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SEVERAL ORDER PARAMETERS IN CRITICAL REGION

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1. Here we consider a set of order parameters in the critical region. We define order parameters as quasi-averages in the sense of N.N.Bogolubov, Jr. Some generalized functional relations for order parameters will be derived and applied to analyse the critical behaviour[1/].

Consider a many-body system Γ/θ with Hamiltonian Γ and temperature modulus $\theta = kT$. Given a set of n operator order parameters S_α, which we suppose to be hermitian[2/]:

$$S_\alpha = S_\alpha^+ , \qquad \alpha = 1, 2, \ldots, n . \tag{1}$$

We shall also suppose that S_α are of the "quasi-additive" type and satisfy the conditions due to N.N.Bogolubov, Jr.:

$$|S_\alpha| \le K_1 , \quad |S_\alpha \Gamma - \Gamma S_\alpha| \le K_2 , \quad |S_\alpha S_\beta - S_\beta S_\alpha| \le \frac{K_3}{N} , \tag{2}$$

where $|\ldots|$ means the operator norm, K_1, K_2, K_3 are constants, N is the number of particles, proportional to the volume of a system V (N/V = const as $N, V \to \infty$).

The quantities of physical interest are equilibrium Gibbs averages $\langle S_\alpha \rangle_{\Gamma,\theta}$ - numerical order parameters (real numbers). Strictly speaking, we should deal not with common Gibbs averages, but with the averages with spontaneously broken symmetry, i.e., with the "quasi-averages"/2/:

$$\prec S_\alpha \succ_{\Gamma,\theta} = \lim_{\{\tau\} \to 0} \lim_{N \to \infty} \langle S_\alpha \rangle_{\Gamma_\tau, \theta} , \tag{3}$$

where Γ_τ is Hamiltonian Γ with additional small symmetry-breaking terms ("sources"), introduced by small parameters $\{\tau\}$ which should be removed $(\{\tau\} \to 0)$ after the limit $N \to \infty$.

[1/] We shall not discuss here the physical reasoning why the study of several order parameters in the critical region is of interest, for details see, e.g., ref./1/.

[2/] The case of non-hermitian order parameters, which is in fact not more general, is discussed in Appendix B.

We shall denote below the quasi-averages $\prec S_\alpha \succ_{\Gamma,\theta}$ as $S_\alpha[\Gamma/\theta]$.

An effective definition of quasi-averages (the choice of "sources") has been proposed by N.N.Bogolubov, Jr./3,4/. On this basis it is possible to deduce (under the conditions (2)) that the following "self-consistence" equations for the order parameters hold true[3/]:

$$S_\alpha[\Gamma/\theta] = S_\alpha[\Gamma + R_n/\theta], \quad \alpha = 1,\dots,n,$$
$$R_n \equiv N\sum_{\beta=1}^{n} \rho_\beta (S_\beta - S_\beta[\Gamma/\theta])^2; \quad \rho_\beta > 0, \tag{4}$$

where $\rho_\beta > 0$ are arbitrary positive parameters. One can also put (4) into the form:

$$S_\alpha[\Gamma/\theta] = S_\alpha\Big[\Gamma + N\sum_{\beta=1}^{n}(\rho_\beta S_\beta^2 - h_\beta S_\beta)/\theta\Big]_{h_\beta = 2\rho_\beta S_\beta[\Gamma/\theta]}. \quad (\alpha = 1,\dots,n) \tag{4a}$$

Introduce for the order parameters (1) a set of susceptibilities (for definition see Appendix A):

$$\chi_{\alpha\beta}[\Gamma/\theta] \equiv \chi_{S_\alpha S_\beta}[\Gamma/\theta]; \quad \alpha,\beta = 1,\dots,n. \tag{5}$$

The form (5) is bilinear in S_α, S_β and satisfies all the properties of the scalar product:

$$0 \le \chi_{\alpha\alpha}[\Gamma/\theta], \quad \chi_{\alpha\beta}[\Gamma/\theta] = \chi_{\beta\alpha}[\Gamma/\theta],$$
$$|\chi_{\alpha\beta}[\Gamma/\theta]|^2 \le \chi_{\alpha\alpha}[\Gamma/\theta]\,\chi_{\beta\beta}[\Gamma/\theta]. \tag{6}$$

These properties make it possible to introduce "cosine of the angle between operators S_α and S_β with respect to the system Γ/θ" :

$$\cos^2_{\alpha\beta}[\Gamma/\theta] \equiv |\chi_{\alpha\beta}[\Gamma/\theta]|^2/\chi_{\alpha\alpha}[\Gamma/\theta]\,\chi_{\beta\beta}[\Gamma/\theta]. \tag{7}$$

From the susceptibilities (5) one can arrange the $n \times n$ real matrix:

$$\mathcal{X}[\Gamma/\theta] = \| \chi_{\alpha\beta}[\Gamma/\theta] \|. \tag{8}$$

[3/] We represent here the result in the form appropriate for our needs; for details see ref./5/ and Appendix A therein.

