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E17 - 12987

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MANY-BOSON AVERAGES FOR THE DICKE-TYPE MODEL SYSTEMS

Submitted to ТМФ

1979

1. Introduction

Here we consider the calculation of the many-boson averages for a class of many-body model systems with the interaction of substance and finite number of modes of a boson field (the Dicke-type models). The corresponding generalized Hamiltonian is

$$H = \sum_{\alpha=1}^{n} \omega_\alpha \overset{+}{a}_\alpha a_\alpha + \sqrt{N} \sum_{\alpha=1}^{n} \left(\lambda^*_\alpha \overset{+}{L}_\alpha a_\alpha + \lambda_\alpha L_\alpha \overset{+}{a}_\alpha \right) + T - N \sum_{\alpha=1}^{n} \varkappa_\alpha \overset{+}{L}_\alpha L_\alpha , \qquad (1)$$

where $\overset{+}{a}_\alpha, a_\alpha$ are boson operators,

$$a_\alpha \overset{+}{a}_\beta - \overset{+}{a}_\beta a_\alpha = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases} \qquad (2)$$

$$a_\alpha a_\beta - a_\beta a_\alpha = 0 ,$$

$L_\alpha, \overset{+}{L}_\alpha, T = \overset{+}{T}$ are operators of "substance subsystem" or " L-subsystem" [1)] , N is the number of particles in the "substance"; $\omega_\alpha, \varkappa_\alpha$ are real parameters, $\omega_\alpha > 0$, $\varkappa_\alpha \geqslant 0$; $\lambda_\alpha, \lambda^*_\alpha$ are complex parameters. The L-subsystem operators should satisfy only the following sufficiently general conditions:

$$\| L_\alpha \| \leq K_1, \qquad (3a)$$

$$\| L_\alpha T - T L_\alpha \| \leq K_2, \qquad (3b)$$

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1) The name "L-subsystem" comes from the notation of the operators L_α .

$$\begin{cases} \| L_\alpha L_\beta - L_\beta L_\alpha \| \le K_3 / N, \\ \| L^+_\alpha L_\beta - L_\beta L^+_\alpha \| \le K_3 / N, \end{cases} \qquad (3c)$$

where $\| \dots \|$ means the operator norm, K_1, K_2, K_3 are constants independent of N. The Hamiltonian (1) is defined in the space

$$\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_B , \qquad (4)$$

where $\mathcal{H}_B$ is the Fock space of the boson subsystem, $\mathcal{H}_L$ is the Hilbert space of L-subsystem.

One of the best known concrete models covered by (1) is just the Dicke maser model, which has been proposed (in the original form) in 1954. This model represents a great number N of two--level atoms coupled to one mode of a quantized radiation and finds applications in the theory of coherent radiation. In 1973 K.Hepp and E.Lieb have obtained the asymptotically exact (in the thermodynamical limit $N \to \infty$) solution of the Dicke model and discovered and described the "superradiant" phase transition in the system. These important results have stimulated the intensive study of the different modifications of the Dicke model and other models of similar mathematical structure in different branches of solid-state physics and statistical mechanics, and also initiated the developement of the mathematically rigorous methods for studying such systems (see, in particular, refs./ 1-26/ and references therein).

The most general results for the basic class of models (1) were obtained in ref. /1/, where the whole class has been examined from a unique standpoint, based, in part, on the ideas of the method of "approximating Hamiltonians" /27/. In particular, it was shown that in the thermodynamical limit $N \to \infty$ the free energy for the system with Hamiltonian (1) coincides with the free energy of a simplified approximating system with the Hamiltonian

$$H_A(C) = T - N \sum_{\alpha=1}^{n} g_\alpha \left(C_\alpha L^+_\alpha + C^*_\alpha L_\alpha \right) + N \sum_{\alpha=1}^{n} g_\alpha |C_\alpha|^2, \quad g_\alpha \equiv \varkappa_\alpha + |\lambda_\alpha|^2 / \omega_\alpha , \qquad (5)$$

where complex variational parameters C_α should be defined from the condition of absolute minimum of the limit $(N \to \infty)$ free energy of the system (5) [2]. Denote these values as $\bar{C}_\alpha$

$$f_\infty \left[H_A(\bar{C}) \right] = \operatorname{abs} \min_C f_\infty \left[H_A(C) \right] ,^{3)} \qquad (5a)$$

$$C_\alpha = \bar{C}_\alpha .$$

So /1/ :

$$\left| f_N[H] - f_\infty \left[H_A(\bar{C}) \right] \right| \xrightarrow[N \to \infty]{} 0 . \qquad (6)$$

