сообщения
объединенного
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исследований
дубна

E17-81-35

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SOME CONSEQUENCES OF THE LOW SYMMETRY OF THE PHONON VISCOSITY TENSOR FOR DIELECTRIC CRYSTALS

1981

Recently, Enz has published the review paper [1] referred here as I, devoted to the two-fluid hydrodynamic description of ordered systems. However, his discussion of the viscosity of dielectric crystals is oversimplified. The assumed form of components of the viscosity tensor for an isotropic (polycrystalline) medium

$$\gamma_{ij,k\ell} = \gamma [\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} + (\nu-1)\delta_{ij}\delta_{k\ell}] \qquad (1.3.86)$$

has the complete Voigt symmetry. Thus, these components are invariant under interchange of indices i and j and also under interchange of pairs ij and $k\ell$. This form is proper for a rarefied gas of real particles. However, the phenomenological expressions (1.3.43, 3.50) and also the expressions obtained from the Chapman-Enskog theory [2] (or derived from the linearized Boltzmann-Peierls equations with the use of the Zwanzig projection operator [3] yield the viscosity tensor which is less symmetric, namely it is invariant only under interchange of pairs of indices

$$\gamma_{ij,k\ell} = \gamma_{k\ell,ij} \qquad (1)$$

For an isotropic medium the viscosity tensor depends on *three* scalar coefficients γ, a, d

$$\gamma_{ij,k\ell} = a\,\delta_{ij}\delta_{k\ell} + \gamma\delta_{ik}\delta_{j\ell} + a\delta_{i\ell}\delta_{jk} \qquad (2)$$

This form follows from the invariance under all rotations belonging to the orthogonal group and symmetry (1) (cf., for example, table Д 20 given by Sirotin and Shaskolskaya[4]). In the Voigt notation the tensor γ (2) defines the real, symmetric matrix 9×9, which we shall call Γ.
This matrix is the direct sum of two matrices

$$\Gamma = \mathbf{A} \oplus \mathbf{B}.$$

The 3×3 matrix $\mathbf{A}$ is equal to

$$\mathbf{A} = (a+\gamma+d)\,\mathbf{I}_3 + d\mathbf{T} + d\mathbf{T}^{-1},$$

where $\mathbf{I}_3$ is the unit 3x3 matrix, and

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} .$$

It is easy to check that $T^3 = 1_3$, hence, the corresponding eigenvalues are $\exp(i\frac{2\pi k}{3})$, where $k = 0,1,2$. The 6x6 matrix B is a bit simpler

$$B = \gamma\, 1_6 + a\, 1_3 \quad \sigma^x ,$$

where σ^x is the Pauli matrix

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The structure of the matrix Γ allows to write down immediately its eigenvalues. Most of them are degenerate. The eigenvalues which differ are

$$a + \gamma + 3d, \quad a + \gamma, \quad \gamma - a .$$

Hence, the condition of the positivity of Γ yields three inequalities

$$(\gamma + a) > 0, \quad (\gamma - a) > 0, \quad a + \gamma + 3d > 0. \tag{3}$$

Although two forms of the viscosity tensor (1. 3.86) and (2) are quite different in both cases, the matrix λ^2_N depends only on γ and the ratio $\gamma = \frac{a + d}{\gamma}$, i.e.,

$$(\lambda^2_N(\hat{q}))_{ij} = \frac{\tau_J}{\rho_p} \sum_{m,n} \hat{q}_m \gamma_{im,jn} \hat{q}_n = \frac{\tau_J \gamma}{\rho_p} (\delta_{ij} + \nu \hat{q}_i \hat{q}_j).$$

The matrix λ^2_N has two degenerate eigenvalues $\lambda^2_{N,1} = \lambda^2_{N,2} = \frac{\tau_J \gamma}{\rho_p}$, which are not necessarily positive, and one positive eigenvalue $\lambda^2_{N,3} = \frac{\tau_J \gamma}{\rho_p} (1+\nu) = \frac{\tau_J}{\rho_p}(a+\gamma+d)$. But the quantity $(\lambda^2_N)_\ell$, which defines the relaxation time τ_N,

$$(\lambda^2_N)_\ell = \hat{q}_i (\lambda^2_N)_{ij}\, \hat{q}_j = \frac{\tau_J \gamma}{\rho_p}(\nu + 1) = \frac{\tau_J}{\rho_p}(\gamma + a + d)$$

is a positive number. This follows from the first and last of inequalities (3).

Since $(\lambda_N^2)_\ell$ depends on two scalar coefficients γ and ν and is positive, the Enz results are correct and his discussion of the Poisseuille flow remains valid.

One can ask why the viscosity tensor for a crystal has a lower symmetry than that for a rarefied gas (Lifshitz, Pitaevakii [5]). The above-mentioned more formal expressions show that the reason is that in opposite to the velocity and the momentum of a particle, the quasimomentum and the group velocity for a phonon are generally not proportional.

The lower symmetry of the viscosity tensor yields another interesting difference for the case of a rerefied gas. One can introduce the tensors of the first and second viscosity (Lifshitz, Pitaevskii [5]). For an isotropic medium the tensor of the first viscosity depends on *two* constants γ_1, γ_2.

$$\gamma^{(T)}_{ij,k\ell} = [\gamma_1 \delta_{ik}\delta_{j\ell} + \gamma_2 \delta_{i\ell}\delta_{jk} - \frac{1}{3}(\gamma_1 + \gamma_2)\delta_{ij}\delta_{k\ell}],$$

and does not depend on *one* scalar coefficient as for a rarefied gas. This fact is frequently overlooked (e.g., Rogers [6, 7]).

The second viscosity part of the viscosity tensor depends only on one scalar coefficient, exactly as for a rarefied gas (Lifshitz, Pitaevskii [5]).

The author would like to thank The Polish Academy of Science for supporting this work.

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Received by Publishing Department
on January 20 1981.