2272/2-81

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T.Paszkiewicz

FIRST AND SECOND PHONON VISCOSITIES FOR DIELECTRIC CRYSTALS

1981

Frequently, hydrodynamic phenomena in dielectric crystals are discussed in terms of kinetic coefficients proper for an isotropic gas real particles. For example, Rogers used the equation for macroscopic drift velocity V of a gas of phonons, which contains two scalar coefficients of the first and second viscosity ζ and η, respectively [1,2]. We shall show that for dielectric crystals the tensor of the first viscosity depends on two scalar coefficients. For this purpose we shall use the results obtained by Gurevich [3] (cf. also Lifshitz, Pitaevskii [4]).

1. VISCOSITY TENSOR

Since we are interested in purely hydrodynamic phenomena we confine our discussion to massive perfect crystals at low temperatures. Under such circumstances the only important processes which change the total quasimomentum of the gas of phonons are the Umklapp processes. At low temperatures such processes are weak. Moreover we suppose that

$$I_U h k_\alpha \sim -\epsilon h k_\alpha ,$$

where ϵ is equal to the Knudsen number Kn which is small in the hydrodynamic regime

$$\epsilon \sim Kn \ll 1.$$

Here I_U is the Umklapp part of the linearized collision operator I, and $h\vec{k}$ is the quasimomentum of a phonon (cf., for example, Beck [5]). Since the dissipation of quasimomentum is weak, the hydrodynamic flows in a dielectric crystal are described in terms of two inhomogeneous time-dependent fields. One of them is the scalar field of temperature $T(\vec{r},t)$, the second one is the vectorial field of the drift velocity $\vec{V}(\vec{r},t)$. In order to describe the phonon part of dissipation processes one introduces the deviation function $\delta f(K;\vec{r},t)$. The argument $K = (\vec{k},j)$ labels the wave vector $\vec{k}$ and the polarization j of acoustic phonons. For the sake of convenience one takes δf in the form of product

$$\delta f(K;\vec{r},t) = m(K)g(K;\vec{r},t),$$

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where $m(K)$ is the derivative of the Planck equilibrium function $f_0(K)$

$$m(K) = -\frac{\partial f_0(K)}{\partial \omega(K)} = \hbar \beta f_0(K)(f_0(K)+1),$$

where $\beta = (k_B T_0)^{-1}$, T_0 being the temperature of the equilibrium state.

The deviation function obeys the kinetic equation

$$I_N[g(K;\vec{r},t)] = \{\frac{\omega(k)}{T_0} v_{K\nu} - s k_\mu (\rho-1)_{\mu\nu} \} \frac{\partial T(r,t)}{\partial r_\nu} + \{k_\mu v_{K\nu} - \frac{s\omega(k)}{c_V} \delta_{\mu\nu}\} \times$$

$$\times \frac{\partial V_\mu(r,t)}{\partial r_\nu} + \{I_U[k_\nu] - k_\gamma(\rho-1)_{\gamma\lambda} D_{\lambda\gamma}\rho_{\gamma\nu}\} V_\nu(\vec{r},t). \tag{1}$$

The presence of the last term differs our kinetic equation (1) from that obtained by Gurevich[3]. This term gives rise to the current of energy. However, the relaxation time approximation[5] in which this term vanishes, indicates that this term is small. Let us explain our notation. The group velocity $\vec{v}_K$ is the derivative of the frequency $\omega(K)$, s is the entropy per unit volume, c_V is the specific heat and $\rho_{\mu\nu}$ is a component of the phonon mass density tensor. Finally, a component $D_{\mu\nu}$ of the tensor D is proportional to the matrix element of the Umklapp part of the linearized collision operator I_U

$$D_{\mu\nu} = \frac{\hbar}{v}(g_\mu, I_U g_\nu)(\rho^{-1})_{\mu\nu} ; \quad (\vec{g} \equiv \vec{k}),$$

