объединенный
институт
ядерных
исследований
дубна

E17 - 12928

N.Angelescu

THE CRITICAL BEHAVIOUR OF KAC-TYPE MODELS FOR SEMI-INFINITE SYSTEMS

1979

E17 - 12928

N.Angelescu

THE CRITICAL BEHAVIOUR OF KAC-TYPE MODELS FOR SEMI-INFINITE SYSTEMS

Contributed to the Summer school "Recent Advances in Statistical Mechanics", Poiana Brasov, Romania, 1979.

Ангелеску Н.

E17 - 12928

Критическое поведение моделей типа Каца для полубесконечной системы

В докладе изложены результаты подробного и строгого рассмотрения критического поведения в некотором классе моделей самосогласованного поля для полубесконечной системы. Показано, что критический показатель намагниченности первого слоя ($\beta_1 = 1$) отличается от показателя объемной намагниченности ($\beta = 1/2$). Доказано автомодельное поведение распределений как намагниченности в полубесконечной системе, так и объемной и поверхностной свободной энергии. Получены также критические показатели локальной и объемной восприимчивости.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1979

Angelescu N.

E17 - 12928

The Critical Behaviour of Kac-Type Models for Semi-Infinite Systems

A detailed and rigorous description of the oritical behaviour within a class of mean-field models of a semi-infinite ferromagnetic spin system is given. It is shown that the critical exponent of the first layer magnetization equals 1, in contradistinction with the bulk magnetization which equals 1/2. The scaling properties of the magnetization profile, and of the bulk- and surface-free energies are proved. Finally, the critical exponents of the (global and local) susceptibilities are derived.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1979

1. INTRODUCTION

This talk is a report on joint work performed with Bundaru, Costache and Nenciu [1-5], devoted to a (rigorous) detailed characterization of the critical behaviour for a class of mean-field models of a semi-infinite ferromagnet. As a rule, mean-field models are not sensitive to the geometry of the system, they being a long-range limit. One can, however, produce a mean-field model able to account for surface effects by increasing the interaction range very anisotropically (cf. Costache's idea in ref. [3] for Ising, extended to all D-vector models by Pearce [6]). For a slab of n layers within which we take the longrange limit, this will result in a system of self-consistency equations for the n-dimensional order parameter. Letting $n \to \infty$, every fixed component of the order parameter approaches a limit; the sequence of these limits is supposed to characterize within our model the nontranslationally invariant state of the semi-infinite system. The problem under discussion is the behaviour of this state near the critical point.

This kind of problems received much attention in the last years (see, for instance, ref. [7] and refs. therein), partly due to the existence of experimental data from local measurements of the low-order correlations near a surface. The scaling phenomenological description of the critical behaviour has been extended to account for surface quantities as well. For a ferromagnetic lattice spin system, this description goes as follows [8] :

Let $\psi_n(\beta,H,h)$ be the free energy per spin of a slab of n layers in a uniform magnetic field H, with an added magnetic field h, localized on the boundary layers. For $n \to \infty$, one expects:

$$\psi_n(\beta, H, h) = \psi^B(\beta, H) + \frac{2}{n}\psi^S(\beta, H, h) + O(n^{-2}). \tag{1}$$

Remark that ψ^B, ψ^S have critical point $\beta = \beta_c$, $H = 0$, while ψ_n may have a different critical temperature, $\beta_c^{(n)} \to \beta_c$; the asymptotic expansion (1) is expected to hold for $(\beta,H) \neq (\beta_c,0)$. One then assumes a "scaling" behaviour for the singular parts (to be identified in each case) of $\psi^B$and $\psi^S$for $t \equiv (\beta - \beta_c)/\beta_c \to \pm 0$ and $H, h \to 0$:

$$\psi^{B,sing}(\beta, H) \sim |t|^{2-\alpha}\phi_{\pm}^B(|t|^{-\Delta}H), \tag{2}$$

$$\psi^{S,sing}(\beta,H,h) \sim |t|^{2-\alpha_S}\phi_{\pm}^S(|t|^{-\Delta}H, |t|^{-\Delta_1}h). \tag{3}$$

