сообщения
объединенного
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E17-80-756

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RELATIONS FOR CRITICAL AMPLITUDES

1980

1. Here we continue discussing of the critical behaviour in the framework of a recently proposed method, involving, in particular, the introduction of a special auxiliary critical system /1-3/. In the "mathematical aspect" the method is based, in particular, on a version of the theory of "quasi-averages" proposed by N.N.Bogolubov, Jr./4/ (for discussion see ref./1/).

In the present paper, closely related to ref./3/, we consider relations for the parameters of the critical singularities - indices of power asymptotics, indices of logarithmic corrections and critical amplitudes*.

For the basic indices we get the well-known scaling equalities, for logarithmic indices and amplitudes the relations obtained are new and previously unknown. The results are compared with experimental data**.

The reader, being interested only in the final results, may familiarize himself with notation (10), relations in tables 1,2,4 and concluding remarks in section 8.

2. Let us briefly account here some results obtained in ref./3/ necessary below (as an introduction to ref./3/ see refs./1,2/).

We denote an arbitrary system with Hamiltonian Γ and temperature $\theta = kT$ as Γ/θ, operator order parameters as A, B, ..., and corresponding susceptibilities as χ_{AB}, ... Numerical order parameters (i.e., averages $\langle A \rangle_{\Gamma/\theta}$, ...) will be denoted $A(\Gamma/\theta)$, ... , and, in general, we denote any quantity F depending on the system Γ/θ as $F(\Gamma/\theta)$ or $F[\Gamma/\theta]$.

We shall consider a conventional "ferromagnetic" system with Hamiltonian H and critical temperature θ_c (H/θ_c is the critical system). For a nonzero magnetic field $h > 0$ the Hamiltonian of the system is $H - hNS$, where $S = S^+$ is

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*We call below the index of power asymptotics the "basic index" or "index", and the index of logarithmic correction "logarithmic index". For the asymptotics $A\varepsilon^{-\alpha}|\ln\varepsilon|^{p_\alpha}$, $\varepsilon \to +0$, α is the basic index, p_α is the logarithmic index, A is the critical amplitude.

**By "experimental data" we shall assume also the results of numerical calculations for concrete models (e.g., for the Ising lattices).

the magnetization operator (per particle)*, N is the number of particles. We shall discuss here only systems with the one-component order parameter ($n=1$) (e.g., the Ising lattices).

Let us consider, in addition to the basic order parameter S, extra order parameter A and assume that both order parameters vanish at the critical point: $A(H/\theta_c)=S(H/\theta_c)=0$.

So, H/θ_c is the critical system. Consider the system in the ordered phase with $\theta=\theta_c(1-\varepsilon)<\theta_c, h=0$, i.e., the system $H/\theta_c(1-\varepsilon), \varepsilon>0, \varepsilon\to+0$. In ref./3/ we have considered an arbitrary system in the ordered phase $H+V/\theta_c$, where V is some "ordering" and "weak" variation of the Hamiltonian. In the case $H/\theta_c(1-\varepsilon)$ $V_\varepsilon=\varepsilon H/(1-\varepsilon)\cong\varepsilon H$ and the condition for V_ε to be "ordering" and "weak" means that $|Y[H/\theta_c(1-\varepsilon)]|>0$, $\chi^{-1}_{YY}[H/\theta_c(1-\varepsilon)]>0; Y=A,S$, for $\varepsilon>0$, and that system $H/\theta_c(1-\varepsilon)$ can be removed into the critical point by introducing, into the Hamiltonian, of the "disordering" term ΔNS^2 by appropriate value of Δ /2,3/ :

$$\frac{H_\varepsilon}{\theta_c}\equiv\frac{H+\Delta(\varepsilon)NS^2}{\theta_c(1-\varepsilon)}=\left\{\begin{matrix}\text{the critical}\\ \text{system}\end{matrix}\right\}. \quad (1)$$

Making use of the results of ref./3/ one can easily write relations connecting behaviour of the order parameters and susceptibilities in the systems $H/\theta_c(1-\varepsilon)$ and $H_\varepsilon-hNS/\theta_c$, where H_ε/θ_c is auxiliary critical system (1).

Introduce short notation $F(\varepsilon)\equiv F[H/\theta_c(1-\varepsilon)]$, $F^{(\varepsilon)}(h)\equiv F[H_\varepsilon-hNS/\theta_c]$, $F=A,S,\chi_{AA},\ldots$; denote as $h(\varepsilon)$ the effective magnetic field (see below (8)). Taking into account formulas (37)-(44) in ref. /3/, one gets, in particular:

$$A(\varepsilon)=A^{(\varepsilon)}(h(\varepsilon)), \quad (2)$$

$$S(\varepsilon)=S^{(\varepsilon)}(h(\varepsilon)), \quad (3)$$

$$\chi_{SS}(\varepsilon)=1/2\Delta(\varepsilon)(\delta_{SS}[\varepsilon]-1), \quad (4)$$

$$\chi_{AS}(\varepsilon)=\frac{\delta_{SS}[\varepsilon]}{\delta_{SS}[\varepsilon]-1}\chi^{(\varepsilon)}_{AS}(h(\varepsilon)), \quad (5)$$

