicated here.

We have prepared a programme for the computation of the IF on Fortran-IV. Wigner coefficients (20) are taken from [4]. The matrices S^j, eq. (7), for $O\boxtimes\Theta$ and $T\boxtimes\Theta$ are taken from [24], while for O(T) we have calculated in [19].

For a more convenient use of the tables of IF we have also given a compatibility table (Table 4) and multiplication tables for the coreps (Tables 5-7).

The IF in the tables are given in the form of a decimal fraction, with 8 figures after the decimal point. The decimal point and the zero of the integers are not printed. We have also tabulated some "accident" zeroes, and in this case only the figure "O" is printed. The sign " - " means that IF is negative. The change in the sign of the IF after the transition $\alpha_1\alpha_2 \rightarrow \alpha_2\alpha_1$ is indicated by an asterisk after the decimal fraction.

For every couple of corepresentations $\alpha_1\alpha_2$ the nonzero IF are tabulated into columns by submatrices $\chi^{\alpha_1\alpha_2\beta}$. In every column under the row with $\alpha_1 \times \alpha_2$ and β, the indices of the IF in the following manner are given:

$\alpha_1 \times \alpha_2$	β
$\beta_1 s_{\beta_1} \beta_2 s_{\beta_2} r_\beta \quad \alpha r_\alpha s_\beta$	$(\alpha_1\beta_1 s_{\beta_1}, \alpha_2\beta_2 s_{\beta_2}; \beta r_\beta \Vert \alpha r_\alpha \beta s_\beta)$.

For example in Table 1 we can find

2 x 3/2 ß=6

31811 7/211 - 77459667 *

which means

$$(231,\ 3/281; 61 \Vert 7/2161) = -(3/281,\ 231; 61 \Vert 7/2161) = -0.77459667$$

The given tables have been used for the examination of the tables of CGC for the coreps of the cubic groups, calculated in [19] by hand.

TABLE 1. Isosclar factors from $SO(3)\boxtimes\Theta$ to $O\boxtimes\Theta$, $T_d\boxtimes\Theta$, $O_h(O)$ and $O_h(T_d)$

1/2x1/2		ß=1
61611	011	1
1/2x1/2		ß=4
61611	111	1
1x1/2		ß=6
41611	1/211	1
1x1/2		ß=8
41611	3/211	1
1x1		ß=1
41411	011	1
1x1		ß=3
41411	211	1
1x1		ß=4
41411	111	1
1x1		ß=5
41411	211	1
3/2x1/2		ß=3
81611	211	1
3/2x1/2		ß=4
81611	111	1
3/2x1/2		ß=5
81611	211	1
3/2x1		ß=6
81411	1/211	1
3/2x1		ß=7
81411	5/211	1
3/2x1		ß=8
81411	3/211	1
81411	5/211	O
81412	3/211	O
81412	5/211	1
3/2x3/2		ß=1
81811	011	1
3/2x3/2		ß=2
81811	311	1
3/2x3/2		ß=3
81811	211	1
3/2x3/2		ß=4
81811	111	1
81811	311	O
81812	111	O
81812	311	1
3/2x3/2		ß=5
81811	211	1
81811	311	O
81812	211	O
81812	311	1
2x1/2		ß=7
51611	5/211	1
2x1/2		ß=8
31611	3/211	63245553
31611	5/211	-77459667*
51611	3/211	77459667
51611	5/211	63245553*
2x1		ß=2
51411	311	1
2x1		ß=3
51411	211	1
2x1		ß=4
31411	111	63245553
31411	311	77459667
51411	111	77459667
51411	311	-63245553
2x1		ß=5
31411	211	81649658
31411	311	57735027*
51411	211	57735027
51411	311	-81649658*
2x3/2		ß=6
31811	1/211	63245554
31811	7/211	-77459667*
51811	1/211	77459667
51811	7/211	63245554*
2x3/2		ß=7
31811	5/211	92582010
31811	7/211	37796447*
51811	5/211	37796447
51811	7/211	-92582010*
2x3/2		ß=8
31811	3/211	63245554
31811	5/211	41403933*
31811	7/211	65465367
51811	3/211	77459667
51811	5/211	-33806171*
51811	7/211	-53452248
51812	3/211	O
51812	5/211	84515426
51812	7/211	53452248*
2x2		ß=1
31311	011	63245551
31311	411	77459672
51511	011	77459672
51511	411	-63245551
2x2		ß=2
31311	311	1
2x2		ß=3
31311	211	75592889
31311	411	65465371
51511	211	65465371
51511	411	-75592889
2x2		ß=4
31511	111	63245552
31511	311	-31622782
31511	411	70710678*
51511	111	44721364
51511	311	89442708
51511	411	O
2x2		ß=5
31511	211	53452254
31511	311	-70710678*
31511	411	46291013
51511	211	65465368
51511	311	O
51511	411	-75592891
5/2x1/2		ß=2
71611	311	1
5/2x1/2		ß=3
81611	211	1*
5/2x1/2		ß=4
81611	311	-1*
5/2x1/2		ß=5
71611	211	-74535599
71611	311	66666667*
81611	211	-66666667*
81611	311	-74535559
5/2x1		ß=6
81411	7/211	1
5/2x1		ß=7
71411	5/211	48795004
71411	7/211	-87287158*
81411	5/211	-87287158*
81411	7/211	-48795004
5/2x1		ß=8
71411	3/211	57735027
71411	5/211	61721340*
71411	7/211	-53452248
81411	3/211	81649658*
81411	5/211	-43643578
81411	7/211	37796447*
81412	3/211	O
81412	5/211	65465367*
81412	7/211	75592895
5/2x3/2		ß=1
81811	411	-1*
5/2x3/2		ß=2
81811	311	-1*
5/2x3/2		ß=3
71811	211	84515426
71811	411	-53452248
81811	211	53452248
81811	411	84515426
5/2x3/2		ß=4
71811	111	57735027
71811	311	-70710678
71811	411	-40824830*
81811	111	-81649658*
81811	311	-50000000*
81811	411	-28867513
81812	111	O