In ref. /6/ a method for the correct definition of quasi--averages for the class of models (1) was under discussion. It was proposed to introduce quasi-averages on the basis of the Hamiltonian

$$H_\tau = H + 2N \sum_{\alpha=1}^{n} \tau_\alpha \omega_\alpha \left(\frac{a^+_\alpha}{\sqrt{N}} + \frac{\lambda^*_\alpha}{\omega_\alpha} \bar{C}^*_\alpha \right) \left(\frac{a_\alpha}{\sqrt{N}} + \frac{\lambda_\alpha}{\omega_\alpha} \bar{C}_\alpha \right), \quad \tau_\alpha > 0 , \qquad (7)$$

where H is the Hamiltonian (1). Then quasi-averages should be defined by the rule /6/

$$\langle \dots \rangle_H = \lim_{\tau_\alpha \to 0} \lim_{N \to \infty} \langle \dots \rangle_{H_\tau} , \qquad (7a)$$

where $\langle \dots \rangle$ is the Gibbs average, and the order of the limit procedures in (7a) being essential.

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2) Note that Hamiltonian (5), in contrast to that of (1), does not contain boson operators and is defined in the space $\mathcal{H}_L$.

3) The definition of the free energy: $f[\mathfrak{H}] = -\frac{\theta}{N} \ln \operatorname{Tr} e^{-\mathfrak{H}/\theta}, \; \mathfrak{H} = \mathfrak{H}^+$.

2. Many-Boson Averages and Substitution Rule

Relation (6) means that Hamiltonians H (1) and $H_A(\bar{C})$ (5) are thermodynamically equivalent to each other on the level of free energy. But if one raises the question of such an equivalence on the level of equilibrium averages (and quasi-averages). one immediately clash with the difficulties for the averages which contain boson operators. Since the approximating Hamiltonian $H_A(\bar{C})$ does not involve boson operators $\overset{+}{a}_\alpha, a_\alpha$ the boson averages taken over $H_A(\bar{C})$ do not exist.

The solution of this problem, as is shown below, proceeds as follows: when calculating any averages over Hamiltonians H and H_τ all boson operators can be replaced by the operators of the L-subsystem, after one can pass to averaging over Hamiltonian $H_A(\bar{C})$, using standard methods /27,28/ .

Consider the generalized operator containing boson operators:

$$\mathfrak{A} = \dots X \dots \frac{\overset{+}{a}_\alpha}{\sqrt{N}} \dots X' \dots \frac{a_\beta}{\sqrt{N}} \dots X'' \dots , \tag{8}$$

where X, X', X'' are bounded in norm operators of the L-subsystem acting in $\mathcal{H}_L$, while under dots one can imply $a^\#_\alpha/\sqrt{N}$ or any operators of L-subsystem of an X type, taken in arbitrary amount and situated in arbitrary order. We shall show that when averaging such operators (8) the boson operators $a^\#_\alpha/\sqrt{N}$ can be replaced by the operators $L^\#_\alpha$ in accordance with the following rules: for the Gibbs averages over Hamiltonian H (as $N \to \infty$):

$$\frac{a^\#_\alpha}{\sqrt{N}} \rightleftarrows -\frac{\lambda^\#_\alpha}{\omega_\alpha} L^\#_\alpha , \tag{9a}$$

and for quasi-averages [4)]

$$\frac{a^\#_\alpha}{\sqrt{N}} \rightleftarrows -\frac{\lambda^\#_\alpha}{\omega_\alpha} \bar{C}^\#_\alpha , \quad L^\#_\alpha \rightleftarrows \bar{C}^\#_\alpha . \tag{9b}$$

[4)] We want to note that these "substitution rules" as well as in general the thermodynamical equivalence of the Hamiltonians (1) and (5), show that the boson operators in the models under consideration demonstrate "C-number" features. That resembles, to some extent, the situation in the N.N.Bogolubov model for superfluidity proposed in 1947 /29/ .