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*For the Ising model, for instance, $S=N^{-1}\sum_{i=1}^{N}\sigma_i$, $\sigma_i=\pm1$.

$$\chi_{AA}(\varepsilon) = \frac{\delta_{SS}[\varepsilon]}{\delta_{SS}[\varepsilon] - \cos^2{}_{AS}(\varepsilon)}\, \chi^{(\varepsilon)}_{AA}(h(\varepsilon)), \quad (6)$$

$$\frac{\chi_{AS}(\varepsilon)}{\chi_{SS}(\varepsilon)} = \frac{\delta_{SS}[\varepsilon]}{\delta_{AS}[\varepsilon]}\, \frac{A(\varepsilon)}{S(\varepsilon)}, \quad (7)$$

with the effective field

$$h(\varepsilon) = 2\Delta(\varepsilon) S(\varepsilon) \equiv S(\varepsilon)/(\delta_{SS}[\varepsilon]-1)\chi_{SS}(\varepsilon), \quad (8)$$

here the superscript (ε) means that the corresponding quantities relate to the auxiliary critical system (1).

The above relations include also the "index-functions" $\delta_{SS}[\varepsilon]$, $\delta_{AS}[\varepsilon]$, defined as follows. Introduce for the original critical system H/θ_c the function /3/

$$\delta_{YS}(h) = Y(h)/h\,(dY(h)/dh), \quad Y(h) \equiv Y[H - hNS/\theta_c].$$

Then $\delta_{YS}[\varepsilon] \equiv \delta^{(\varepsilon)}_{YS}(h = h(\varepsilon))$ where superscript (ε) indicates that H/θ_c in $\delta_{YS}(h)$ should be replaced by H_ε/θ_c (1); $h(\varepsilon)$ see in (8). Note that if $Y(h) \sim h^{1/\delta_{YS}}$ (δ_{YS} = constant), then $\delta_{YS}(h) \to$ $\to \delta_{YS}$ as $h \to 0$. We shall assume below the power asymptotics for order parameters and susceptibilities (see (10)) and consider "index-functions" $\delta_{YS}[\varepsilon]$ in (2)-(8) to be constants, despite that H/θ_c is replaced here by H_ε/θ_c .

Formula (6) involves also the "squared cosine of the angle between A and S ", $\cos^2{}_{AS}$. For arbitrary system Γ/θ and order parameters A and B we define /3/ : $\cos^2{}_{AB} \equiv |\chi_{AB}|^2/\chi_{AA}\chi_{BB}$, and independently of the system Γ/θ :

$$0 \le \cos^2{}_{AB}(\Gamma/\theta) \equiv \frac{|\chi_{AB}(\Gamma/\theta)|^2}{\chi_{AA}(\Gamma/\theta)\chi_{BB}(\Gamma/\theta)} \le 1 \quad (9)$$

(For more details see ref./3/).

3. When studying the critical behaviour one usually deals with two "external fields" h and θ and two corresponding "order parameters" S and H (Hamiltonian). It is convenient to introduce, instead of H , a correctly normilized "temperature order parameter" (singular part of the energy)

$L=-N^{-1}(H-\langle H\rangle_{H/\theta_c})$. We thus have quantities L, S, χ_{SS}, $\chi_{SL}\equiv$ $\equiv\chi_{LS},\chi_{LL}$ to be considered for $h\equiv 0$, $\theta\neq\theta_c$ (we shall deal with $\theta<\theta_c$ only) and for $h>0$, $\theta\equiv\theta_c$. So we have 10 functions of interest, among which only 6 are independent (4 susceptibilities appear to be derivatives of the order parameters).

Note that in the critical region with vanishing errors: $\chi_{LL}=\theta_c C,\ \chi_{SL}=\partial S/\partial\varepsilon$, where C is the specific heat for a fixed magnetic field, $C\equiv C_h$.

Following the well-known experimental and theoretical results (see, e.g., ref. /5/), let us suppose for the order parameters and susceptibilities the power asymptotics with possible logarithmic corrections to hold true:

$$S(\varepsilon)=B\varepsilon^{\beta}|\ln\varepsilon|^{P_\beta},\quad S(h)=\left(\frac{h}{D}\right)^{\frac{1}{\delta}}|\ln h|^{P_\delta},$$
$$L(\varepsilon)=\frac{A_-\varepsilon^{1-\alpha}}{1-\alpha}|\ln\varepsilon|^{P_\alpha},\quad L(h)=Z\,h^{\zeta}|\ln h|^{P_\zeta}, \tag{10}$$
$$\chi_{SS}(\varepsilon)=\Gamma_-\varepsilon^{-\gamma}|\ln\varepsilon|^{P_\gamma},\quad \chi_{LL}(h)=E\,h^{-\epsilon}|\ln h|^{P_\epsilon}.$$

Differentiating order parameters in (10), one also gets:

$$\chi_{LL}(\varepsilon)=A_-\varepsilon^{-\alpha}|\ln\varepsilon|^{P_\alpha},\quad \chi_{SS}(h)=\frac{h^{\frac{1}{\delta}-1}}{\delta D^{1/\delta}}|\ln h|^{P_\delta},$$
$$\chi_{SL}(\varepsilon)=\beta B\varepsilon^{\beta-1}|\ln\varepsilon|^{P_\beta},\quad \chi_{SL}(h)=\zeta Z h^{\zeta-1}|\ln h|^{P_\zeta}, \tag{10a}$$

where $\alpha,\beta,\gamma,\delta,\zeta,\epsilon$ are critical indices, $P_\alpha,P_\beta,P_\gamma,P_\delta,P_\zeta,P_\epsilon$ are logarithmic indices, A_-,B,Γ_-,D,Z,E are critical amplitudes*. Here we have used the short notation $F(\varepsilon)\equiv F[H/\theta_c(1-\varepsilon)]$, $F(h)\equiv F[H-hNS/\theta_c]$.