51812	3/211	0	51511	411	41833002*	41812	11/211	-14213381
51812	5/211	-66815311*	51511	511	44721360	41812	11/212	56183323
51812	7/211	-30860676	3x5/2		β=6	51711	3/211	-34503278*
51812	9/211	-66143783*	21711	1/211	37796447	51711	5/211	42257713
51812	9/212	-14433760*	21711	7/211	66666667*	51711	7/211	-57505463*
3x2		β=1	21711	9/211	-31782086	51711	9/211	-03003125
51511	411	-1*	21711	11/211	55829053*	51711	9/212	22936745
3x2		β=2	41811	1/211	-65465367*	51711	11/211	-47673130*
41511	311	-1*	41811	7/211	57735027	51711	11/212	30151135*
3x2		β=3	41811	9/211	-27524094*	51811	3/211	55634864
21311	211	-59761431*	41811	11/211	-40291148	51811	5/211	10482849*
21311	411	58554004*	51711	1/211	-43643578*	51811	7/211	-32810304
21311	511	-54772256	51711	7/211	19245009	51811	9/211	-26446928*
41511	211	-32732683	51711	9/211	73397585*	51811	9/212	48364202*
41511	411	-80178373	51711	11/211	48349378	51811	11/211	20695934
41511	511	-50000000*	51811	1/211	-48795004	51811	11/212	-09349470
51511	211	73192506*	51811	7/211	-43033148*	51812	3/211	0
51511	411	11952286*	51811	9/211	-53339646	51812	5/211	55717103
51511	511	-67082040	51811	11/211	54056248*	51812	7/211	48606968*
3x2		β=4	3x5/2		β=7	51812	9/211	-54756294
21511	111	37796447	41711	5/211	79681908	51812	9/212	05120921
21511	311	-57735027	41711	7/211	-37796447*	51812	11/211	-42946544*
21511	411	-44721360*	41711	11/211	-47140452*	51812	11/212	46747350*
21511	511	-54772256	41811	5/211	08908708*	3x3		β=1
21511	512	15430335	41811	7/211	84515426	21211	011	37796447
41311	111	50709255	41811	11/211	52704628	21211	411	-73854895
41311	311	51639778	51811	5/211	59761431	21211	611	55829053
41311	411	0	51811	7/211	37796447*	41411	011	65465367
41311	511	0	51811	11/211	-70710678*	41411	411	63960214
41311	512	69006556	3x5/2		β=8	41411	611	40291148
41511	111	41403933	21811	3/211	-37796447*	51511	011	65465367
41511	311	15811389	21811	5/211	46291005	51511	411	-21320072
41511	411	61237243*	21811	7/211	25197632*	51511	611	-72524068
41511	511	-50000000	21811	9/211	42766860	3x3		β=2
41511	512	-42257712	21811	9/212	35176324	41511	311	-70710678*
51311	111	37796447*	21811	11/211	52223297*	41511	611	70710678
51311	311	-57735027*	21811	11/212	0	3x3		β=3
51311	411	44721360	41711	3/211	-46291005	41411	211	53452248
51311	511	54772256*	41711	5/211	-06299408*	41411	411	38223540
51311	512	15430335*	41711	7/211	-15430335	41411	511	0
51511	111	-53452248*	41711	9/211	-60436722*	41411	611	75377836
51511	311	-20412415*	41711	9/212	-30772873*	41511	211	-59761431*
51511	411	47434165	41711	11/211	49746834	41511	411	17094086*
51511	511	-38729834*	41711	11/212	22473329	41511	511	70710678
51511	512	54554473*	41811	3/211	46291005*	41511	611	33709994*
3x2		β=5	41811	5/211	50395262	51511	211	0
41311	211	26726124	41811	7/211	15430335*	51511	411	89188258
41311	311	-57735027*	41811	9/211	04029114	51511	511	0
41311	411	65465367	41811	9/212	-43082021	51511	611	-45226701
41311	511	40824829*	41811	11/211	07106691*	3x3		β=4
41511	211	75592894	41811	11/212	56183321*	21511	111	37796447*
41511	311	20412415*	41812	3/211	0	21511	311	40824829*
41511	411	23145503	41812	5/211	-25197634*	21511	411	-47673130
41511	511	-57735027*	41812	7/211	46201005	21511	511	0
51511	211	0	41812	9/211	-28203804*	21511	512	-43643578*
51511	311	79056941	41812	9/212	55391172*	21511	611	52223297
81812	311	-50000000x	81811	011	81649652	51611	7/211	80178372*
81812	411	86602541	81811	411	57735021	3x1		β=1
5/2x3/2		β=5	5/2x5/2		β=2	41411	411	1
71811	211	28171808	81811	311	1	3x1		β=2
71811	311	-52704628*	5/2x5/2		β=3	51411	311	1*
71811	411	-80178373	71811	211	34503274*	3x1		β=3