For the simplest case of one- and two-boson operators in averages the rules (9) have been obtained in refs./1,6/. In particular,

$$\left\langle \frac{a_\alpha}{\sqrt{N}} \right\rangle_H = -\frac{\lambda_\alpha}{\omega_\alpha} \bar{C}_\alpha , \tag{10}$$

$$\left\langle \frac{\overset{+}{a}_\alpha a_\alpha}{N} \right\rangle_H = \frac{|\lambda_\alpha|^2}{\omega_\alpha^2} \langle \overset{+}{L}_\alpha L_\alpha \rangle_H \left(1 + O\left(\tfrac{1}{N}\right)\right),$$

$$O\left(\tfrac{1}{N}\right) \xrightarrow[N \to \infty]{} 0 .$$

These relations follow from the intermediate results established in refs. /1,6/ :

$$\sum_{\alpha=1}^{n} \omega_\alpha \langle \overset{+}{B}_\alpha B_\alpha \rangle_H \xrightarrow[N \to \infty]{} 0 , \tag{11}$$

$$\sum_{\alpha=1}^{n} \omega_\alpha \left[\langle \overset{+}{B}_\alpha B_\alpha \rangle_{H_\tau} + \tau_\alpha \langle \overset{+}{D}_\alpha D_\alpha \rangle_{H_\tau} \right] \xrightarrow[\substack{N \to \infty \\ \tau_\alpha > 0}]{} 0 , \tag{12}$$

where

$$B_\alpha = \frac{a_\alpha}{\sqrt{N}} + \frac{\lambda_\alpha}{\omega_\alpha} L_\alpha ,$$

$$D_\alpha = \frac{a_\alpha}{\sqrt{N}} + \frac{\lambda_\alpha}{\omega_\alpha} \bar{C}_\alpha .$$

However, the extension of the substitution rules (9) to the general case of the many-boson operators (8) appears to be non-trivial [5)], since we need here new methods, which should enable us majorate the many-boson averages with a high number of boson operators by means of averages with a lesser number of operators. To derive the rules (9) we shall use the commutation relations (2), general structure of the Hamiltonians H (1) and H_τ (7) and relations (11) (12) [6)] . We shall apply also a general ine-

[5)] The problem of such an extension was formulared in ref. /1/.

[6)] Note also that one can derive the relation (11), using only the commutation relations (2) and the structure of the Hamiltonian (1), see ref. /7/ .

quality for equilibrium averages, which we derive in the next section.

3. Auxiliary Inequality

Let Γ be the Hamiltonian of a system, θ be the temperature modulus ($\theta =$ KT), $\langle \dots \rangle_\Gamma$ be equilibrium Gibbs average

$$\langle \dots \rangle_\Gamma = \mathrm{Tr}\,(\dots e^{-\Gamma/\theta}) / \mathrm{Tr}\, e^{-\Gamma/\theta}. \tag{13}$$

Introduce the auxiliary quadratic form

$$(A, B)_\varepsilon = \int_0^1 e^{\tau \frac{\varepsilon}{\theta}} \langle A(\tau) B \rangle_\Gamma \, d\tau , \tag{14}$$

where

$$A(\tau) = e^{\tau \frac{\Gamma}{\theta}} A \, e^{-\tau \frac{\Gamma}{\theta}} , \tag{14a}$$

and where ε is a nonzero real parameter.

<u>Lemma.</u> Let operators A, R and Hamiltonian Γ satisfy the relation

$$A\Gamma - \Gamma A = \varepsilon A + R , \tag{15}$$

where ε is a nonzero real number (positive or negative), and let B be an arbitrary operator. Then the following relations hold true [7]

$$\langle BA \rangle_\Gamma = \frac{\langle AB - BA \rangle_\Gamma}{e^{\varepsilon/\theta} - 1} - \frac{(R, B)_\varepsilon}{\theta (e^{\varepsilon/\theta} - 1)} , \tag{16a}$$

$$\left| \langle BA \rangle_\Gamma - \frac{\langle AB - BA \rangle_\Gamma}{e^{\varepsilon/\theta} - 1} \right| \le \frac{1}{2\theta \left| e^{\varepsilon/\theta} - 1 \right|} \otimes \tag{16b}$$

$$\otimes \sqrt{\left(\langle R R^+ \rangle_\Gamma + e^{\varepsilon/\theta} \langle R^+ R \rangle_\Gamma \right) \left(\langle B^+ B \rangle_\Gamma + e^{\varepsilon/\theta} \langle B B^+ \rangle_\Gamma \right)} .$$

[7] Condition (15) is not a restriction. For $[A, \Gamma] = K$ one can always put $K = \varepsilon A + R_\varepsilon$, where $R_\varepsilon = K - \varepsilon A$. Then relations (16) remain valid, and one can regard ε as a variational parameter, which should be chosen in accordance with a concrete problem.