It is known that critical indices in (10) are not all independent. There are 4 relations among 6 indices following from the scaling hypothesis, which can be chosen, e.g., in the form:

a) $\gamma=\beta(\delta-1)$ c) $\zeta(\gamma+\beta)=1-\alpha$

b) $\alpha+2\beta+\gamma=2$ d) $\epsilon(\gamma+\beta)=\alpha$. (11)

We shall discuss relations (11) in the framework of our approach and also derive new relations for logarithimic indices and critical amplitudes.

4. The simplest (and too rough) method is the following

*Subscript (-) in A_-,Γ_- means the region $\theta<\theta_c$. Sometimes instead of A_- A_-/α is used for $\alpha>0$ (and A_- for $\alpha=0$) (note that such a notation is in ref. /3/). For the parameters of the asymptotics $L(h)$, $\chi_{SL}(h),\chi_{LL}(h)$ there are no generally accepted notations; we originate from the order of the Greek alphabet.

(see also ref./3/). Put in (2)-(8) $A = L$ and assume that asymptotics like on the right-hand side of (10),(10a) remain valid if one replaces the critical system H/θ_c by the auxiliary critical system H_ε/θ_c (1); neglect the dependence H_ε/θ_c upon ε (as $\varepsilon \to 0$), but for precaution supply all quantities related to H_ε/θ_c by the superscript 0 ; replace everywhere $\delta_{YS}[\varepsilon]$ by the constants δ°_{YS}; $\delta^\circ_{SS} \equiv \delta^\circ$, $\delta^\circ_{LS} \equiv 1/\zeta^\circ$.

For the effective field (8) one then obtains:

$$h(\varepsilon) = \frac{S(\varepsilon)}{(\delta^\circ - 1)\chi_{SS}(\varepsilon)} = \frac{B}{\Gamma_-(\delta^\circ - 1)}\,\varepsilon^{\gamma+\beta}\,|\ln\varepsilon|^{P_\beta - P_\gamma} \tag{12}$$

Substituting asymptotics (10) and (12) into (2),(3),(6),(7), one obtains for the basic indices:

$$\begin{aligned} &\text{a)}\quad \gamma = \beta(\delta^\circ - 1) && \text{c)}\quad \zeta^\circ(\gamma+\beta) = 1-\alpha \\ &\text{b)}\quad \alpha + 2\beta + \gamma = 2 && \text{d)}\quad \epsilon^\circ(\gamma+\beta) = \alpha\,, \end{aligned} \tag{13}$$

for the logarithmic indices:

$$\begin{aligned} &\text{a)}\ \delta^\circ P^\circ_\delta = P_\gamma + P_\beta(\delta^\circ - 1) && \text{c)}\ P^\circ_\zeta = P_\alpha + \zeta^\circ(P_\gamma - P_\beta) \\ &\text{b)}\ P_\alpha + P_\gamma = 2P_\beta && \text{d)}\ P^\circ_\epsilon = P_\alpha + \epsilon^\circ(P_\beta - P_\gamma), \end{aligned} \tag{14}$$

and also 4 equalities for amplitudes. In particular, it follows from (3) in addition to (a) in (13),(14) that:

$$\left((\delta^\circ - 1)\Gamma_- D^\circ B^{\delta^\circ - 1}\right)^{\frac{1}{\delta^\circ}} (\gamma+\beta)^{-P^\circ_\delta} = 1\,, \tag{15}$$

and from (7) in addition to (b):

$$\beta^2 B^2 / A_- \Gamma_- = 1 \tag{16}$$

(here we have taken into account that $\zeta^\circ\delta^\circ/(1-\alpha) \equiv 1/\beta$, see (13)). Two other relations for amplitudes follow from (2) and (6) (see (49) in ref./3/), but these relations, as well as (16), are to be modified (see below).

For futher needs it seems to be convenient to make now a stop and consider, independently, functions $\cos^2_{SL}(\varepsilon) \equiv \cos^2_{SL}[H/\theta_c(1-\varepsilon)]$ and $\cos^2_{SL}(h) \equiv \cos^2_{SL}[H - hNS/\theta_c]$ for the asymptotics (10). We have:

$$\cos^2_{SL}(\varepsilon) = \frac{\beta^2 B^2}{A_- \Gamma_-}\,\varepsilon^{\gamma+2\beta+\alpha-2}\,|\ln\varepsilon|^{2P_\beta - P_\gamma - P_\alpha}, \tag{17a}$$

$$\cos^2{}_{SL}(h) = \frac{\delta \zeta_s^2 Z^2 D^{1/\delta}}{E} h^{\epsilon + 2\zeta_s - 1/\delta - 1} |\ln h|^{2P_{\zeta_s} - P_\delta - P_\epsilon} \quad (17b)$$

Taking into account that functions (17) satisfy general inequalities (9), one can easily obtain restrictions on critical parameters presented in tables 1 and 2. Each table can be read in arbitrary sequence; blank spaces mean the absence of restrictions*.