In view of (6) this matrix is symmetric, and all its eigenvalues, trace, and determinant are non-negative, independent of the system Γ/θ.

Starting with (4) we now obtain some functional relations for susceptibilities. "Switching on" external fields in the system Γ/θ in (4), i.e., making therein the substitution

$$\Gamma/\theta \to \Gamma - N\sum_{\beta=1}^{n} h_\beta S_\beta/\theta, \tag{9}$$

differentiating the resulting relations with respect to h_β and putting then $h_\beta \to 0$, we get:

$$\chi_{\alpha\beta}[\Gamma/\theta] = \chi_{\alpha\beta}[\Gamma + R_n/\theta] + \sum_{\gamma=1}^{n} 2\rho_\gamma \chi_{\alpha\gamma}[\Gamma + R_n/\theta]\,\chi_{\gamma\beta}[\Gamma/\theta] \tag{10}$$

(for R_n see (4)).

Using (8) one can rewrite these relations into the matrix form:

$$\mathcal{X}[\Gamma/\theta] = \mathcal{X}[\Gamma + R_n/\theta] + \mathcal{X}[\Gamma + R_n/\theta]\,2\hat{\rho}\,\mathcal{X}[\Gamma/\theta], \tag{11}$$

where $\hat{\rho}$ is a diagonal matrix with the diagonal elements ρ_α, $\alpha = 1,\dots,n$.

If $\mathcal{X}^{-1}[\Gamma/\theta]$ and $\mathcal{X}^{-1}[\Gamma + R_n/\theta]$ are non-zero matrices, one can put (11) into the form:

$$\mathcal{X}^{-1}[\Gamma + R_n/\theta] = \mathcal{X}^{-1}[\Gamma/\theta] + 2\hat{\rho}. \tag{11a}$$

2. Up to now system Γ/θ was arbitrary. Consider now system near its critical point.

For the sake of simplisity let us consider a conventional "ferromagnetic" system with Hamiltonian H and critical temperature θ_c. For non-zero magnetic field $h > 0$ the Hamiltonian of the system will be

$$H_h = H - hNS, \quad S = S^+, \tag{12}$$

where S is the magnetization operator of the system (per particle), N is the number of particles. We suppose that for $\theta < \theta_c$ the spontaneous ordering appears:

$$S[H/\theta_c(1-\varepsilon)] = \begin{cases} >0, & \varepsilon > 0 \\ \to 0, & \varepsilon \to 0 \end{cases}; \quad S\left[\frac{H}{\theta_c(1+\varepsilon)}\right] \equiv 0, \quad \varepsilon > 0. \tag{13}$$

One can regard (13) as the condition of criticality of the system H/θ_c. However, it is more convenient to us to take the formal critical-point condition in the form

$$S[H-\tau NS^2/\theta_c] = \begin{cases} >0, & \tau > 0 \\ \to 0, & \tau \to 0. \end{cases} \tag{14}$$

Here the "ferromagnetic" term $(-\tau NS^2)$ models the renormalization of the Hamiltonian, which is equivalent to the lowering of temperature below θ_c in (13). Note that in view of (4) in the disordered phase (analogous to $\theta > \theta_c$) we have

$$S[H+\rho NS^2/\theta_c] \equiv S[H/\theta_c] = 0, \quad \rho > 0. \tag{15}$$

Let us now introduce, in addition to the basic order parameter S, two extra order parameters A and B, normed by the condition[4/]:

$$A[H/\theta_c] = B[H/\theta_c] = S[H/\theta_c] = 0. \tag{16}$$

We shall also suppose that the external field $h>0$ removes the system from the critical point, so that all order parameters become non-zero and all susceptibilities become finite:

$$Y(h) \equiv Y[H-hNS/\theta_c] = \begin{cases} \neq 0, & h > 0 \\ \to 0, & h \to 0, \end{cases} \tag{17a}$$

$$|\chi^{-1}_{XY}(h)| \equiv |\chi^{-1}_{XY}[H-hNS/\theta_c]| > 0, \quad h > 0, \quad X, Y = A, B, S. \tag{17b}$$

