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DYNAMO IN HELICAL MHD TURBULENCE

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1 Introduction

Quantum field theory method including renormalization group (RG) approach has been successfully used for the theoretical explanation of various phenomena in developed turbulence (see [1] and wherein references).

In this paper the quantum field model of helical MHD is investigated. As a starting point we consider the Navier-Stokes equation for the velocity field and the equation for magnetic field that are driven by gaussian random forces with a given 2×2 matrix D of the hydrodynamic, magnetic and mixed noise correlators, respectively.

In ref. [2] (see also [3]) the multiplicative renormalizability of the quantum field model of non-helical MHD turbulence has been proved and the RG approach has been applied to study the asymptotic behaviour of the model considered. The existence of two infrared-stable fixed points has been established. These points induce the existence of two critical regimes: the magnetic regime and the kinetic one (the later being of the Kolmogorov type).

The critical properties of the helical MHD are not known in the case of arbitrary noise matrix D. To provide the multiplicative renormalizability and consequent application of RG it is necessary to extend the theory adding the extra dissipative terms with new helical Prandtl numbers [4]. Therefore, also a critical behaviour of the helical MHD is more complicated. A priori, the existence of the former stable regime of the Kolmogorov type is not clear. In the following the existence of the critical regimes mentioned above for ordinary MHD is demonstrated for the helical one.

There is an additional problem in the helical MHD: the instability of the theory which is induced by the exponential increasing of the magnetic fluctuations in the large scales range (see [5], for example). The elimination of this instability leads to formation of a large-scale magnetic field known as the turbulent dynamo. Removal of the instability in quantum field formulation of helical MHD can be achieved by means of a nice and very well known spontaneous symmetry breaking mechanism followed by the creation of homogeneous stationary magnetic field. The special case, when only the hydrodynamic noise does not vanish, was analyzed in [6].

In this paper we have calculated the value of spontaneous magnetic field $\mathbf{c}$ in one loop approximation for matrix D of noises in generic form. This value has been found from the conditions of overall exponential instabilities elimination in the steady state.

2 The formulation of the problem

The interaction of electrically neutral conductive turbulent incompressible fluid (with the unit magnetic permeability) with the magnetic field is described by the MHD equations driven by random forces [2]):

$$\begin{aligned} \nabla_t \mathbf{v} &= \nu\Delta\mathbf{v} - (\mathbf{b}\partial)\mathbf{b} - \partial p + F^{\mathbf{v}} \\ \nabla_t \mathbf{b} &= \nu'\Delta\mathbf{b} - (\mathbf{b}\partial)\mathbf{v} + F^{\mathbf{b}}, \end{aligned} \tag{1}$$

where $\nabla_t = \partial_t + (\mathbf{v}\partial)$ is a covariant derivative. The first equation is the well-known Navier-Stokes equation for the transversal velocity field $\mathbf{v}(x) = v_i(\mathbf{x}, t)$ with the additional nonlinear contribution of the Lorentz force (the longitudinal contribution is ascribed to pressure p). The second equation for magnetic field $\mathbf{b}(x) = b_i(\mathbf{x}, t)$ (it is connected with magnetic induction $\mathbf{B}$ by the relation $\mathbf{b} = \mathbf{B}/\sqrt{4\pi\varrho}$, where ϱ is a fluid density) follows from the Maxwell equations for continuous medium. The magnetic diffusion coefficient ν' is connected with the coefficient of molecular viscosity by relation $\nu' = u\nu$ with dimensionless magnetic Prandtl number (PN) u^{-1}.

The random forces are assumed to have a Gaussian distribution with $<F> = 0$ and they are given by 2×2 matrix of the noise correlators $D = <FF>$. The matrix elements are: the hydrodynamic $D^{\mathbf{vv}}$ noise, the magnetic $D^{\mathbf{bb}}$ one and the mixed $D^{\mathbf{vb}}$ one.