T a b l e 1. Allowed relations between temperature critical characteristics and $\cos^2{}_{SL}(\varepsilon = +0)$.

$\alpha + 2\beta + \gamma = 2$		$\alpha + 2\beta + \gamma > 2$
$P_\alpha + P_\gamma = 2P_\beta$	$P_\alpha + P_\gamma > 2P_\beta$	
$\frac{\beta^2 B^2}{A_- \Gamma_-} = \cos^2{}_{SL}(\varepsilon = +0) \leq 1$		
$\cos^2{}_{SL}(\varepsilon = +0) > 0$	$\cos^2{}_{SL}(\varepsilon = +0) = 0$	

T a b l e 2. Allowed relations between field critical characteristics and $\cos^2{}_{SL}(h = 0)$

$\epsilon + 2\zeta_s - 1/\delta = 1$		$\epsilon + 2\zeta_s - 1/\delta > 1$
$P_\delta + P_\epsilon = 2P_{\zeta_s}$	$P_\delta + P_\epsilon > 2P_{\zeta_s}$	
$\delta \zeta_s^2 Z^2 D^{1/\delta} / E = \cos^2{}_{SL}(h=0) \leq 1$		
$\cos^2{}_{SL}(h = 0) > 0$	$\cos^2{}_{SL}(h = 0) = 0$	

As follows from table 1, one should distinguish between two different situations:

a) $\cos^2{}_{SL}(\varepsilon = +0) > 0$ b) $\cos^2{}_{SL}(\varepsilon = +0) = 0$.(18)

*The reader can easily see that table 1 contains, in particular, the equality due to Essam and Fisher $\alpha + 2\beta + \gamma = 2$ /6/ (see (11b)) and inequality due to Rushbrooke $\alpha + 2\beta + \gamma \geq 2$ /7/; table 2 contains analogous relations for the field characteristics discussed by Coopersmith /8/ .

In the case (18a) the equalities (b) in (13),(14) do hold true and deliberately $\beta^2 B^2/A_- \Gamma_- \leq 1$. In the case (18b) at least one of the equalities (13b), (14b) is violated. Analogous conclusions follow from table 2 for the field parameters.

We thus see that relations (13b), (14b) and (16) taken together mean that

$$\cos^2_{SL} (\varepsilon = +0) = 1 \quad . \tag{19}$$

Therefore it is interesting to consider $\cos^2_{SL} (\varepsilon = +0)$ for concrete systems. The values of $\cos^2_{SL} (\varepsilon = +0)$ for the Ising lattices of different space dimensions (d) are represented in table 3*.

Here Q_A means the dimensionless amplitude ratio:

$$Q_A = \beta^2 B^2/A_- \Gamma_- , \tag{20}$$

under the condition (18a) Q_A coincides with $\cos^2_{SL} (\varepsilon = +0)$; "yes" and "no" indicate whether or not (13b),(14b) hold true; $\times$ means the absence of logarithmic corrections.

T a b l e 3. Values of $\cos^2_{SL} (\varepsilon = +0)$ and other parameters for the Ising lattices of different dimensions.

Ising	(13b)	(14b)	Q_A	$\cos^2_{SL}(\varepsilon=+0)$	$1-\cos^2_{SL}(\varepsilon=+0)$	$\frac{A_+}{A_-}$
Molecular field ($d=\infty$)	yes	x	1	1	0	0
$d = 4$	yes	yes	0.75	0.75	0.25	0.25
$d = 3$	yes	x	0.53	0.53	0.47	0.51
$d = 2$	yes	no	I.85	0	1	1

Note that for the $d=4$ Ising model $P_\alpha = P_\beta = P_\gamma = 1/3$ and (14b) holds true, but for $d = 2$ $P_\alpha = 1$, $P_\beta = P_\gamma = 0$ and (14b) is violated, here $\cos^2_{SL} (\varepsilon) \cong 1.85 \, |\ln \varepsilon|^{-1} \to 0$ as $\varepsilon \to 0$. The two last columns of talble 3 will be discuss below.

*The most of data for concrete systems used in this paper, as well as numerious references to original papers, can be found in ref./9/ ; for $d = 4$, see also ref. /10/ .

We thus see that in real cases $\cos^2_{SL}(\varepsilon = +0)$ does not reach its maximum (19),

$$\cos^2_{SL}(\varepsilon = +0) < 1, \qquad d = 2,3,4, \qquad (21)$$

while $\cos^2_{SL}(\varepsilon = +0) = 1$ corresponds to the molecular-field approximation ("$d = \infty$").