Consider now general relations (10) for the case of three order parameters

$$\chi_{XY}[\Gamma/\theta] = \chi_{XY}[\Gamma + R_3/\theta] + \sum_{(Z)} 2\rho_Z \chi_{XZ}\left[\frac{\Gamma+R_3}{\theta}\right]\chi_{ZY}\left[\frac{\Gamma}{\theta}\right], \tag{18}$$

[4/] We shall often call "order parameter" both averages and operators.

$$R_3 = N\sum_{(Z)} \rho_Z (Z - Z[\Gamma/\theta])^2, \quad \rho_Z > 0, \tag{18a}$$

where X, Y, Z run over $\{A, B, S\}$. Choose here $\Gamma/\theta = H - \tau NS^2/\theta_c$ and put $\rho_S = \tau > 0, \rho_A = \rho_B = 0$[5/]. Besides, instead of variables X, Y, we shall write directly A, B, keeping in mind that A, B may coincide with each other and with S ($A = B$, $A = B = S$, etc.).

Then we obtain the basic relation in the form:

$$\chi_{AB}[H-\tau NS^2/\theta_c] = \chi_{AB}[H-h(\tau)NS/\theta_c] + 2\tau\,\chi_{AS}[H-\tau NS^2/\theta_c]\,\chi_{SB}[H-h(\tau)NS/\theta_c], \tag{19}$$

$$h(\tau) \equiv 2\tau S[H-\tau NS^2/\theta_c]. \tag{19a}$$

In view of (4) we have also for the order parameters:

$$Y[H-\tau NS^2/\theta_c] = Y[H-h(\tau)NS/\theta_c], \quad Y = A, B, S. \tag{20}$$

So, we have the relations which connect the characteristics of the system in the ordered phase $H - \tau NS^2/\theta_c$ with those of the system with external effective field $h(\tau)$ (19a). In order to represent these relations in a more appropriate form, let us introduce "index-function":

$$\delta_{YS}(h) = \frac{Y(h)}{h} \Big/ \frac{dY(h)}{dh}, \quad Y(h) \equiv Y[H-hNS/\theta_c], \tag{21}$$

$$\delta^*_{YS}(\tau) \equiv \delta_{YS}(h = h(\tau)), \quad Y = A, B, S, \tag{22}$$

for $h(\tau)$ see (19a).

The function $\delta_{YS}(h)$ characterizes the asymptotical behaviour of $Y(h)$ as $h \to 0$ and coincides with the corresponding critical index in the case of quasi-power asymptotics. Let

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[5/] Here $\rho_A = 0$, $\rho_B = 0$ is meant in the sense of the limit $\rho_A \to 0$, $\rho_B \to 0$. Such a limit procedure is possible if no one of susceptibilities in (18) becomes $\pm\infty$. This condition holds true due to (17b) if $\tau > 0$ (see below).

$$Y(h) = h^{\frac{1}{\delta_{YS}}} \varphi_{YS}(h), \qquad (23)$$

where δ_{YS} is constant (critical index), and $\varphi_{YS}(h)$ varies as $h \to 0$ strictly slower than by the power law. Then one can easily derive

$$\frac{1}{\delta_{YS}(h)} = \frac{1}{\delta_{YS}} + \omega_{YS}(h), \qquad (24)$$

$$\omega_{YS}(h) \equiv \frac{d\varphi_{YS}(h)}{dh} \Big/ \frac{\varphi_{YS}(h)}{h} \xrightarrow[h \to 0]{} 0, \qquad (24a)$$

((24a) is just the condition that $\varphi_{YS}(h)$ varies slower than by the power law).

Note also that in view of (20) for (22) the relation is valid:

$$\delta^*_{YS}(\tau) = \frac{Y[\tau]}{S[\tau]} \frac{1}{2\tau \chi_{YS}[H - h(\tau)NS/\theta_c]}, \qquad (25)$$

where $Y = A, B, S$; for $h(\tau)$ see (19a). Here we have used a short notation of the form: $F[\tau] \equiv F[H - \tau NS^2/\theta_c]$, where F is order parameter or susceptibility. We shall also use this notation below.