81811	211	-95949723*	71811	411	-61721341*	41411	211	80178373
81811	311	-15474612	71811	511	70710678	41411	411	59761431
81811	411	-23541180*	81811	211	87287154	51411	211	-59761431*
81812	211	0	81811	411	48794987	51411	411	80178373*
81812	311	-83562903*	81811	511	0	3x1		β=4
81812	411	54929422	5/2x5/2		β=4	41411	311	-61237243
5/2x2		β=6	71711	111	28171813	41411	411	79056941*
71511	1/211	57735027	71711	311	70272837	51411	311	79056941*
71511	7/211	76980036*	71711	411	0	51411	411	61237243
71511	9/211	-27216552	71711	511	52704634	3x1		β=5
81311	1/2	-63245554*	71711	512	-38604400	21411	211	48795004
81311	7/211	21081851	71811	111	-50395263*	21411	311	-57735027*
81311	9/211	-74535600*	71811	311	-07856739*	21411	411	-65465367
81511	1/211	51639778*	71811	411	70710678	41411	211	-53452248
81511	7/211	-60246408	71811	511	47140452*	41411	311	-79056941*
81511	9/211	-60858062*	71811	512	13280310*	41411	411	29880715
5/2x2		β=7	81811	111	64241600	51411	211	69006556*
81311	5/211	37796447*	81811	311	-43143351	51411	311	-20412415
81311	7/211	92582010	81811	411	0	51411	411	69436507*
81511	5/211	92582010*	81811	511	50847518	3x3/2		β=6
81511	7/211	-37796447	81811	512	37765032	41811	7/211	40824830*
5/2x2		β=8	81812	111	0	41811	9/211	91287093
71311	3/211	53452248	81812	311	55470022	51811	7/211	-91287093
71311	5/211	-26726124*	81812	411	0	51811	9/211	40824830*
71311	7/211	-56343617	81812	511	-13867500	3x3/2		β=7
71311	9/211	-54554472*	81812	512	82041271	41811	5/211	-80178373
71311	9/212	16666667*	5/2x5/2		β=5	41811	7/211	-59761431*
71511	3/211	21821789	71811	211	69006551*	51811	5/211	59761431*
71511	5/211	-65465367*	71811	311	52704620	51811	7/211	-80178373
71511	7/211	23002185	71811	411	-15430333*	3x3/2		β=8
71511	9/211	08908708*	71811	511	47140449	21811	3/211	37796447
71511	9/212	-68041382*	81811	211	21821781	21811	5/211	46291005*
81311	3/211	33806170*	81811	311	0	21811	7/211	-53452248
81311	5/211	67612340	81811	411	97590063	21811	9/211	-32732684*
81311	7/211	17817416*	81811	511	0	21811	9/212	50000000*
81311	9/211	-34503278	81812	211	0	41811	3/211	65465368
81311	9/212	-52704628	81812	311	66666667	41811	5/211	-40089186*
81511	3/211	-74322335*	81812	411	0	41811	7/211	46291005
81511	5/211	-07688517	81812	511	-74535600	41811	9/211	09449112*
81511	7/211	-25663962*	3x1/2		β=6	41811	9/212	43301270*
81511	9/211	-52313735	41611	7/211	1 *	41812	3/211	0
81511	9/212	-31964200	3x1/2		β=7	41812	5/211	-40089186
81512	3/211	0	21611	5/211	65465367	41812	7/211	-61721341*
81512	5/211	-19221293*	21611	7/211	-75592895*	41812	9/211	66143783
81512	7/211	72939681	51611	5/211	75592895*	41812	9/212	14433758
81512	9/211	-54929422*	51611	7/211	65465367	51811	3/211	-65465367*
81512	9/212	35959720*	3x1/2		β=8	51811	5/211	-13363063
5/2x5/2		β=1	41611	5/211	80178372*	51811	7/211	15430335*
71711	011	57735021	41611	7/211	59761430	51811	9/211	-09449111
71711	411	-81649652	51611	5/211	-59761430	51811	9/212	72168783
41411	111	-40089186	51511	611	0	41511	411	39477101*
41411	311	57735027	3x3		β=5	41511	511	-40824830
41411	411	0	21411	211	-48795004	41511	611	-28287619*
41411	511	-68465320	21411	311	-40824829*	41511	612	-48412292*
41411	512	19287919	21411	411	13957263	51511	211	-66815310
41411	611	0	21411	511	-57735027*	51511	311	0
41511	111	51754917*	21411	611	49236596	51511	411	-25482359
41511	311	0	21411	612	0	51511	511	0
41511	411	52223296	41411	211	-13363062	51511	611	-58992488
41511	511	-17677670*	41411	311	0	51511	612	37500000
41511	611	47673129	41411	411	-76447079	3x2		β=5
51511	111	-13363062	41411	511	0	51311	211	59761431*
51511	311	57735027	41411	611	08416568	51311	311	0

```
51511 411  0
51511 511  68465320
51511 512  42433421
41511 512  44821075
```

```
41411 612 -62500000
41511 211 -17251639*
41511 311  57735027
```

```
51311 411 -58554004*
51311 511  54772256
```