The problem (1) is equivalent to the quantum theory with a doubled number of the fields $\Phi = \{\mathbf{v}, \mathbf{b}, \mathbf{v}', \mathbf{b}'\}$ and the functional action [7, 8]:

$$\begin{aligned} S(\Phi) &= \frac{\mathbf{v}'D^{\mathbf{vv}}\mathbf{v}'}{2} + \frac{\mathbf{b}'D^{\mathbf{bb}}\mathbf{b}'}{2} + \frac{\mathbf{v}'D^{\mathbf{vb}}\mathbf{b}'}{2} + \frac{\mathbf{b}'D^{\mathbf{bv}}\mathbf{v}'}{2} + \mathbf{v}'[-\partial_t \mathbf{v} + \nu_0 \Delta \mathbf{v} \\ &\quad -(\mathbf{v}\partial)\mathbf{v} + (\mathbf{b}\partial)\mathbf{b}] + \mathbf{b}'[-\partial_t \mathbf{b} + u_0\nu_0 \Delta \mathbf{b} - (\mathbf{v}\partial)\mathbf{b} + (\mathbf{b}\partial)\mathbf{v}], \end{aligned} \tag{2}$$

where $\mathbf{v}', \mathbf{b}'$ are some auxiliar vectorial fields. Hereafter in the similar expressions, the integration over $\mathbf{x}$, t and the traces over the vector indices are implied. As it is usual in QFT, the action (2) is considered to be unrenormalized with the bare constants marked by the subscript "0". The basic objects of the study are the Green functions of the fields Φ (the correlation functions and response functions in the terminology of the original problem (1)). They can be determined as functional derivatives with respect to an external sources $A = \{A^{\mathbf{v}}, A^{\mathbf{b}}, A^{\mathbf{v}'}, A^{\mathbf{b}'}\}$ of the generating functional $G(A) = \int D\Phi \exp[S(\Phi) + A\Phi]$ i.e. they are the functional averaged values of the corresponding number of the fields ϕ with a weight $\exp[S(\phi)]$. Here, $D\Phi$ denotes the functional measure of the integration over the fields Φ with all normalization coefficients.

We have to choose a concrete form of D in the wave vector-frequency $(\mathbf{k}, \omega)$ representation. The noises are transversal for the incompressible liquid. The action for helical MHD can possess scalar terms as well as pseudoscalar ones. Hence, the tensor structure of all noises is a linear combination of both tensor and pseudotensor. Then the correlators have the form:

$$\begin{aligned} D^{\mathbf{vv}}_{js} &= g_0 \nu_0^3 k^{1-2\epsilon} \mathbf{P}^1_{js}, \qquad D^{\mathbf{bb}}_{js} = g_0' \nu_0^3 k^{1-2a\epsilon} \mathbf{P}^2_{js}, \\ D^{\mathbf{bv}}_{js} &= D^{\mathbf{vb}}_{js} = g_0'' \nu_0^3 k^{1-(1+a)\epsilon} \mathbf{P}^3_{js}. \end{aligned} \tag{3}$$

Here, $\mathbf{P}^r_{js} = P_{is} + i\rho_r \varepsilon_{jsl} k_l/k$, where $P_{is} = \delta_{is} - k_i k_s/k$ stands for transversal projector and ε_{isl} is Levi-Civita pseudotensor. Dimensionless real parameters $\rho = \{\rho_1, \rho_2, \rho_3\}$ satisfy the conditions $|\rho| \leq 1$, $\rho_3^2 \leq |\rho_1\rho_2|$. The scalar parts of the noises explicitly written in (3) are in a standard power form [2]. The parameters g_0, g_0', g_0'' play the role of the bare coupling constants, and a, ϵ are free parameters of the theory. The

value $\epsilon = 2$ corresponds to the Kolmogorov energy pumping from infra-red region of the small $\mathbf{k}$.