<u>Proof.</u> Making use of (15), we have

$$\theta \frac{dA(\tau)}{d\tau} = -\varepsilon A(\tau) - R(\tau) , \tag{17}$$

and hence

$$e^{\varepsilon/\theta} \langle A(\tau) B \rangle_{\tau=1} - \langle AB \rangle = -\frac{1}{\theta} \int_0^1 e^{\tau \frac{\varepsilon}{\theta}} \langle R(\tau) B \rangle d\tau \tag{18}$$

(We omit here and below index Γ in averages). Taking into account that $\langle A(\tau) B \rangle_{\tau=1} = \langle BA \rangle$ (a consequence of the definitions, see (13) and (14a)), we get the inequality in (16).

To prove the inequality in (16), let us turn to the spectral formulas for binary averages

$$\langle X(t) Y \rangle = \int_{-\infty}^{+\infty} J_{XY}(\omega) e^{i\omega t} d\omega , \tag{19}$$

$$\langle Y X(t) \rangle = \int_{-\infty}^{+\infty} J_{XY}(\omega) e^{\frac{\omega}{\theta}} e^{i\omega t} d\omega , \tag{19a}$$

$$X(t) = e^{i\Gamma t} X e^{-i\Gamma t} .$$

In particular, for $t = -i\tau/\theta$, where τ is a real parameter, $X(t)$ (19a) transforms into $X(\tau)$ (14a) , and we get the spectral representation for the bilinear form (14)

$$(X, Y)_\varepsilon = \int_0^1 e^{\tau \frac{\varepsilon}{\theta}} \langle X(\tau) Y \rangle d\tau =$$

$$= \theta \int_{-\infty}^{+\infty} J_{XY}(\omega) \frac{e^{\frac{\varepsilon + \omega}{\theta}} - 1}{\varepsilon + \omega} d\omega . \tag{20}$$

We shall also need the N.N.Bogolubov inequality [30]. Let a bilinear form $Z(X,Y)$ of operators X, Y satisfy the conditions

$$Z(X,\overset{+}{X}) \geqslant 0\,,\quad [Z(X,Y)]^{*} = Z(\overset{+}{Y},\overset{+}{X}). \tag{21}$$

then the inequality holds true

$$|Z(X,Y)|^{2} \leq Z(X,\overset{+}{X})\, Z(\overset{+}{Y},Y). \tag{21a}$$

Taking into account properties of the spectral density $J_{XY}(\omega)$ (for details see [30]), one can easily verify that the bilinear form (20) satisfies the conditions (21) and hence [8]:

$$|(X,Y)_{\varepsilon}|^{2} \leq (X,\overset{+}{X})_{\varepsilon}\,(\overset{+}{Y},Y)_{\varepsilon}\ . \tag{22}$$

On the other hand, taking into account the elementary inequality

$$(e^{x}-1)/x \leq \frac{1}{2}(1+e^{x})$$

and representation (20), we obtain:

$$(X,\overset{+}{X})_{\varepsilon} \leq \frac{1}{2}\int_{-\infty}^{+\infty} J_{X\overset{+}{X}}(\omega)\left(1+e^{\frac{\varepsilon+\omega}{\theta}}\right)d\omega = \tag{23}$$

$$= \frac{1}{2}\left(\langle X\overset{+}{X}\rangle + e^{\varepsilon/\theta}\langle \overset{+}{X}X\rangle\right).$$

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[8] Note, that for $\varepsilon = 0$ the form (20) coincides with the Green function in the energy representation by zero argument [30]:

$$(X,Y)_{\varepsilon=0} = -2\pi\theta \langle\langle X;Y\rangle\rangle_{E=0}\ .$$

The inequality (22) then transforms into a well-known inequality for Green functions due to N.N.Bogolubov, which has been used to prove $1/q^{2}$-theorems [30].

Applying the bounds (22) and (23) to $(R,B)_{\varepsilon}$, we get just the inequality in (16). So, our lemma is proved.