Making use of the "index-functions" and taking into account (25) and basic relations (19),(20) we obtain, in particular:

$$\chi_{SS}[\tau] = \frac{1}{\delta^*_{SS}(\tau) - 1} \cdot \frac{1}{2\tau}, \quad \tau > 0; \qquad (26)$$

$$\chi_{AS}[\tau] = \frac{\delta^*_{SS}(\tau)}{\delta^*_{AS}(\tau)} \frac{A[\tau]}{S[\tau]} \left\{ \frac{1}{\delta^*_{SS}(\tau) - 1} \frac{1}{2\tau} \right\}; \qquad (27)$$

$$\chi_{AB}[\tau]\left(1 - \frac{1}{\delta^*_{SS}(\tau)} \cdot \frac{\chi_{AS}[\tau]\chi_{BS}[\tau]}{\chi_{AB}[\tau]\,\chi_{SS}[\tau]}\right) = \chi_{AB}\left[\frac{H - h(\tau)NS}{\theta_c}\right]; \qquad (28)$$

$$\chi_{AA}[\tau]\left(1 - \frac{1}{\delta^*_{SS}(\tau)} \cos^2_{AS}[\tau]\right) = \chi_{AA}[H - h(\tau)NS/\theta_c], \qquad (29)$$

$$\cos^2_{AS}[\tau] \equiv |\chi_{AS}[\tau]|^2 / \chi_{AA}[\tau]\chi_{SS}[\tau]; \qquad (29a)$$

$$\frac{\chi_{AS}[\tau]}{\chi_{BS}[\tau]} = \frac{\delta^*_{BS}(\tau)}{\delta^*_{AS}(\tau)} \frac{A[\tau]}{B[\tau]}. \qquad (30)$$

Note that here A and B may coincide with each other or S; one can interchange A and B or replace through (26) to (30) A by B or S, etc.

Formulas (26) to (30) are exact. For the case of quasi-power asymptotics (23) the index-functions $\delta^*_{YS}(\tau)$ tend to constants as $\tau \to 0$, and one can use these formulas to analyse the asymptotical behaviour of the order parameters and susceptibilities for the system

$$H - \tau NS^2/\theta_c, \quad \tau > 0. \qquad (31)$$

However, this system is not "experimentally observable" and it would be desirable to find formulas for an arbitrary system in the ordered phase, e.g.,for the system $H/\theta_c(1-\varepsilon)$, $\varepsilon > 0$.

Such a generalization of the above results under some additional conditions appears to be possible. So, we have critical system H/θ_c; consider a system in the ordered phase $H+V/\theta_c$, where V is a variation of the Hamiltonian of the <u>ordering</u> type, so that

$$|S[H+V/\theta_c]| = \begin{cases} > 0, & V \neq 0, \\ \to 0, & V \to 0. \end{cases} \qquad (32)$$

Suppose also that the variation V is of the <u>weak</u> type, so that there exists such a finite value of the parameter $\Delta = \Delta(V) > 0$ that by introducing into the Hamiltonian of the "disordering" term $+\Delta NS^2$ by $\Delta = \Delta(V)$ we turn the system $H+V/\theta_c$ into the critical point:

$$\frac{H_V}{\theta_c} \equiv \frac{H + V + \Delta(V)NS^2}{\theta_c} = \left\{ \begin{matrix} \text{the critical system} \\ \text{with respect to } S \end{matrix} \right\},$$

$$\Delta(V) = \begin{cases} > 0, & V \neq 0 \\ \to 0, & V \to 0 \end{cases}. \qquad (33)$$

It is sufficiently now to take in (26)-(30), instead of H/θ_c, the system (33) and put $\tau = \Delta(V)$. Then $H - \tau NS^2/\theta_c$ passes to $H + V/\theta_c$ and $H - h(\tau)NS/\theta_c$ passes to $H_V - h(V)NS/\theta_c$, where H_V/θ_c is the critical system (33) and

$$h(V) \equiv 2\Delta(V) S[H + V/\theta_c]. \qquad (34)$$

As a simple example of the system $H+V/\Theta_c$ one can take the initial one for $\Theta < \Theta_c$:

$$H/\theta_c(1-\varepsilon), \quad \varepsilon > 0. \tag{35}$$

It is quite clear that for a wide class of systems, e.g., for the Ising-type lattice models, the temperature variation will be of the "weak ordering type" in the above sense.

Introduce short notation:

$$F[V] \equiv F[H+V/\theta_c], \; F = A, B, S, \chi_{AB}, \text{ etc.,} \tag{36}$$

$$\delta_{YS}[V] = \delta_{YS}^{(V)}(h = 2\Delta(V)S[H+V/\theta_c]), \tag{37}$$

where $\delta_{YS}^{(V)}(h)$ is the function (21) for the system $H_V - hNS/\theta_c$. On the basis of (26) to (30) we then obtain:

$$\chi_{SS}[V] = \frac{1}{\delta_{SS}[V]-1} \, \frac{1}{2\Delta(V)}; \tag{38}$$

$$\chi_{AB}[V] = \frac{\delta_{SS}[V]}{\delta_{SS}[V]-C_{ABS}[V]} \chi_{AB}\left[\frac{H_V - h(V)NS}{\theta_c}\right], \tag{39}$$