TABLE 2. Isoscalar factors from O⊗Θ to T ⊗ Θ and T_h(T)

```
2x2        β=1
11111 111  1
3x2        β=2
21111 311 -1*
3x3        β=1
21211 111  1
21212 211  1
3x3        β=2
21211 311  1
4x2        β=4
41111 511 -1
4x3        β=4
41211 411  1
41212 511 -1*
4x4        β=1
41411 111  1
4x4        β=2
41411 311  1
4x4        β=4
41411 411  1
41412 511  1
5x2        β=4
41111 411 -1
5x3        β=4
41211 511 -1
41212 411  1*
5x4        β=1
41411 211 -1
5x4        β=2
41411 311 -1*
5x4        β=4
41411 511  1
41412 411 -1*
5x5        β=1
41411 111  1
5x5        β=2
41411 311 -1
5x5        β=4
```

```
41411 411 -1
41412 511  1*
6x2        β=5
51111 711  1*
6x3        β=6
51211 811  1
6x4        β=5
51411 611  1
6x4        β=6
51411 811  1
6x5        β=5
51411 711  1*
6x5        β=6
51411 811  1*
6x6        β=1
51511 111  1
6x6        β=4
51511 411  1
7x2        β=5
51111 611 -1*
7x3        β=6
51211 811  1
7x4        β=5
51411 711 -1
7x4        β=6
51411 811 -1
7x5        β=5
51411 611 -1*
7x5        β=6
51411 811  1*
7x6        β=1
51511 211 -1*
7x6        β=4
51511 511  1*
7x7        β=1
51511 111  1
7x7        β=4
51511 411 -1
```

```
8x2        β=6
61111 811 -1*
8x3        β=5
61211 611  1
61212 711  1
8x3        β=6
61211 811  1
8x4        β=5
61411 611  1
61412 711  1
8x4        β=6
61411 811  1
61412 822  1
8x5        β=5
61411 711  1*
61412 611 -1*
8x5        β=6
61411 811  1*
61412 821  1*
8x6        β=2
61511 311  1
8x6        β=4
61511 411  1
61512 511  1
8x7        β=2
61511 311 -1
8x7        β=4
61511 511  1*
61512 411 -1*
8x8        β=1
61611 111  1
61612 211  1
8x8        β=2
61611 311  1
8x8        β=4
61611 411  1
61612 421  1
61614 521  1
61613 611  1
```