4. Proof of the Substitution Rules

Consider first the rule (9a) for the case of common averages $\langle\dots\rangle_{H}$. We shall assume N to be finite, taking the limit $N\to\infty$ at the very end of calculations. Note first of all, that in virtue of (11)

$$\langle \overset{+}{B}_{\alpha} B_{\alpha}\rangle_{H} \xrightarrow[N\to\infty]{} 0, \tag{24a}$$

$$\langle B_{\alpha} \overset{+}{B}_{\alpha}\rangle_{H} \xrightarrow[N\to\infty]{} 0 \tag{24b}$$

(here (24b) follows from (24a) and (2), (3b)). Let us also take into account the general inequality [30] (see (21a)):

$$|\langle XY\rangle|^{2} \leq \langle X\overset{+}{X}\rangle\langle \overset{+}{Y}Y\rangle\ . \tag{25}$$

Let us now fix in $\mathfrak{A}$ one of the creation operators $\overset{+}{a}_{\alpha_0}$ and rewrite (8) as

$$\mathfrak{A} = \mathfrak{A}'\,\frac{\overset{+}{a}_{\alpha_0}}{\sqrt{N}}\,\mathfrak{A}'', \tag{26}$$

where $\mathfrak{A}'$ and $\mathfrak{A}''$ are operators analogous to $\mathfrak{A}$. We want to show that $\overset{+}{a}_{\alpha_0}/\sqrt{N}$ in (26) can be replaced by $\overset{+}{L}_{\alpha}$ in accordance with (9a). We put (26) into the form

$$\mathfrak{A} = \frac{\overset{+}{a}_{\alpha_0}}{\sqrt{N}}\,\mathfrak{A}_1 + \left[\mathfrak{A}', \frac{\overset{+}{a}_{\alpha_0}}{\sqrt{N}}\right]_{-}\mathfrak{A}'', \tag{26a}$$

$$\mathfrak{A}_1 \equiv \mathfrak{A}'\mathfrak{A}'',$$

and note that the second term here does not contain $\overset{+}{a}_{\alpha_0}/\sqrt{N}$ (besides it contains the additional factor $1/N$ (if not being zero) and, hence, it does not give any contribution to the first term in the limit $N\to\infty$), so we have only to prove that

$$\left|\left\langle \frac{\overset{+}{a}_{\alpha_0}}{\sqrt{N}}\mathfrak{X}_1\right\rangle_H - \left\langle\left(-\frac{\lambda^*_{\alpha_0}}{\omega_{\alpha_0}}\overset{+}{L}_{\alpha_0}\right)\mathfrak{X}_1\right\rangle_H\right| \xrightarrow[N\to\infty]{} 0, \qquad (27)$$

or, in an equivalent form,

$$\left|\left\langle \overset{+}{B}_{\alpha_0}\mathfrak{X}_1\right\rangle_H\right| \xrightarrow[N\to\infty]{} 0. \qquad (27a)$$

Making use of (25), we obtain:

$$\left|\left\langle \overset{+}{B}_{\alpha_0}\mathfrak{X}_1\right\rangle_H\right| \le \sqrt{\left\langle \overset{+}{B}_{\alpha_0}B_{\alpha_0}\right\rangle_H \left\langle \overset{+}{\mathfrak{X}}_1\mathfrak{X}_1\right\rangle_H}, \qquad (28)$$

where $\mathfrak{X}_1$ is an operator of the same structure as $\mathfrak{X}$ (8). Taking into account that the boson operators all commute with the operators of L-subsystem, one can put $\mathfrak{X}_1$ into the form:

$$\mathfrak{X}_1 = X_L A_s, \qquad (29)$$

where X_L is a bounded in norm operator of L-subsystem and A_s is a pure boson operator of the form

$$A_s = \underbrace{\frac{a^{\#}_{\alpha_1}}{\sqrt{N}}\cdots\frac{a^{\#}_{\alpha_s}}{\sqrt{N}}}_{s}. \qquad (29a)$$

Then

$$\left\langle \overset{+}{\mathfrak{X}}_1\mathfrak{X}_1\right\rangle_H = \left\langle \overset{+}{A}_s\overset{+}{X}_L X_L A_s\right\rangle_H \le \|X_L\|^2\left\langle \overset{+}{A}_s A_s\right\rangle_H. \qquad (30)$$

Taking into account (11), (28) and (30), we see that the problem of substantiation of (27) reduces to the proof of the following key inequality

$$\left\langle \overset{+}{A}_s A_s\right\rangle_H \le const_s, \quad N\to\infty, \qquad (31)$$

for all finite numbers $s = 1, 2, \ldots$.