3 Renormalization

In standard way, we solve the primary infrared problem for the physical value $\epsilon = 2$ by transfer to the region of small values ϵ, where the ultraviolet (UV) divergences appear. They can be eliminated by the addition of the appropriate counterterms to the action (2) [4]. The counterterms are formed of the superficial UV divergences, which are present in one-particle irreducible Green functions [9]. The following 1-PI Green functions possess the UV divergences: $<\mathbf{v}'\mathbf{v}>$, $<\mathbf{b}'\mathbf{b}>$, $<\mathbf{v}'\mathbf{b}>$, $<\mathbf{b}'\mathbf{v}>$, $<\mathbf{v}'\mathbf{bb}>$. Corresponding diagrams are shown in Fig. 1 and Fig. 2 and counterterms have the form: $\nu\mathbf{v}'\Delta\mathbf{v}$, $\nu\mathbf{b}'\Delta\mathbf{b}$, $\mathbf{v}'(\mathbf{b}\partial)\mathbf{b}$, $\nu\mathbf{v}'\Delta\mathbf{b}$ and $\nu\mathbf{b}'\Delta\mathbf{v}$. The last two ones are not present in the primary action (2). For this reason, it is necessary to consider the extended theory with the additional cross dissipative terms $v\nu\mathbf{v}'\Delta\mathbf{b}$, $w\nu\mathbf{b}'\Delta\mathbf{v}$ with new helical magnetic Prandtl numbers v^{-1}, w^{-1}. Besides UV divergences mentioned above which appear themselves like the poles of ϵ (ϵ−UV divergences), another divergences proportional to the UV cutoff Λ can appear in the Green functions $<\mathbf{v}'\mathbf{v}>$, $<\mathbf{v}'\mathbf{b}>$, $<\mathbf{b}'\mathbf{v}>$, $<\mathbf{b}'\mathbf{b}>$. They acquire the form of $\Lambda\mathbf{v}'\,rot\,\mathbf{v}$, $\Lambda\mathbf{v}'\,rot\,\mathbf{b}$, $\Lambda\mathbf{b}'\,rot\,\mathbf{v}$ and $\Lambda\mathbf{b}'\,rot\,\mathbf{b}$. These $\Lambda - UV$ divergences generate the instability of the theory, that

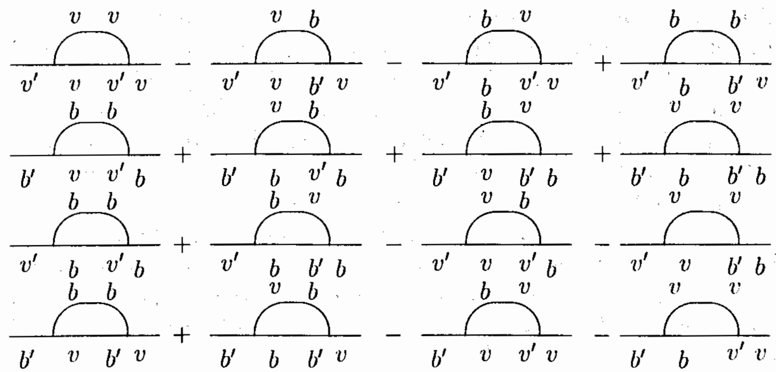

Figure 1: One-loop Feynman diagrams which are UV-divergent. Only diagrams related to the Green functions $<\mathbf{v}'\mathbf{v}>$, and $<\mathbf{b}'\mathbf{b}>$ (first and second line) can contain $\Lambda - UV$ divergences

cause exponential growth in time of corresponding response functions. Therefore, their direct insertion into the action (2) is not allowed and we have to find an effective way to eliminate them. We will consider the new vacuum state with zero mean values of fields $\mathbf{v}$, $\mathbf{v}'$, $\mathbf{b}'$, and non-vanishing $<\mathbf{b}> \equiv \mathbf{c} \neq 0$. In quantum field theory appearance of non-zero vacuum value of field is associated with spontaneous symme-