5. To explain the disagreement between (19) and (21), we propose here a phenomenological scheme. Let us consider that in the critical region (for $\theta < \theta_c$) the temperature order parameter L can be represented as the sum of two "orthogonal" components L_1, L_2 where L_1 is "parallel" to S and L_2 is "orthogonal" to S,

$$L = L_1 + L_2, \qquad L_1 \parallel S,\ L_2 \perp S,\ L_1 \perp L_2, \qquad (22)$$

and the parallelism and orthogonality are assumed in the sense that $\chi_{SL_2} = 0$, $\chi_{L_1L_2} = 0$ (but $\chi_{SL_1} \neq 0$, $\chi_{L_1L_1} \neq 0$, $\chi_{L_2L_2} \neq 0$), $\cos^2_{SL_1} = 1$, $\cos^2_{SL_2} = 0$ (all quantities are taken for the system $H/\theta_c(1-\varepsilon)$, $\varepsilon \to +0$) *. Then $\chi_{SL} = \chi_{SL_1}$, $\chi_{LL} = \chi_{L_1L_1} + \chi_{L_2L_2}$ and we have**:

$$a)\ \chi_{L_1L_1}(\varepsilon) = \chi_{LL}(\varepsilon)\cos^2_{SL}(\varepsilon), \quad b)\ \chi_{L_2L_2}(\varepsilon) = \chi_{LL}(\varepsilon)(1 - \cos^2_{SL}(\varepsilon)). \qquad (23)$$

We shall also assume that the term $\Delta(\varepsilon)NS^2$ in (1) "acts" only on L_1 and does not affect L_2 (and thus the system H_ε/θ_c (1) is not critical with respect to L_2 and coincides with $H/\theta_c(1-\varepsilon)$), and, on the contrary, $L_1 \parallel S$, and one may hope that with respect to L_1 the properties of the systems H_ε/θ_c and H/θ_c are the same or similar.

It has been shown at the end of ref. 3 that in disordered phase the

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*Note that χ_{AB} possesses the properties of the scalar product on the set of order parameters A, B, ... /3/ .

**The well-known thermodynamical relation for the specific heat for a fixed magnetic field $C \equiv C_h$ and fixed magnetization C_m in our notation can be written in the form $C_m(\Gamma/\theta) = C_h(\Gamma/\theta)(1 - \cos^2_{SL_\Gamma}(\Gamma/\theta))$, $L_\Gamma \equiv \Gamma/N$. Taking into account (23) we see that in fact $\chi_{L_2L_2} \equiv \theta C_m$.

term $\Delta N S^2$ does not effect order parameters different from S, therefore one should expect to have in the disordered phase the operator $L_2 \perp S$ only, $L = L_2$ (note also that $\cos^2_{SL} \equiv 0$ for $\theta > \theta_c$).

In the ordered phase we have in addition an "inner molecular field" (selfconsistent magnetic field) originating from the spontaneous ordering of the interacting magnetic moments. So, let us interpret L_1 as the "molecular field energy" and L_2 as the "proper temperature energy".

In the framework of such a phenomenological scheme one can expect that the specific heat at fixed magnetization C_m (see footnote ** on page 8) in the system $\theta < \theta_c$, $H/\theta_c(1-\varepsilon)$, should coincide with the specific heat $C \equiv C_h$ in the system $H/\theta_c(1+\varepsilon)$, $\theta > \theta_c$. If so, one can easily see from the relation for C_m and C_h (footnote ** on page 8) that critical specific-heat amplitudes A_+ ($\theta > \theta_c$) and A_- ($\theta < \theta_c$) should satisfy the relation:

$$\frac{A_+}{A_-} = 1 - \cos^2_{SL}(\varepsilon = +0). \qquad (24)$$

As is seen from table 3, this relation really holds true for the Ising model (we must note that for $d=3$ data available are rough).

Now we are able to explain the disagreement of (19) with (21). The fact is that the choice $A = L$ in (2)-(8) is incorrect; one should put $A = L_1$. Then instead of (19) one gets the identity $\cos^2_{SL} = \cos^2_{SL}$ (see also below).

Let us turn now to relations (2),(6), from which the equalities (c) and (d) in (13),(14) have been derived.

In the framework of our phenomenological scheme let us assume that switching on a nonzero field $h > 0$ at $\theta \equiv \theta_c$ "excites" only the energy L_1 and $\chi_{LL}(h)$ is approximated by $\chi^0_{L_1 L_1}(h)$. Putting in (6) $A = L_1$ and taking into account (23a) we get:

$$\chi_{LL}(\varepsilon)\cos^2_{SL}(\varepsilon) = \frac{\delta^\circ}{\delta^\circ - 1}\, \chi^{(0)}_{LL}(h(\varepsilon)). \qquad (25)$$

One can also put $A = L_1$ in (2) and express $L_1(\varepsilon)$ through $L(\varepsilon)$. If one assumes that $\chi_{L_1 L_1}(\varepsilon) \equiv \partial L_1(\varepsilon)/\partial \varepsilon$ and takes into account (23a), rewriting the right-hand side of this equality in the case of the asymptotics (10), then $L_1(\varepsilon) =$

$= \frac{1-\alpha}{\gamma+2\beta-1} L(\varepsilon) \cos^2_{SL}(\varepsilon)$ and in view of (2) one obtains:

$$L^{(\varepsilon)}(h(\varepsilon)) = \frac{1-\alpha}{\gamma+2\beta-1} L(\varepsilon) \cos^2_{SL}(\varepsilon). \qquad (26)$$

Note that if one puts in (7) $A = L_1$ and takes into account the above relation for $L_1(\varepsilon)$ and $L(\varepsilon)$, then (7) becomes identity.