These formulae are to be understood as follows: there exist numbers α, α_S, Δ, Δ_1 such that $|t|^{\alpha-2}\psi^{B,sing}(\beta,|t|^{\Delta}H)$ and $|t|^{2-\alpha_S}\psi^{S,sing}(\beta,|t|^{\Delta}H, |t|^{\Delta_1}h)$ have (non-trivial) limits $\phi_{\pm}^B(H)$ and $\phi_{\pm}^S(H,h)$ as $t \to \pm 0$ for every $H,h \geq 0$. Moreover, the asymptotic expansions (1)-(3) are supposed to be such that the critical exponents of other quantities can be obtained in terms of α, α_S, Δ, Δ_1 by taking appropriate derivatives of the r.h.s. of (2) or (3) (e.g., the surface layer magnetization of the semi-infinite system : $\lim_{n\to\infty}\frac{n}{2}\frac{\partial\psi_n}{\partial h}$ should behave like $\frac{\partial}{\partial h}$ of the r.h.s. of (3), giving the critical exponent $\beta_1 = 2 - \alpha_S - \Delta_1$). This implies certain relations among critical exponents, which represent the main physical content of the scaling theory.

One sometimes assumes [9], in order to describe the cross-over from surface- to bulk-critical behaviour, that the magnetization profile scales, i.e., when measuring magnetizations in units of the order of the bulk magnetization and distances in correlation lengths, the magnetization profile converges.

Our results can be shortly characterized as a proof of these statements within our class of mean-field models. However, we do not have a proof of all critical exponent relations, but only check their validity up to the susceptibilities. As far as we know, the only rigorous results of this kind have been obtained for the two-dimensional Ising ferromagnet by Mc Coy and Wu [10] (who calculated the critical exponent of the 1st row magnetization (=1/2) different from that of the bulk spontaneous magnetization (=1/8)) and by Bariev [11] (who proved the scaling of the magnetization profile).

2. THE KAC-TYPE MODEL FOR A SLAB WITH n ROWS

Consider a rectangle of n rows (labelled i, j, ...) and N columns (labelled μ, ν ...) of Ising spins (classical D-vector spins and their spherical limit [6,12] can be treated on equal footing, but I shall restrict myself for simplicity to the Ising case). Let the energy of a given configuration $S = (S_{i\mu})_{\substack{i=1\dots n\\ \mu=1\dots N}}$ be:

$$\mathcal{H}_{\gamma,n,N}(H, S) = -\sum_{\substack{1\le i<j\le n\\ 1\le\mu<\nu\le N}} J_{ij}^{(n)} \gamma e^{-\gamma|\mu-\nu|} S_{i\mu} S_{j\nu} - \sum_{1\le i\le n} H_i \sum_{1\le\mu\le N} S_{i\mu} . \tag{4}$$

Here $\gamma > 0$ is a parameter controlling the interaction range, an homogeneous magnetic field, $H_i \geq 0$, has been imposed on row $i\,(i = 1\dots n)$, and the interaction matrix is supposed to have a direct product form, of which one factor is an exponential interaction along the row (Kac potential) and the other is chosen as:

$$J_{ij}^{(n)} = 4\delta_{ij} - \delta_{|i-j|,1} , \quad i, j = 1\dots n . \tag{5}$$

Otherwise: every spin interacts via a Kac potential with spins in the same and in the nearest-neighbouring rows. Let $Z_{\gamma,n,N}(\beta, H)$ be the corresponding partition function. The free energy per spin of our model is defined as:

$$\psi_n(\beta, H) = -\lim_{\gamma\to+0} \lim_{N\to\infty} (Nn\beta)^{-1} \log Z_{\gamma,n,N}(\beta, H) . \tag{6}$$

The correlation functions (state) of the model are defined as limits; for instance $(H \neq 0)$:

$$m_i^{(n)}(\beta, H) = \lim_{\gamma\to+0} \lim_{N\to\infty} \langle S_{i\mu} \rangle_{\gamma,N,n,H} \quad \text{(row magnetization)} \tag{7}$$

$$\chi_{ij}^{(n)}(\beta, H) = \lim_{\gamma\to+0} \lim_{N\to\infty} N^{-1} \sum_{1\le\mu,\nu\le N} \langle S_{i\mu} S_{j\nu} \rangle^{T}_{\gamma,N,n,H} \quad \text{(susceptibility matrix)} \tag{8}$$