$$C_{ABS}[V] \equiv \chi_{AS}[V]\chi_{BS}[V]/\chi_{AB}[V]\chi_{SS}[V];$$

$$\chi_{AS}[V] = \frac{\delta_{SS}[V]}{\delta_{SS}[V]-1} \chi_{AS}[H_V - h(V)NS/\theta_c]; \tag{40}$$

$$\chi_{AA}[V] = \frac{\delta_{SS}[V]}{\delta_{SS}[V]-\cos^2_{AS}[V]} \chi_{AA}[H_V - h(V)NS/\theta_c]; \tag{41}$$

$$\chi_{AS}[V]/\chi_{BS}[V] = \delta_{BS}[V]A[V]/\delta_{AS}[V]B[V]; \tag{42}$$

and also (see (20))

$$Y[H+V/\theta_c] \equiv Y[H_V - h(V)NS/\theta_c], \tag{43}$$

$$Y = A, B, S,$$

where for the field $h(V)$ in (39) to (43) we have, taking into account (38):

$$h(V) = 2\Delta(V)S[H+V/\theta_c] = S[V]/(\delta_{SS}[V]-1)\chi_{SS}[V]. \tag{44}$$

So, we have obtained a set of exact relations for the case of weak ordering variation of the Hamiltonian V, which we can specify, for example, as the temperature variation. These relations, however, involve the auxiliary critical system H_V/θ_c. If we accept the physical assumption that pussing from H/θ_c to H_V/θ_c does not change the behaviour of the index-functions $\delta_{YS}(h)$ (or at least their limit values), we then get a set of functional relations establishing the connection between the critical characteristics of the systems $H+V/\theta_c$ and $H-hNS/\theta_c$.

These relations demonstrate the existence of interrelations in the critical behaviour of the order parameters and susceptibilities. Under concrete assumptions on the critical asymptotics of these quantities one gets relations for the parameters of these asymptotics (critical indices and amplitudes).

3. Let us consider in more detail the consequences of our relations for the case of two order parameters S and L corresponding to the external magnetic field and variation of temperature: $H-hNS/\theta_c(1-\varepsilon)$. Note that the temperature order parameter L is proportional to the Hamiltonian ($L \sim H/N$) and χ_{LL} is the specific heat of a system.

In this case there are six independent functions of interest: $S(\varepsilon), L(\varepsilon), \chi_{SS}(\varepsilon), S(h), L(h), \chi_{LL}(h)$, for which we suppose the power asymptotics with logarithmic corrections to hold true:

$$S(\varepsilon) = B\varepsilon^{\beta}|\ln\varepsilon|^{P_\beta},$$

$$L(\varepsilon) = \frac{A_-}{\alpha(1-\alpha)}\varepsilon^{1-\alpha}|\ln\varepsilon|^{P_\alpha}, \tag{45}$$

$$\chi_{SS}(\varepsilon) = \Gamma_-\varepsilon^{-\gamma}|\ln\varepsilon|^{P_\gamma},$$

$$S(h)=(h/D)^{1/\delta}\,|\ln h|^{P_\delta},$$
$$L(h)=Z h^{\zeta}\,|\ln h|^{P_\zeta}, \quad (\zeta\equiv 1/\delta_{LS}), \qquad (45)$$
$$\chi_{LL}(h)=E h^{-\epsilon}\,|\ln h|^{P_\epsilon};$$

where the critical indices α, β, γ, δ, ζ, ϵ are positive numbers, $\delta>1$, $1>\alpha>0$; the logarithmic indices P_α, P_β, P_γ, P_δ, P_ζ, P_ϵ are arbitrary real numbers; the critical amplitudes A_-, B, Γ_-, D, Z, E are positive numbers. Here we have used a short notation of the form: $F(\varepsilon)\equiv F[H/\theta_c(1-\varepsilon)]$, $F(h)\equiv F[H-hNS/\theta_c]$.

We shall assume the temperature variation to be <u>weak</u>, so, we can introduce the auxiliary critical system (see (33)):

$$\frac{H_\varepsilon}{\theta_c}\equiv\frac{H+\Delta(\varepsilon)NS^2}{\theta_c(1-\varepsilon)}=\left\{\begin{matrix}\text{the critical}\\ \text{system}\end{matrix}\right\}. \qquad (46)$$

We shall also assume that the asymptotics like in the last three relations in (45) remain valid for the case of the system (46).