TABLE 3. Isoscalar factors from O⊗Θ to O(T) and T_d(T)

```
2x2        β=1
11111 111 -1
3x2        β=2
21111 311  1*
3x2        β=3
31111 311 -1*
3x3        β=1
21311 111  70710678
21311 211  70710678 *
31211 111  70710678
31211 211 -70710678 *
3x3        β=2
31311 311  1
```

```
5x5        β=4
41411 411  1
41412 511  1
6x2        β=5
51111 711 -1*
6x3        β=6
51211 811  1
6x3        β=7
51311 811  1
6x4        β=5
51411 611  1
6x4        β=6
51411 811  1
```

```
8x2        β=7
71111 811 -1*
8x3        β=5
61311 611 -70710678
61311 711 -70710678
71211 611 -70710678
71211 711 -70710678
8x3        β=6
61311 811 -1
8x3        β=7
71211 811  1
8x4        β=5
61411 611  70710678
```



```
3x3        β=3
21211 311  1
4x2        β=4
41111 511  1
4x3        β=4
41211 411  70710678
41211 511 -70710678 *
41311 411  70710678
41311 511  70710678 *
4x4        β=1
41411 111 -1
4x4        β=2
41411 311  1
4x4        β=3
41411 311  1
4x4        β=4
41411 411 -1
41412 511  1
5x2        β=4
41111 411 -1
5x3        β=4
41211 411 -70710678 *
41211 511 -70710678
41311 411  70710678 *
41311 511 -70710678
5x4        β=1
41411 211  1
5x4        β=2
41411 311 -1*
5x4        β=3
41411 311  1*
5x4        β=4
41411 411  1*
41412 511 -1*
5x5        β=1
41411 111  1
5x5        β=2
41411 311  1
5x5        β=3
41411 311  1
71511 411 -70710678
71511 511  70710678*
8x8        β=1
61711 111  70710678
61711 211  70710678*
71611 111  70710678
71611 211 -70710678*
8x8        β=2
71711 311  1
8x8        β=3
```

```
6x4        β=7
51411 811  1
6x5        β=5
51411 711  1*
6x5        β=6
51411 811  1*
6x5        β=7
51411 811 -1*
6x6        β=1
51511 111  1
6x6        β=4
51511 411  1
7x2        β=5
51111 611 -1*
7x3        β=6
51211 811  1
7x3        β=7
51311 811 -1
7x4        β=5
51411 711 -1
7x4        β=6
51411 811 -1
7x4        β=7
51411 811  1
7x5        β=5
51411 611  1*
7x5        β=6
51411 811 -1*
7x5        β=7
51411 811 -1*
7x6        β=1
51511 211 -1*
7x6        β=4
51511 511  1*
7x7        β=1
51511 111 -1
7x7        β=4
51511 411  1
8x2        β=6
61111 811  1*
61611 311  1
8x8        β=4
61611 411 63245551
61611 421 31622773
61611 511 0
61611 521 70710678*
61711 411-31622773
61711 421 63245551
61711 511 70710678*
```

```
61411 711 -70710678
71411 611  70710678
71411 711  70710678
8x4        β=6
61411 811  44721352
61411 821  89442714*
71411 811  89442714
71411 821 -44721352*
8x4        β=7
61411 811  89442714
61411 821 -44721352*
71411 811 -44721352
71411 821 -89442714*
8x5        β=5
61411 611  70710678*
61411 711 -70710678*
71411 611 -70710678*
71411 711 -70710678*
8x5        β=6
61411 811 -1*
71411 821 -1*
8x5        β=7
61411 821  1*
71411 811 -1*
8x6        β=2
61511 311  1
8x6        β=3
71511 311  1
8x6        β=4
61511 411  70710678
61511 511  70710678*
71511 411  70710678
71511 511 -70710678*
8x7        β=2
61511 311 -1
8x7        β=3
71511 311  1
8x7        β=4
61511 411  70710678
61511 511  70710678*
61711 521  0
71611 411 -31622773
71611 421  63245551
71611 511 -70710678*
71611 521  0
71711 411 -63245551
71711 421 -31622773
71711 511  0
71711 521  70710678*
```