v being the volume of the unit cell. The normal or Umklapp part of the collision operator is introduced with the help of the linearized collision integrals[5]. For an arbitrary function of K, say $\mu(K)$, one defines

$$(I_i\mu)(K) = I_i[\mu(K)], \quad (i = N, U).$$

The frequency is denoted by $g_0(K)$; and the α-th component of quasimomentum, by $g(K)$. The scalar product is introduced with the help of the real bilinear form

$$(\mu,\nu) = \frac{1}{N} \sum_K \mu(K) m(K) \nu(K),$$

N being the number of unit cells of a crystal. One can show that collision operators are nonpositive real symmetric operators (cf., for example[5])

$$(I_i\mu,\nu) = (I_i\nu,\mu), \quad (I_i\mu,\mu) \leq 0, \quad (i = N,U).$$

Thus, the tensor D, describing the lattice part of the dissipation of the total quasimomentum of the gas of phonons, is nonpositive.

We shall confine ourselves only to the discussion of the viscosity. With the use of the explicit expression for the density of quasimomentum

$$\Pi_{\alpha\beta}(\vec{r},t) = -\sum_K h k_\alpha v_{K\beta} \delta f(K;\vec{r},t)$$

one obtains the expression for components of the viscosity tensor

$$\eta_{\alpha\beta,\mu\nu} = -Nh(g_\alpha v_\beta, \zeta_{\mu\nu}), \tag{2}$$

where $\zeta_{\mu\nu}(K)$ is an even function of $\vec{k}$, being a solution of the kinetic equation

$$I_N[\zeta_{\mu\nu}(K)] = \{ k_\mu v_{K\nu} - \frac{\omega(K)s}{c_V} \delta_{\mu\nu} \}.$$

The function $\zeta_{\mu\nu}(K)$ obeys the subsidiary condition which guarantees the uniqueness of solutions of the above equation

$$(g_0, \zeta_{\mu\nu}) = 0. \tag{3}$$

The condition (3) allows us to write $\eta_{\alpha\beta,\mu\nu}$ in a symmetric form

$$\eta_{\alpha\beta,\mu\nu} = -Nh(I_N \zeta_{\alpha\beta} \zeta_{\mu\nu}).$$

Thus, the tensor η is nonnegative and symmetric in pairs of indices

$$\eta_{\alpha\beta,\mu\nu} = \eta_{\mu\nu,\alpha\beta} \tag{4}$$

The form (2) explains the reason for a lower symmetry of the viscosity tensor of dielectric crystals in comparison with that for gas of real particles. For a gas (cf. ref.[4] formula (8.8) with suitable changes of notation)

$$\eta_{\alpha\beta,\mu\nu} \sim -(p_\alpha v_\beta, \zeta_{\mu\nu}) = -\frac{1}{m}(p_\alpha p_\beta, \zeta_{\mu\nu}).$$

The viscosity tensor is in addition invariant under interchange of indices within the pairs. This happens because the momentum of a particle is proportional to its velocity. For a crystal, the quasimomentum cannot be related to the group velocity in such a simple way.

2. FIRST AND SECOND VISCOSITIES

Now, we shall follow the discussion given by Lifshitz and Pitaevskii[4]. We shall confine our considerations to the case of isotropic (polycrystalline) media.

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Let us solve the kinetic equation (1) with all terms vanishing but the term proportional to the derivative of a velocity. This equation can be cast to the form with terms giving the first and second viscosities to be easily distinguished

$$I_N[g] = k_\mu v_{K\gamma} \left(\frac{\partial V_\mu}{\partial r_\gamma} - \frac{1}{3} \operatorname{div} \vec{V} \delta_{\mu\gamma} \right) + \left(\frac{1}{3} \vec{k} \vec{v} - \frac{\omega(K)s}{c_V} \right) \operatorname{div} \vec{V}. \tag{5}$$