In order to prove (31), let us use the general inequality (16) by $B = \overset{+}{A}$

$$\left\langle A\overset{+}{A}\right\rangle_\Gamma \le \frac{\left\langle A\overset{+}{A} - \overset{+}{A}A\right\rangle_\Gamma}{e^{\varepsilon/\theta}-1} + \frac{1}{2\theta|e^{\varepsilon/\theta}-1|}\cdot \qquad (32)$$

$$\cdot\sqrt{\left(\left\langle R\overset{+}{R}\right\rangle_\Gamma + e^{\varepsilon/\theta}\left\langle \overset{+}{R}R\right\rangle_\Gamma\right)\left(\left\langle A\overset{+}{A}\right\rangle_\Gamma + e^{\varepsilon/\theta}\left\langle \overset{+}{A}A\right\rangle_\Gamma\right)},$$

where

$$[A,\Gamma]_- = \varepsilon A + R, \quad \varepsilon\ne 0. \qquad (32a)$$

Let H be the Hamiltonian (1), A_s be the operator (29a), then we have

$$[A_s, H]_- = \varepsilon_s A_s + R_s, \qquad (33)$$

$$\varepsilon_s = \sum_{i=1}^{s}(\pm 1)_i\,\omega_{\alpha_i}, \qquad (33a)$$

$$R_s = \sum_{i=1}^{s}(\pm 1)_i\left(\frac{a^{\#}_{\alpha_1}}{\sqrt{N}}\cdots\cancel{\frac{a^{\#}_{\alpha_i}}{\sqrt{N}}}\cdots\frac{a^{\#}_{\alpha_s}}{\sqrt{N}}\right)\lambda^{\#}_{\alpha_i}L^{\#}_{\alpha_i}, \qquad (33\text{ b})$$

where $(\pm 1)_i$ equals +1 for α_i corresponding to a_{α_i} and (-1) for α_i corresponding to $\overset{+}{a}_{\alpha_i}$.

We assume below that $\varepsilon_s\ne 0$. If for given A_s $\varepsilon_s\ne 0$ we represent $\overset{+}{A}_s A_s$, using the commutation relations (2), in the ordered form

$$\overset{+}{A}_s A_s = \sum_{\ell=0}^{s} k_\ell\left(\frac{1}{N}\right)^{s-\ell}\overset{+}{A}{}'_\ell A'_\ell, \quad k_\ell\ge 0,$$

where operators A'_ℓ involve only the annihilation operators

$$A'_\ell = \frac{a_{\alpha_1}}{\sqrt{N}} \cdots \frac{a_{\alpha_\ell}}{\sqrt{N}}, \quad A'_{\ell=0} = const,$$

with $\varepsilon'_\ell > 0$. So, we see that one can always consider the case when in (38) $\varepsilon_S \neq 0$

Denote

$$M_S = \max_{\{\alpha_i\}} \langle \overset{+}{A}_S A_S \rangle_H, \tag{34}$$

where maximum is taken over all possible sets of S operators $a^{\#}_{\alpha_i}$ in A_S (29a). We have to prove boundedness of M_S for every S. We shall do it by induction. As it follows from (11), M_S is bounded for $S = 1,2$.

Let

$$M_{S_1} \leq const \quad \text{for} \quad S_1 = 1,2,\ldots,S-1. \tag{35}$$

Let us show that then also

$$M_S \leq const. \tag{36}$$

Note that in view of (2), (3), (25) and (35)

$$\langle R_S \overset{+}{R}_S \rangle_H \leq const\, M_{S-1}, \tag{37}$$

$$\langle \overset{+}{R}_S R_S \rangle_H \leq const\, M_{S-1},$$

$$|\langle A_S \overset{+}{A}_S - \overset{+}{A}_S A_S \rangle_H| \leq const\, \frac{M_{S-1}}{N}$$

(where constants depend, in particular, on $M_1,\ldots,M_{S-2}$). Then using inequality (32) (for the case $\Gamma = H$, $A = A_S$, $R = R_S$, see (33)) we obtain [9] .