Let us now specify the system $H+V/\theta_c$ in (38),(41)-(44) as $H/\theta_c(1-\varepsilon)$, and the system H_V/θ_c as (46), and put $A=L$, $B=S$. We then get (under the above assumptions) a set of relations for the critical parameters in (45). Namely, we obtain for critical indices:

$$\gamma=\beta(\delta^\circ-1), \qquad \zeta^\circ(\gamma+\beta)=1-\alpha, \qquad (47)$$
$$\gamma+2\beta+\alpha=2, \qquad \epsilon^\circ(\gamma+\beta)=\alpha,$$

for logarithmic indices:

$$\delta^\circ P^\circ_\delta=P_\gamma+P_\beta(\delta^\circ-1), \qquad P^\circ_\zeta=P_\alpha+\zeta^\circ(P_\gamma-P_\beta), \qquad (48)$$
$$P_\alpha+P_\beta=2P_\gamma, \qquad P^\circ_\epsilon=P_\alpha+\epsilon^\circ(P_\beta-P_\gamma),$$

and for critical amplitudes:

$$(\delta^\circ-1)\Gamma_- D^\circ B^{\delta^\circ-1}=(\gamma+\beta)^{\delta^\circ P^\circ_\delta}, \qquad (49)$$
$$\frac{\alpha(1-\alpha)Z^\circ}{A_-}\left(\frac{B}{(\delta^\circ-1)\Gamma_-}\right)^{\zeta^\circ}(\gamma+\beta)^{P^\circ_\zeta}=1,$$
$$\alpha\beta^2B^2/A_-\Gamma_-=1,$$
$$\frac{\alpha\delta^\circ E^\circ}{(\delta^\circ-1)A_-}\left(\frac{B}{(\delta^\circ-1)\Gamma_-}\right)^{-\epsilon^\circ}(\gamma+\beta)^{P^\circ_\epsilon}=1.$$

Here the superscript $^\circ$ means that the corresponding parameters belong to the auxiliary critical system (46).

The relations (47) (if we omit the superscript $^\circ$) are well-known quasi-phenomenological scaling laws[6]. These relations are confirmed by numerous experimental studies and by results for the Ising model and some other simple concrete models[6]. So, we have an argument in favour of our assumption that the critical behaviour of the systems H/θ_c and H_V/θ_c (46) is similar (and even the same on the level of critical indices). We can also hope that the relations (48) for the logarithmic indices remain hold true even if we omit superscript $^\circ$ therein[7]. However, for the critical amplitudes (see (49)) it seems to be wrong in the general case. Nevertheless, one can use (49) for approximate estimates.

For an additional discussion of the relations for critical indices see Appendix C.

<u>4.</u> Let us give a few remarks on the disordered phase. As the simplest example of the system in the disordered phase, we shall take

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[6] To be more exact, we must note that in full measure this is just so only for the two left equalities in (47).

[7] It is worth to note that in practice the logarithmic corrections are rare, more often $P_\alpha=P_\beta=P_\gamma=P_\delta=P_\zeta=P_\epsilon=0$.

$$H+\rho NS^2/\theta_c\ ,\ \rho>0\ , \tag{50}$$

where H/θ_c is our critical "ferromagnetic" system with

$$\chi_{SS}[H/\theta_c]=+\infty\ ,\ S[H/\theta_c]=0\ . \tag{51}$$

(11a): Consider first the case of only one order parameter S in

$$\chi^{-1}_{SS}[\Gamma+\rho N(S-S[\Gamma/\theta])^2/\theta]=\chi^{-1}_{SS}[\Gamma/\theta]+2\rho. \tag{52}$$

Passing here to the limit $\Gamma/\theta\rightarrow H/\theta_c$ and taking into account (51) we get (see also/5/):

$$\chi_{SS}[H+\rho NS^2/\theta_c]=\frac{1}{2\rho}\ ,\ S[H/\theta_c]=0. \tag{53}$$

In the general case of n order parameters S_α we can suppose that the inverse matrix of susceptibilities χ^{-1}(see(8)) becomes zero at the critical point:

$$\chi^{-1}[\Gamma/\theta]=\begin{cases}\neq 0\ ,\ \Gamma/\theta\neq H/\theta_c\ ,\\ \rightarrow 0\ ,\ \Gamma/\theta\rightarrow H/\theta_c\ .\end{cases} \tag{54}$$

Then in view of (11a) we obtain:

$$\chi^{-1}[H+R_n/\theta_c]=2\hat{\rho}\qquad ,\text{i.e.,} \tag{55}$$

$$\chi_{\alpha\beta}[H+N\sum_{\gamma=1}^{n}\rho_\gamma(S_\gamma-S_\gamma[H/\theta_c])^2/\theta_c]=\frac{1}{2\rho_\alpha}\Delta_{\alpha\beta}\ , \tag{55a}$$

where $\Delta_{\alpha\beta}$ is the Kronecker symbol.