In order to find the first viscosity, let us consider the equation which follows from Eq. (5) for the divergenceless velocity $\vec{V}'(\vec{r},t)$

$$I_N[g] = \left(k_\mu v_{K\gamma} - \frac{1}{3} \vec{k} \vec{v} \delta_{\mu\gamma} \right) \frac{\partial V'_\mu}{\partial r_\gamma}. \tag{6}$$

Since the derivative of the velocity component can take an arbitrary value we look for the solution of Eq. (6) in the form

$$g(K;\vec{r},t) = \zeta^{(I)}_{\mu\nu}(K) \frac{\partial V'_\mu}{\partial r_\nu} \quad .$$

The tensorial function $\zeta^{(I)}_{\mu\nu}(K)$ obeys the equation

$$I_N[\zeta^{(I)}_{\mu\nu}(K)] = \left(k_\mu v_{K\nu} - \frac{1}{3} \vec{k} \vec{v} \delta_{\mu\nu} \right) . \tag{7}$$

Because one can subtract the term $\frac{1}{3} \delta_{\mu\nu} \operatorname{Tr} \zeta(K)$ from $\zeta_{\mu\nu}(K)$ without changing the function $g(K;\vec{r},t)$, one can assume that $\zeta_{\mu\nu}(K)$ is traceless. Hence, the trace of both sides of Eq. (7) vanishes. This means that for an isotropic medium the subsidiary condition

$$(\zeta^{(I)}_{\mu\nu}, g_0) = 0$$

is automatically fulfilled. The tensor of the first viscosity

$$\eta^{(I)}_{\alpha\beta,\mu\nu} = -hN(g_\alpha v_\beta, \zeta^{(I)}_{\mu\nu})$$

contracted over the second pair of indices vaniches

$$\eta^{(I)}_{\alpha\beta,\mu\mu} = -hN(g_\alpha v_\beta, \zeta^{(I)}_{\mu\mu}) = 0. \tag{8}$$

The same property has $\eta^{(I)}$ contracted over the first pair of indices.

One can easily check that $\eta^{(I)}$ is a nonnegative tensor, symmetric in pairs of indices,

$$\eta^{(I)}_{\alpha\beta,\mu\nu} = -hN\left(g_\alpha v_\beta - \frac{1}{3} g v \delta_{\alpha\beta}, \zeta^{(I)}_{\mu\nu} \right) = -hN(I_N \zeta^{(I)}_{\alpha\beta}, \zeta^{(I)}_{\mu\nu}),$$

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$$\eta^{(I)}_{\alpha\beta,\alpha\beta} \leq 0.$$

The most general form of the viscosity tensor $\eta^{(I)}$, compatible with the rotational symmetry and the condition (8), is

$$\eta^{(I)}_{\alpha\beta,\mu\nu} = \eta\delta_{\alpha\mu}\delta_{\beta\nu} + a\delta_{\alpha\nu}\delta_{\beta\mu} - \frac{1}{3}(\eta+a)\delta_{\alpha\beta}\delta_{\mu\nu}. \tag{9}$$

This form is different from that for a gas[4]

$$\eta^{(I)}_{\alpha\beta,\mu\nu} = \eta[\delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu} - \frac{2}{3}\delta_{\alpha\beta}\delta_{\mu\nu}]. \tag{10}$$

In order to find the second viscocity we should solve the equation for a scalar, even in $\vec{K}$ function $\zeta^{(II)}(K)$

$$I_N[\zeta^{(II)}(K)] = (\frac{1}{3}\vec{k}\vec{v}_K - \frac{\omega(k)s}{c_V}).$$

The second viscosity tensor is proportional to the unit tensor, with the nonnegative coefficient of proportionality

$$\eta^{(II)} = -Nh(\frac{1}{3}\vec{g}\vec{v}, \zeta^{(II)}) = -Nh(I_N\zeta^{(II)}, \zeta^{(II)}) \geq 0.$$

In the derivation of the last equation, we have used the subsidiary condition

$$(g_0, \zeta^{(II)}) = 0.$$

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