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[9] Here ε_S corresponds to a set of $\{\alpha_i\}$, which realises maximum in the definition (34). We assume $\varepsilon_S \neq 0$ (see the discussion after formula (33)).

$$M_S \leq |e^{\varepsilon_S/\theta} - 1|^{-1} \left\{ C_1 \frac{M_{S-1}}{N} + \right.$$
$$\left. + C_2 \sqrt{M_S M_{S-1}} \right\}, \tag{38}$$

where constants C_1 and C_2 depend on $M_1, \ldots, M_{S-2}$. It follows directly from (38) that under the assumption (35) the bound (36) is also valid. So, by the induction principle, the bound (31) is proved for all S and, hence, the substitution rule (9a) is proved for the case of the operator $\overset{+}{a}_\alpha/\sqrt{N}$. One can prove this rule for $a_\alpha/\sqrt{N}$ in an analogous way, using (24b) instead of (24a). Applying this rule step by step, one can replace all the boson operators in $\langle \mathfrak{A} \rangle_H$ by the operators $L^{\#}_\alpha$.

In the case of quasi-averages the proof of substitution rule (9b) is quite analogous to that presented above. One should do all necessary operations by finite N and $\tau_\alpha > 0$, passing to the limit $N \to \infty$, and then $\tau_\alpha \to 0$, at the very end of calculations. One should use here instead of (24) the analogous bounds for B_α and D_α which follow from (12). The problem then reduces to prove that averages of the type (31), with averaging taken over H_τ (7), are bounded. Since the transition from H to H_τ means only the renormalization of parameters $\omega_\alpha \to (1+\tau_\alpha)\omega_\alpha$ and redefinition of $\{L_\alpha\}$, it does not destroy the general structure of the Hamiltonian we have used. So, the proof represented above for the case of averages over H is also valid for averages over H_τ .

So, the substitution rules (9) are completely proved. In accordance with these rules, for example, we have [10]

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[10] Simple illustrations for the relation (39b) in the concrete case of the Dicke model one can find in ref. /2/ (see Theorems 3.3 and 3.13 therein).

$$\left\langle \frac{\overset{+}{a}_{\alpha_1}}{\sqrt{N}} \cdots \frac{\overset{+}{a}_{\alpha_s}}{\sqrt{N}} \frac{a_{\alpha_s}}{\sqrt{N}} \cdots \frac{a_{\alpha_1}}{\sqrt{N}} \right\rangle_H = \tag{39a}$$

$$= \left| \frac{\lambda_{\alpha_1} \cdots \lambda_{\alpha_s}}{\omega_{\alpha_1} \cdots \omega_{\alpha_s}} \right|^2 \left\langle \overset{+}{L}_{\alpha_1} \cdots \overset{+}{L}_{\alpha_s} L_{\alpha_s} \cdots L_{\alpha_1} \right\rangle_H \otimes$$

$$\otimes \left(1 + O\left(\frac{1}{N}\right)\right), \; O\left(\frac{1}{N}\right) \xrightarrow[N\to\infty]{} 0,$$

$$\left\langle \left(\prod_{i=1}^{s} \frac{a^{\#}_{\alpha_i}}{\sqrt{N}} \right) \left(\prod_{j=1}^{r} L^{\#}_{\alpha_j} \right) F \right\rangle_H = \tag{39b}$$

$$= (-1)^s \left(\prod_{i=1}^{s} \frac{\lambda^{\#}_{\alpha_i}}{\omega_{\alpha_i}} \bar{C}^{\#}_{\alpha_i} \right) \left(\prod_{j=1}^{r} \bar{C}^{\#}_{\alpha_j} \right) \langle F \rangle_H ,$$

where F is an arbitrary operator of L - subsystem. So, arbitrary equilibrium averages and quasi-averages with boson operators can be always represented through the "pure" L-subsystem averages.

In conclusion we note that from the physical point of view the substitution rules (9) mean that the boson subsystems in the systems of the class considered (see (1)) are completely driven by the corresponding L-subsystems, and every re-arranging in L-subsystem necessitates the corresponding re-arranging in the boson subsystem (see also a discussion in /1/). This does not depend on any concrete features of the models, being a direct consequence of the general structure of the Hamiltonian and of the commutation relations for bosons.

A c k n o w l e d g e m e n t s: We wish to express our gratitude to the members of the V.K.Fedyanin seminar for valuable discussion.

N o t e a d d e d : Just recently (November, 1979) the communication /31/ has appeared, where the many-boson correlation averages for the concrete case of the Dicke model were considered. The boundedness of the many-boson averages was proved there by a method generalizing that of ref. /7/ .

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Received by Publishing Department
on December 5 1979.