APPENDIX A.

Let Γ/θ be an arbitrary system, N be the number of particles in the system, $A=A^+$ and $B=B^+$ be hermitian operators (order parameters). Then the generalized susceptibility $\chi_{AB}[\Gamma/\theta]$ is defined as follows:

$$\chi_{AB}[\Gamma/\theta]=\left(\frac{\partial}{\partial h}\langle B\rangle_{\Gamma-hNA/\theta}\right)_{h=0}=\left(\frac{\partial}{\partial h}\langle A\rangle_{\Gamma-hNB/\theta}\right)_{h=0}, \tag{A1}$$

and the following representation holds true:

$$\chi_{AB}[\Gamma/\theta]=\frac{N}{\theta}\int_0^1\langle\tilde{A}(\tau)\tilde{B}\rangle_{\Gamma/\theta}\,d\tau; \tag{A2}$$

$$\tilde{A}=A-\langle A\rangle_{\Gamma/\theta}\ ,\ \tilde{B}=B-\langle B\rangle_{\Gamma/\theta};\ \tilde{A}(\tau)=e^{\tau\frac{\Gamma}{\theta}}\tilde{A}e^{-\tau\frac{\Gamma}{\theta}},$$

where $\langle\cdots\rangle_{\Gamma/\theta}$ means the equilibrium Gibbs average for system Γ/θ 8/:

$$\langle\ldots\rangle_{\Gamma/\theta}=\frac{\mathrm{Tr}\,(\ldots e^{-\Gamma/\theta})}{\mathrm{Tr}\,e^{-\Gamma/\theta}}\ . \tag{A3}$$

In the case of non-hermitian operators $A\neq A^+$, $B\neq B^+$ the susceptibility is defined analogously:

$$\chi_{AB}[\Gamma/\theta]=\left[\frac{\partial}{\partial h^*}\langle B\rangle_{\Gamma-N(h^*A+hA^+)/\theta}\right]_{h^*=0}, \tag{A4}$$

$$\chi_{A^+B}[\Gamma/\theta]=\left[\frac{\partial}{\partial h}\langle B\rangle_{\Gamma-N(h^*A+hA^+)/\theta}\right]_{h^*=0},\ \text{etc.}$$

The representation (A2) remains valid also in this case. Making use of (A2) one can show/2/ that the bilinear form $\chi_{AB^+}[\Gamma/\theta]\equiv\chi_{B^+A}[\Gamma/\theta]$ can be treated as the scalar product of the operators (A,B).

8/In the above considerations susceptibility appears as a derivative not of the common averages, but of the quasi-averages (see (3)). However, this difference does not matter much, since for "regular" systems the operations of differentiating and passing to the limits in (3) commute, while "singular" points (e.g. the critical point)are considered by the limit procedure in final results.

APPENDIX B.

One can easily extend the scheme described above to the case of non-hermitian order parameters S_α, S^+_α $(S_\alpha \neq S^+_\alpha)$, $\alpha = 1,\dots,n$. Conditions (2) should be suplemented by $|S_\alpha S^+_\beta - S^+_\beta S_\alpha| \leq K_3/N$. As before, the identities (4) for quasi-averages hold true, being now the complex identities; now in (4)

$$R_n = \sum_{\gamma=1}^{n} \rho_\gamma (S_\gamma - S_\gamma[\Gamma/\theta])(S^+_\gamma - S^+_\gamma[\Gamma/\theta]), \quad \rho_\gamma > 0. \tag{A5}$$

Introducing a set of susceptibilities $\chi_{\alpha\beta^*} \equiv \chi_{\beta^*\alpha} = \chi_{S_\alpha S^+_\beta}$, $\chi_{\alpha\beta} \equiv \chi_{\beta\alpha} = \chi_{S_\alpha S_\beta}$, $\chi_{\alpha^*\beta^*} \equiv \chi_{\beta^*\alpha^*} = \chi_{S^+_\alpha S^+_\beta}$, and with the reasoning analogous to those when substantiating (10), we obtain the relations generalizing (10)(we omit Θ for simplicity):

$$\chi_{\alpha\beta^*}(\Gamma) = \chi_{\alpha\beta^*}(\Gamma + R_n) + \sum_{\gamma=1}^{n} \rho_\gamma [\chi_{\alpha\gamma^*}(\Gamma+R_n)\chi_{\gamma\beta^*}(\Gamma) + \chi_{\alpha\gamma}(\Gamma+R_n)\chi_{\gamma^*\beta^*}(\Gamma)],$$
$$\chi_{\alpha\beta}(\Gamma) = \chi_{\alpha\beta}(\Gamma+R_n) + \sum_{\gamma=1}^{n} \rho_\gamma [\chi_{\alpha\gamma}(\Gamma+R_n)\chi_{\gamma^*\beta}(\Gamma) + \chi_{\alpha\gamma^*}(\Gamma+R_n)\chi_{\gamma\beta}(\Gamma)], \tag{A6}$$

(plus complex conjugated relations). These relations can be also written in the matrix form (see (11)).