Substituting the asymptotics (10),(17a) into relations (25), (26) we obtain equalities for critical parameters presented in the second (c) and third (d) lines of table 4. Four types of relations in this table (a,b,c,d) correspond to the four equalities in (11). The first line (a) will be considered below; line (b) in the "general case" of table 4 is absent, this line should contain parameters α, P_α, A_-, the relations of which with other parameters are governed by table 1. For the important special case $\cos^2_{SL}(\varepsilon=+0) > 0$ (see (18b)) we have relations presented in the last three lines of table 4 (the relation (a) remains unchanged). /In table 4 we omit the superscript (0) indicating the auxiliary critical system (1)/.

The verification of the equalities (c) and (d) (or (c') and (d')) of table 4 is of considerable interest, but because of the lack of data on the "field" asymptotics we are now unable to do this.

It is interesting to note that in the $d = 2$ Ising model $P_\alpha = 1$, $P_\beta = P_\gamma = 0$ and from table 4(d) it follows that the specific heat $\chi_{LL}(h)$ is nonsingular as $\theta = \theta_c$, $h > 0$, $h \to 0$ (in spite of $\chi_{LL}(\varepsilon) \sim |\ln \varepsilon| \to \infty$ as $\varepsilon \to 0$ $(h = 0,\ \theta \to \theta_c - 0)$). One can easily see this also directly from (25), where $\chi_{LL}(\varepsilon) \sim |\ln\varepsilon|$, $\cos^2_{SL}(\varepsilon) \sim |\ln \varepsilon|^{-1}$. So, we have here $\chi_{LL}(h=0) = E = const$, $E/A_- = 1.7255...$

6. We now have only to discuss relations (13a),(14a),(15). Since the original equality (3) does not involve L (explicitly), we may hope that the peculiarities discussed above in view of decomposition $L = L_1 + L_2$ would not be so essential for relations (13a),(14a) (see also (a) in table 4). For concrete

T a b l e 4. The main relations for critical parameters

	Type	Basic Indices	Logarithmic Indices	Critical amplitudes
General case $\cos^2_{SL}(\varepsilon=+0)\geqslant 0$	a	$\gamma=\beta(\delta-1)$	$\delta P_\delta = P_\gamma + P_\beta(\delta-1)$	$\left((\delta-1)\Gamma_- D B^{\delta-1}\right)^{\frac{1}{\delta}}(\gamma+\beta)^{-P_\delta} \gtrless 1 \ (\approx 1)$
	c	$\zeta = \dfrac{\gamma+2\beta-1}{\gamma+\beta}$	$P_\zeta = P_\beta\dfrac{\gamma+1}{\gamma+\beta} - \dfrac{1-\beta}{\gamma+\beta}P_\gamma$	$Z = \dfrac{\beta\gamma B}{\gamma+2\beta-1}\left(\dfrac{\beta B}{\gamma\Gamma_-}\right)^{\frac{1-\beta}{\gamma+\beta}}\left(\dfrac{1}{\gamma+\beta}\right)^{P_\beta+\frac{(1-\beta)(P_\beta-P_\gamma)}{\gamma+\beta}}$
	d	$\epsilon = \dfrac{2-\gamma-2\beta}{\gamma+\beta}$	$P_\epsilon = P_\beta\dfrac{\gamma+2}{\gamma+\beta} - P_\gamma\dfrac{2-\beta}{\gamma+\beta}$	$E = \dfrac{\beta\gamma^2 B}{\gamma+\beta}\left(\dfrac{\beta B}{\gamma\Gamma_-}\right)^{\frac{2-\beta}{\gamma+\beta}}\left(\dfrac{1}{\gamma+\beta}\right)^{P_\beta+\frac{(2-\beta)(P_\beta-P_\gamma)}{\gamma+\beta}}$
Special case $\cos^2_{SL}(\varepsilon=+0)=0$	b′	$\alpha+2\beta+\gamma=2$	$P_\alpha+P_\gamma = 2P_\beta$	$\dfrac{\beta^2 B^2}{A_-\Gamma_-} = \cos^2_{SL}(\varepsilon=+0) \leq 1$
	c′	$\zeta = \dfrac{1-\alpha}{\gamma+\beta}$	$P_\zeta = P_\alpha + \zeta(P_\gamma - P_\beta)$	$Z = \dfrac{\beta^2 B^2}{(1-\alpha)\Gamma_-}\left(\dfrac{(\delta-1)\Gamma_-}{B}\right)^{\zeta}(\gamma+\beta)^{-P_\zeta}$
	d′	$\epsilon = \dfrac{\alpha}{\gamma+\beta}$	$P_\epsilon = P_\alpha + \epsilon(P_\beta - P_\gamma)$	$E = \dfrac{\delta-1}{\delta}\,\dfrac{\beta^2 B^2}{\Gamma_-}\left(\dfrac{B}{(\delta-1)\Gamma_-}\right)^{\epsilon}(\gamma+\beta)^{-P_\epsilon}$

systems these relations are usually valid*. Passing to amplitudes and taking into account (15) we see that it would be convenient to introduce dimesionless ratios

$$Q_B = ((\delta-1)\Gamma_- D B^{\delta-1})^{\frac{1}{\delta}}(\gamma+\beta)^{-P_\delta}, \tag{27}$$

$$R_B \equiv Q_B^{\delta} = (\delta-1)\Gamma_- D B^{\delta-1}(\gamma+\beta)^{-\delta P_\delta}. \tag{27a}$$