This limiting model is a solvable (mean-field) model, as is proved in ref. /3,6/ , i.e., the free energy (6) can be calculated as the minimum of a known function:

$$\psi_n(\beta, H) = \min_{R^n} \mathcal{G}_n(\beta, H; \cdot), \tag{9}$$

$$\mathcal{G}_n(\beta, H; m) = \frac{1}{2n} \sum_{1 \le i \le n} [m_i (J^{(n)} m)_i - \frac{2}{\beta} \mathcal{F}(\beta (J^{(n)} m + H)_i)]. \tag{9'}$$

Here $\mathcal{F}(x)$ is the free energy of one spin in the magnetic field x at $\beta = 1$; for Ising model $\mathcal{F}(x) = \log \cosh x$. The analytic form of $\mathcal{F}$ will not be used, but the following properties (shared by all models) will be essential:

a) $\mathcal{F}$ is C^∞, even, and has linear increase at infinity;

b) $\mathcal{F}'|_{R_+}$ is strictly increasing, strictly concave and $\mathcal{F}'(\infty) = 1$;

c) $\mathcal{F}''(0) = 1$, $\mathcal{F}^{IV}(0) = -2$ (this number is model-dependent).

Due to a), the minimum is attained in (9) at a stationary point of $\mathcal{G}_n$, i.e., at a solution of the following nonlinear system:

$$m_i = \mathcal{F}'(\beta (J^{(n)} m + H)_i), \quad i = 1 \dots n. \tag{10}$$

If one defines $\Phi_{n,\beta,H} : R^n \to R^n$ by $\Phi_{n,\beta,H}(m)_i = \mathcal{F}'(\beta (J^{(n)} m + H)_i)$, one can view stationary points of $\mathcal{G}_n$ as fixed points of $\Phi_{n,\beta,H}$.

For $x \in R^n$, we agree to write $x \succcurlyeq 0$ instead of $x_i \ge 0$, $\forall i = 1 \dots n$; and $x \succ 0$ for $x_i > 0$, $\forall i$. It is clear from the definition that $\Phi_{n,\beta,H}$ is increasing w.r.t. $\succcurlyeq$ and that $\Phi_{n,\beta,H}|_{R_+^n}$ is (component-wise) strictly concave. This gives the following solution to the minimum problem:

Proposion 2.1.

Let $H \succcurlyeq 0$, $H \neq 0$. Then $\Phi_{n,\beta,H}$ has in R_+^n a unique fixed point, $m^{(n)}(\beta, H)$. This is the unique point at which $\mathcal{G}_n(\beta, H; \cdot)$ attains minimum.

The proof exhibits $m^{(n)}(\beta, H)$ as the limit of the sequence $\{(\Phi_{n,\beta,H})^{\circ k}(0);\ k = 1, 2, \dots\}$, which is increasing, because $\Phi_{n,\beta,H}(0) \succcurlyeq 0$, $\neq$ 0. This allows to derive properties of $m^{(n)}(\beta, H)$:

Proposition 2.2.

(i) $m^{(n)}(\beta,H) \succ 0$ is strictly increasing and concave of H on $\mathbf{R}^n_+ \setminus \{0\}$.

(ii) Let

$$m^{(n)}(\beta, 0) = \lim_{H \to 0 (H \in \mathbf{R}^n_+)} m^{(n)}(\beta, H),$$

then $m^{(n)}(\beta,0)=0$ for $\beta \leq \beta_c^{(n)} \equiv 1/\lambda_{max}(J^{(n)})$ and $m^{(n)}(\beta,0) \succ 0$ for $\beta > \beta_c^{(n)}$.

This latter point identifies $\beta_c^{(n)}$ as the critical inverse temperature of the slab:

$$\beta_c^{(n)} = (4 + 2\cos\frac{\pi}{n+1})^{-1} ;$$

so, we shall sketch the proof. For $\beta \leq \beta_c^{(n)}$, $\Phi_{n,\beta,H}$ is a contraction, so it has one fixed point in all $\mathbf{R}^n$. For $\beta > \beta_c^{(n)}$, one can find $x \succ 0$ such that $(\Phi_{n,\beta,H})^{\circ k}(x)$ be an increasing sequence: take x to be the eigenvector of $J^{(n)}$ corresponding to $\lambda_{max}(J^{(n)})$ normalized so that $x_i < \xi$, $\forall i$, where $\xi > 0$ is the solution of $\mathcal{F}'(\beta\lambda_{max}\xi) = \xi$; ; indeed, then:

$$\Phi_{n,\beta,H}(x)_i = \mathcal{F}'(\beta\lambda_{max}x_i + \beta H_i) \geq \mathcal{F}'(\beta\lambda_{max}x_i) \geq x_i .$$

Let us remark finally that $m^{(n)}(\beta,H)$ can be identified with the vector of row magnetizations (7), and $\|\partial m_i^{(n)}(\beta,H)/\partial H_j\|_{i,j=1\ldots n}$ with the susceptibility matrix (8).