It should be noted that the relations (A6) are not more general, in principle, than (10) for hermitian order parameters. Since one can always write $S_\alpha = S^1_\alpha + i S^2_\alpha$, where S^1_α and S^2_α are hermitian, (A6) are equivalent to a set of relations for hermitian S^1_α, S^2_α ($\alpha = 1,\dots, n$) analogous to (10).

The simplification occurs in the special case when in view of the symmetry of the system Γ/θ $\chi_{\alpha\beta} \equiv \chi_{\alpha^*\beta^*} \equiv 0$ for all α, β, and only $\chi_{\alpha\beta^*} \equiv \chi_{\beta^*\alpha} \neq 0$. Then (A6) reduces to

$$\chi_{\alpha\beta^*}(\Gamma) = \chi_{\alpha\beta^*}(\Gamma+R_n) + \sum_{\gamma=1}^{n} \rho_\gamma \chi_{\alpha\gamma^*}(\Gamma+R_n)\chi_{\gamma\beta^*}(\Gamma) \tag{A7}$$
$$\alpha = 1,\dots,n.$$

These relations coincide with (10) (with evident replacements $\chi_{\alpha\beta} \to \chi_{\alpha\beta^*}$ and R_n (4) $\to$ R_n (A5)) with the only diffe-

rence: instead of $2\rho_\gamma$ in (10) we have ρ_γ in (A7).

In a particular case of one order parameter S we can derive, on the basis of (A7), the relation analogous to (53):

$$\chi_{SS^+}[H + \rho N S S^+/\theta_c] = \frac{1}{\rho},$$
$$(\chi_{SS}[H+\rho NSS^+/\theta_c] \equiv \chi_{S^+S^+}[H+\rho NSS^+/\theta] \equiv 0), \tag{A8}$$

where $\rho > 0$, H/θ_c is the critical system, $S[H/\theta_c] = 0$.

APPENDIX C.

Let us return to the relations (38)-(44) (Section 2) for the system $H + V/\theta_c$. Assume that the variation V is characterized by a small parameter $\xi > 0$, $V \equiv V(\xi) \to 0$ as $\xi \to 0$, and that the quasi-power asymptotics hold true:

$$\chi_{XY}(\xi) \sim \xi^{-\lambda_{XY}}, \; Y(\xi) \sim \xi^{\nu_Y}, \; \chi_{SS}(\xi) \sim \xi^{-\gamma},$$
$$S(\xi) \sim \xi^{\beta}, \; \chi_{XY}(h) \sim h^{-\mu_{XY}}, \; Y(h) \sim h^{1/\delta_{YS}},$$
$$S(h) \sim h^{1/\delta_{SS}} \; (\delta_{SS} > 1); \text{ where } F(\xi) \equiv F[V(\xi)],$$
$$X, Y = \{A, B, S\}.$$

Then on the basis of (40) to (44) we obtain the following relations for the critical indices:

$$\lambda_{XS} + \nu_X = \lambda_{YS} + \nu_Y, \quad \lambda_{YS} = (\gamma+\beta)\mathring{\mu}_{YS},$$
$$\nu_Y \mathring{\delta}_{YS} = \gamma + \beta, \qquad \lambda_{YY} = (\gamma+\beta)\mathring{\mu}_{YY}, \tag{A9}$$

where $X, Y = A, B, S$. From (39) we also find that one of the following three possibilities hold true:

a) $\lambda_{AB} = (\gamma+\beta)\mathring{\mu}_{AB}$, $\lambda_{AS} + \lambda_{BS} < \lambda_{AB} + \gamma$; (A10)

b) $\lambda_{AB} + \gamma = \lambda_{AS} + \lambda_{BS}$, $(\gamma+\beta)\mathring{\mu}_{AB} < \lambda_{AB}$;

c) $\lambda_{AB} = (\gamma+\beta)\mathring{\mu}_{AB} = \lambda_{AS} + \lambda_{BS} - \gamma$.

The additional superscript $\circ$ in (A9),(A10) means that the corresponding quantities belong to the auxiliary critical system (33).

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