One can presume that $Q_B = 1$. The values of Q_B for Ising lattices and also experimental data for CO_2 and Xe ** are presented in table 5. One thus sees that in most cases $Q_B > 1$ though close to 1.

T a b l e 5. Parameters Q_B and $2\beta^2(\delta-1)/(1-\alpha)$ for the Ising model and systems CO_2 and Xe

System	Q_B	$\frac{2\beta^2(\delta-1)}{1-\alpha}$
Molecular field ($d=\infty$)	1	1
d =4	1	1
d =3	1.066...	0.89
d =2	1.06350...	0.4375
CO_2	1.06	0.94
Xe	1.09	0.96

We are unable to give an exhaustive explanation of the fact $Q_B > 1$. One of possible interpretations is the following. We assume the spontaneous magnetization to be the sum of two ingredients:

$$S(\varepsilon) = C_o(\varepsilon) + S_1(\varepsilon), \tag{28}$$

where C_o is a "primary" part of magnetization due to "first origins" (say, some imaginary "sum rules",etc.), and the other part S_1 appears as a consequence of the inner molecular field excited by C_c and the term $\Delta(\varepsilon)NS^2$ in (1) compensates only $S_1(\varepsilon)$ in $S(\varepsilon)$***

*Logarithmic corrections are available in the $d=4$ Ising model where $\delta=3$, $P_\alpha=P_\beta=P_\delta=P_\gamma=\frac{1}{3}$ and equalities (a),(b')for logarithmic indices hold true. The best of our knowledge, indices P_{ζ}, P_ϵ (as well as ζ_o, ϵ) have not been calculated for any standard system. If the relations of table 4 are correct, then for the $d=4$ Ising model there should be $\epsilon=0$, $\zeta_o=2/3$, $P_{\zeta}=P_\epsilon=\frac{1}{3}$ (further discussion see in Appendix C).

**Necessary data are taken from tables II-IV of ref. /9/; for CO_2 and Xe there are represented data averaged over three methods of fitting experiment in table IV.

***One can also suppose that on the operator level C_o has " C -number" properties and does not affect χ_{SS}, but leads to "residual" magnetization in system (1), which in such a case should be written as $H + \Delta(\varepsilon)N(S-C_o)^2/\theta_c(1-\varepsilon)$.

Then one should write the equality (3) only for $S_1(\varepsilon) = B_1 \varepsilon^{\beta} |\ln \varepsilon|^{p_\beta}$ and assume $Q_{B(1)} = 1$, where in $Q_{B(1)}$ amplitude B is replaced by $B_1 \lesssim B$; since $Q_{B(1)} \lesssim Q_B$, one gets $Q_B \gtrsim 1$ (table 5). Perhaps, one should also take into account the difference between B and B_1 in other equalities in table 4, which then appear to be approximate relations (however, since really $|B - B_1| \ll B$ the numerical error is expected to be small). Further discussion of the hypothethis (28) see in Appendix B.

Note that if we accept the hypotheses on decompositions (22) and (28), we then get in fact the phenomenological scheme of "three subsystems", what resembles the situation in the theory of superfluidity (C_0 is an analog of the condensate, (S_1, L_1) is an analog of the superfluid component, L_2 is an analog of the normal component).

7. Many authors (details and references see in ref.[9]) have proposed to use for the interpretation of experimental data different dimensionless amplitude ratios: $R = A_+ \Gamma_+ / B^2$, $Q_1 = (B/\Gamma_+ D)^{1/\delta}/B$, $R_\chi \equiv Q_1^{-\delta} = \Gamma_+ D B^{\delta - 1}$, A_+/A_-, Γ_+/Γ_-.
In the framework of our approach more natural dimensionless amplitude ratios are the following:

$$\cos^2_{SL}(\varepsilon = +0), \ \cos^2_{SL}(h=0), \ Q_A = \frac{\beta^2 B^2}{A_- \Gamma_-}, \ \frac{A_+}{A_-},$$
$$Q_B = \left((\delta - 1)\Gamma_- D B^{\delta-1}\right)^{\frac{1}{\delta}} (\gamma + \beta)^{-\beta}, \ R_B = Q_B^{\delta}, \ Q_\Gamma = \frac{\Gamma_+}{\Gamma_-(\delta - 1)} \tag{29}$$

(ratio Q_Γ will be discussed elsewhere). We propose to include ratios (29) in the tables of critical parameters for concrete models with one-component order parameter ($n = 1$). Note that in the molecular-field approximation each ratio in (29) equals 1.