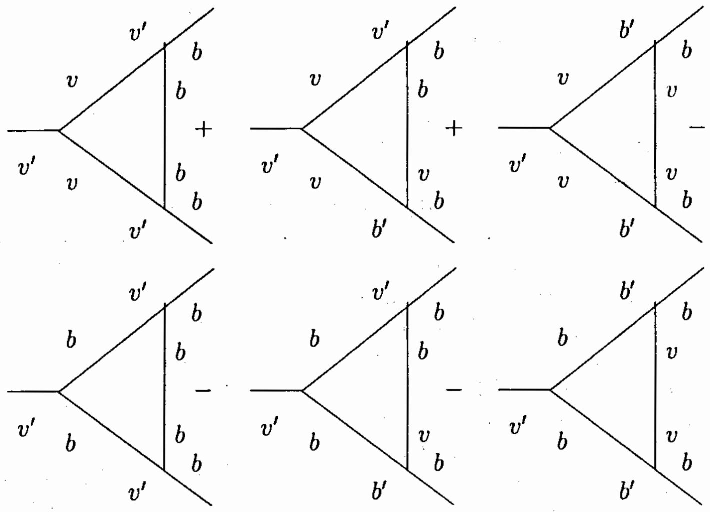

Figure 2: Diagrams related to the Green function $< \mathbf{v'bb} >$.

try breaking. The value of spontaneous mean field is determined from requirement of minimum of potential energy at the tree level. In the case considered here the situation is rather different technically. We will calculate spontaneous magnetic field to achieve the elimination of unstable Λ-terms, which appear at the one-loop level. Possible $\Lambda - UV$ divergences can be found only in diagrams related to the Green functions $< \mathbf{v'v} >$ and $< \mathbf{b'b} >$. In one-loop approximation they are shown in Fig. 1 (first and second line respectively). By direct calculations and/or from the symmetry analysis of the given one-loop Feynman diagrams, one can find that only $< \mathbf{b'b} >$ contains $\Lambda-$terms. Here all $\Lambda-$ divergences can be eliminated by means of the shift of $\mathbf{b}$, namely $\mathbf{b(x)} \to \mathbf{b(x)} + \mathbf{c}$, and, on the other hand, $\epsilon-$UV divergences are compensated by means of five independent renormalization constants $Z_i, i = 1, ...5$ in extended model of helical MHD. As a result one obtains model with renormalized action:

$$
\begin{aligned}
S_R^h(\Phi) &= \frac{\mathbf{v}'D^{\mathbf{vv}}\mathbf{v}'}{2} + \frac{\mathbf{b}'D^{\mathbf{bb}}\mathbf{b}'}{2} + \frac{\mathbf{v}'D^{\mathbf{vb}}\mathbf{b}'}{2} + \frac{\mathbf{b}'D^{\mathbf{bv}}\mathbf{v}'}{2} - \mathbf{v}'[\partial_t \mathbf{v} - \\
& Z_1\nu\Delta\mathbf{v} - Z_4 v\nu\Delta\mathbf{b} + (\mathbf{v}\partial)\mathbf{v} - Z_3(\mathbf{b}\partial)\mathbf{b} - Z_3(\mathbf{c}\partial)\mathbf{b}] - \qquad (4)\\
& \mathbf{b}'[\partial_t\mathbf{b} - Z_2 u\nu\Delta\mathbf{b} - Z_5 w\nu\Delta\mathbf{v} + (\mathbf{v}\partial)\mathbf{b} - (\mathbf{b}\partial)\mathbf{v} - (\mathbf{c}\partial)\mathbf{v}],
\end{aligned}
$$

where all parameters are renormalized couterparts of bare ones. The action (4) generates Green functions without divergences. In this case Feynman rules have the following form