TABLE 4. Compatibility Table

SO(3)⊗Θ	j:	0	1	2	3	1/2	3/2	5/2
O⊗Θ, O_h(O) T_d⊗Θ, O_h(T_d)	β:	1	4	3+5	2+4+5	6	8	7+8
T⊗Θ, T_h(T)	β:	1	4	2+4	1+4+4	5	6	5+6
O(T), T_d(T)	β:	1	4	2+3+4	1+4+4	5	6+7	5+6+7

TABLE 5. Multiplication Table for $O \boxtimes \Theta$, $T_d \boxtimes \Theta$, $O_h(\Theta)$ and $O_h(T_d)$

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	3	5	4	7	6	8
3	3	3	1+3 +2	4+5	4+5	8	8	6+7+8
4	4	5	4+5	1+3+3+4	2+3+4+5	6+8	7+8	$6+7+8^2$
5	5	4	4+5	2+3+4+5	1+3+5+4	7+8	6+8	$6+7+8^2$
6	6	7	8	6+8	7+8	4+1	2+5	3+4+5
7	7	6	8	7+8	6+8	2+5	4+1	3+4+5
8	8	8	6+7+8	$6+7+8^2$	$6+7+8^2$	3+4+5	3+4+5	$2+4^2+5+1+3+5$

TABLE 6. Multiplication Table for $T \boxtimes \Theta$ and $T_h(T)$

	1	2	4	5	6
1	1	2	4	5	6
2	2	1+2+1	4^2	6	5^2+6
4	4	4^2	1+2+4+4	5+6	5^2+6^2
5	5	6	5+6	4+1	$2+4^2$
6	6	5^2+6	5^2+6^2	$2+4^2$	$1+4^3+1+2+4$

TABLE 7. Multiplication Table for O(T) and T_d(T).

·	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	3	1	4	6	7	5
3	3	1	2	4	7	5	6
4	4	4	4	1+2+3+4+4	5+6+7	5+6+7	5+6+7
5	5	6	7	5+6+7	4+1	2+4	3+4
6	6	7	5	5+6+7	2+4	4+3	1+4
7	7	5	6	5+6+7	3+4	1+4	4+2

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