8. Let us formulate some concluding remarks.

a) In addition to the basic indices, when studying critical behaviour, of a considerable interest are logarithmic indices and critical amplitudes.

For logarithmic indices we obtain new relations (table 4) which seem to be exact equalities of the same status as the "scaling" laws for basic indices.

b) Natural dimensionless amplitude ratios seem to be quantities (29), for which there are some theoretical predictions discussed above. For the field amplitudes there are some equalities presented in table 4, which may appear to be only approximate relations.

c) Essential functions in the critical region are $\cos^2_{SL}(\varepsilon)$ and $\cos^2_{SL}(h)$, $\varepsilon, h \to 0$.

d) It seems to be highly actual to calculate the parameters of the "field" asymptotics $L(h)$ and $\chi_{LL}(h)$ (the singular part of energy and specific heat for $\theta \equiv \theta_c$, $h \neq 0$). Such data would make it possible to test some considerations discussed above on the role of molecular field in the critical region. In particular, it is of interest to check the conclusion that in the $d=2$ Ising model the specific heat may not be singular as $\theta \equiv \theta_c$, $h > 0$, $h \to 0$.

APPENDIX A

Proceeding from the hypothesis that $\Delta(\varepsilon) N S^2$ in (1) compensates only $L_1 = L(\varepsilon)\cos^2_{SL}(\varepsilon)$ in $\varepsilon H = -\varepsilon N L + const$, one can try to analyse the equality for averages $\varepsilon \langle L \rangle \cos^2_{SL} = \Delta(\varepsilon)\langle S^2 \rangle$ with averaging over $H/\theta_c(1-\varepsilon)$. Rewriting this equality for the case of asymptotics (10) (note that $\langle S^2 \rangle \equiv \langle S \rangle^2$) we obtain the equality, which reduces to a somewhat surprising relation for critical indices: $2\beta^2(\delta - 1) = 1 - \alpha$. This equality, as one can see from table 5, holds with good accuracy for different systems (except $d=2$ Ising) (for $d=3$ we have taken/9/: $\beta = 5/16$, $\delta = 5$, $\alpha = 1/8$). One can try to explain deviations from 1 by the replacement S^2 to $(S - C_o)^2$ in (1) (see footnote *** on page 12) what leads to relation $2\beta^2(\delta - 1)/(1-\alpha) = (1 - B_o/B)^2$, where $C_o(\varepsilon) = B_o \varepsilon^{\beta} / \ln \varepsilon /^{p_\beta}$.

APPENDIX B.

In the framework of our hypotheses (22),(28) it seems to be natural to assume that by $\theta < \theta_c$ the Hamiltonian contains the effective long-range term $(- \Delta(\varepsilon) N S^2)$. If one accepts as the origin of such a term the "primary" magnetization $C_o(\varepsilon)$, then it would be natural to consider $\Delta(\varepsilon)$ to be physical function of $C_o(\varepsilon)$. Then $\Delta(\varepsilon) = \lambda_o |C_o(\varepsilon)|^{\delta - 1}$, $\lambda_o = 1/2(\delta-1) \Gamma B_o^{\delta - 1}$. So the "coupling constant" $\Delta \sim |C_o|^{\delta - 1}$ and therefore, from the aesthetic considerations the most acceptable is the case when $\delta - 1 = 2, 4, 6, \ldots$, i.e., $\delta = 3, 5, \ldots 2k+1, \ldots$. It is interesting to note that in the Ising model we have just the case: $\delta = 3, 5, 15$ for $d = 4, 3, 2$. It may happen that odd values of δ are not so accidental.

APPENDIX C.

For isotropic systems with the n -component order parameter $(n \geq 2)$ susceptibility is infinite, $\chi_{SS} = \infty$, for all temperatures below θ_c and the straightforward application of the results described above is impossible. On the other hand, the scaling equalities $\gamma = \beta(\delta - 1)$ and $\alpha + 2\beta + \gamma = 2$ remain valid if one takes therein γ for $\theta > \theta_c$. Here we consider, on the same grounds, relations for logarithmic indices (table 4). For $d = 4$ in systems with an n -component order parameter one has (see, e.g., ref. /10/): $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$ and there are logarithmic corrections with indices: $p_\alpha = (4-n)/(n+8)$, $p_\beta = 3/(n+8)$, $p_\gamma = (n+2)/(n+8)$, $p_\delta = 1/3$, where γ and p_γ are taken for $\theta > \theta_c$ (indices α , p_α are the same for $\theta > \theta_c$ and $\theta < \theta_c$). One can easily verify that these values satisfy relations (a) and (b') in talbe 4 (second column), while other relations in table 4 lead to the following predictions: $p_{z_0} = (10-n)/3(n+8)$, $p_\epsilon = p_\alpha = (4-n)/(n+8)$ (with $\alpha = \epsilon = 0$, $z_0 = 2/3$).

Acknowledgements

The author would like to thank Prof. N.N.Bogolubov, Jr. for valuable remarks and the members of the Prof. V.K.Fedyanin seminar for discussion.

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Received by Publishing Department
on December 25 1980.