$$
\begin{aligned}
\Delta_{12}^{vv'} &= \frac{MP_{12}}{LM - SV}, \qquad \Delta_{12}^{vb'} = -\frac{VP_{12}}{LM - SV}, \\
\Delta_{12}^{bv'} &= -\frac{SP_{12}}{LM - SV}, \qquad \Delta_{12}^{bb'} = \frac{LP_{12}}{LM - SV}, \\
\Delta_{12}^{v'v} &= \frac{M^+P_{12}}{L^+M^+ - S^+V^+}, \qquad \Delta_{12}^{b'v} = -\frac{V^+P_{12}}{L^+M^+ - S^+V^+}, \\
\Delta_{12}^{v'b} &= -\frac{S^+P_{12}}{L^+M^+ - S^+V^+}, \qquad \Delta_{12}^{b'b} = \frac{L^+P_{12}}{L^+M^+ - S^+V^+}, \\
\Delta_{12}^{vv} &= \frac{D_{12}^{vv}MM^+ - D_{12}^{vb}MV^+ - D_{12}^{bv}VM^+ + D_{12}^{bb}VV^+}{(L^+M^+ - S^+V^+)(LM - SV)}, \\
\Delta_{12}^{bb} &= \frac{D_{12}^{vv}SS^+ - D_{12}^{vb}SL^+ - D_{12}^{bv}LS^+ + D_{12}^{bb}LL^+}{(L^+M^+ - S^+V^+)(LM - SV)}, \\
\Delta_{12}^{vb} &= \frac{-D_{12}^{vv}MS^+ + D_{12}^{vb}ML^+ + D_{12}^{bv}VS^+ - D_{12}^{bb}VL^+}{(L^+M^+ - S^+V^+)(LM - SV)}, \\
\Delta_{12}^{bv} &= \frac{-D_{12}^{vv}SM^+ + D_{12}^{vb}SV^+ + D_{12}^{bv}LM^+ - D_{12}^{bb}LV^+}{(L^+M^+ - S^+V^+)(LM - SV)}, \qquad (5)
\end{aligned}
$$

where

$$
\begin{aligned}
L &= -i\omega + \nu k^2, \quad M = -i\omega + \nu u k^2, \\
V &= \nu v k^2 - i\gamma, \quad S = \nu w k^2 - i\gamma, \\
D_{mn}^{vv} &= g_1\nu^3 k^{4-d-2\varepsilon}(P_{mn} + i\rho_1\varepsilon_{mnl}\frac{k_l}{k}), \\
D_{mn}^{vb} &= g_3\nu^3 k^{4-d-\varepsilon(1+a)}(P_{mn} + i\rho_3\varepsilon_{mnl}\frac{k_l}{k}), \\
D_{mn}^{bv} &= D_{mn}^{vb}, \\
D_{mn}^{bb} &= g_2\nu^3 k^{4-d-2\varepsilon a}(P_{mn} + i\rho_2\varepsilon_{mnl}\frac{k_l}{k}), \qquad (6)
\end{aligned}
$$

with

$$\gamma = \mathbf{c}\cdot\mathbf{k}. \qquad (7)$$

and vertices are defined by the expressions (using transversality condition of all the fields, a derivative in the vertices can be transferred to auxiliary fields):

$$
\begin{aligned}
\mathbf{v}'\cdot(\mathbf{v}\cdot\partial)\mathbf{v} &= v_i' t_{ijl} v_j v_l/2, \\
\mathbf{v}'\cdot(\mathbf{b}\cdot\partial)\mathbf{b} &= v_i' t_{ijl} b_j b_l/2, \\
\mathbf{b}'\cdot(\mathbf{b}\cdot\partial)\mathbf{v} - \mathbf{b}'\cdot(\mathbf{v}\cdot\partial)\mathbf{b} &= b_i'\bar{t}_{ijl} b_j v_l, \qquad (8)
\end{aligned}
$$

where

$$t_{ijl}^k = i(k_j\delta_{il} + k_l\delta_{ij}), \quad \bar{t}_{ijl}^k = i(k_j\delta_{il} - k_l\delta_{ij}). \qquad (9)$$

Using Feynman rules defined above one can immediately calculate diagrams which are shown in Fig. 1 and Fig. 2. We are interesting in linear part (in momentum unit) of the diagrams. As has been already stressed the response function $< \mathbf{v'v} >$ has no linear part. On the other hand, at the one-loop level the response function $< b'b >$ $\Lambda-$divergent part and part $\sim \mathbf{c}$ are in the form:

$$< b_i'b_j > \sim ik_m\varepsilon_{iml}(g\rho_1 C_1 + g'\rho_2 C_2 + g''\rho_3 C_3) \times \\ \times \left[\nu\Lambda\delta_{jl} - |\mathbf{c}|\frac{3\pi}{8}\sqrt{\frac{(1+u)^2-(v+w)^2}{(1+u)^2(u-vw)}}(\delta_{jl}+e_j e_l)\right], \qquad (10)$$

where $\mathbf{e} \equiv \mathbf{c}/|\mathbf{c}|$, $C_1 = (w(v+w) - u(1+u))/\xi$, $C_2 = (1 + u - v(v+w))/\xi$, $C_3 = 2(uv - w)/\xi$ and $\xi = 6(1+u)^2(u - vw)\pi^2$. From requirement of vanishing of $\Lambda-$term in (10) one determines the value of spontaneous field

$$|\mathbf{c}| = \frac{8\nu}{3\pi}\sqrt{\frac{(1+u)^2-(v+w)^2}{(1+u)^2(u-vw)}}\Lambda. \qquad (11)$$

In such a way we obtain the renormalized Green functions which are finite as $\Lambda \to \infty$ formally, as it is usual in the field theory. But in real problems a natural cutoff exists. In the developed turbulence the Kolmogorov dissipative length $l_D = \Lambda^{-1}$ plays the role of a minimal scale. This length can be expressed in terms of basic phenomenological parameters - viscosity ν and energy dissipation rate ε. Then from (11) and simple dimensional considerations one obtains $|\mathbf{c}| \sim (\nu\varepsilon)^{1/4}$ and it determines order of magnitude of the spontaneous field $\mathbf{c}$.

4 Short remarks to the RG analysis and critical regimes

After successful elimination of all UV divergences we can use the renormalization group procedure and arrive to the set of RG Gell-Mann equations for five invariant charges:

$$s\frac{d\bar{g}_i(s)}{ds} = \beta_{g_i}(\bar{g}(s), \epsilon), \qquad \bar{g}_i(s)\,|_{s=1} = g_i \;\; g_i \equiv g, g', u, v, w, \qquad (12)$$

where $s = p/\mu$ (μ is scale setting parameter). RG β-functions are expressed via renormalization constants Z_i, which have been calculated in one-loop approximation. Note that β-functions are finite in the limit of $\epsilon \to 0$.

The Gell-Mann-Low equations (12) have been solved numerically for the various initial values of the invariant charges g_i. It provides the possibility to analyze the attracting regions of infrared fixed points. There are two infrared-stable fixed points: the Gaussian $g_i^* = 0$ and nontrivial $g^* \neq 0$, $u^* = 1.393$, $g'^* = u^* = v^* = w^* = 0$. The latter provides the existence of asymptotic critical regime of the Kolmogorov type. RG approach improves expressions of simple perturbation theory and leads to the replacement of the original charges in them by the invariant ones, and, in critical regime by their values at the corresponding fixed points.

5 Conclusion

In the paper we have studied correlation and response functions of velocity and magnetic fluctuations. Generally, these functions contain singularities, which can be eliminated by proper renormalization procedure. As a result, RG equations have been obtained and their solution have been found in the range of small wave numbers. This solution, which corresponds to the famous Kolmogorov scaling law, is stabilized by spontaneous appearance of non-zero mean magnetic field when mirror symmetry of the system under consideration is broken.

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