This is the consequence of a general theorem asserting that, for a convergent sequence of convex functions, the derivatives converge to the derivative of the limit wher ever the latter is differentiable. (in fact, for identifying $\chi_{ij}^{(n)}$, one needs GHS inequalities, which are known only for the Ising /18/ and spherical /12/ models).

3. THE SEMI-INFINITE SYSTEM

We fix henceforth the magnetic field distribution to that already used in the description of the scaling theory: a homogeneous magnetic field $H \geq 0$ on all rows, but on the boundary rows where the field is $H+h$. The semi-infinite system is obtained as an $n \to \infty$ limit. Simple monotonicity arguments show that for every i, $\lim_{n\to\infty} m_i^{(n)}(\beta,H,h) = \mathbf{m}_i(\beta,H,h)$ exists. Moreover, taking advantage that every equation in (10) involves only a finite number of rows, these limits satisfy the infinite system. It is convenient to invert $\mathcal{F}'$ in (10) and write this system as a recurrence relation:

$$m_0 = h\ ;\ m_{i+1} = F_{\beta,H}(m_i) - m_{i-1},\quad i = 1,2,\dots, \tag{11}$$

where

$$F_{\beta,H}(x) = \beta^{-1}\mathcal{F}'^{-1}(x) - 4x - H\ . \tag{11'}$$

On this form, it is plain that, where m_1 be known, all magnetizations could be obtained from (11). So, the next task is to derive an equation for $m_1(\beta,H,h)$. It turns out that this is a functional equation for $f_{\beta,H}(\cdot) = m_1(\beta,H,\cdot)$:

$$x + f \circ f(x) = F_{\beta,H} \circ f(x). \tag{12}$$

This follows by remarking that $m_1^{(n-2)}(\beta,H,m_1^{(n)}(\beta,H,h)) = m_2^{(n)}(\beta,H,h) = F_{\beta,H}(m_1^{(n)}(\beta,H,h)) - h$, and exploiting continuity when letting $n \to \infty$. So, there exists at least a positive, increasing, concave solution of eq. (12), $f : [-H, \infty) \to [0, \infty)$. Under these conditions it can be shown that the solution is unique and C^∞ (see Sec.V), and thus we can freely use eq. (11) to derive properties of $f_{\beta,H}$ and of the m_i's, which can be determined as:

$$m_i(\beta,H,h) = (f_{\beta,H})^{\circ i}(h). \tag{13}$$

<u>Proposition 3.1.</u>

$f_{\beta,H}$ is a strict contraction of $[0,\infty)$, whenever $(\beta,H) \neq (\beta_c, 0)$ (in the latter case $f'(0) = 1$).

Indeed, suppose $H > 0$ or $\beta > \beta_c$; then we know that $f_{\beta,H}(0) > 0$; as $F_{\beta,H} - f_{\beta,H}$ is strictly convex, we have

$$F'_{\beta,H} \circ f_{\beta,H}(0) - f'_{\beta,H} \circ f_{\beta,H}(0) >$$

$$> \frac{1}{f_{\beta,H}(0)}[F_{\beta,H} \circ f_{\beta,H}(0) - f_{\beta,H} \circ f_{\beta,H}(0) - F_{\beta,H}(0) + f_{\beta,H}(0)] =$$

$$= \frac{1}{f_{\beta,H}(0)}[H + f_{\beta,H}(0)] \geq 1\ .$$

So, $\forall x \geq 0,\ f'_{\beta,H}(x) \leq f'_{\beta,H}(0) = \dfrac{1}{F'_{\beta,H} \circ f_{\beta,H}(0) - f'_{\beta,H} \circ f_{\beta,H}(0)} < 1.$

If $f_{\beta,0}(0) = 0$, $f'_{\beta,0}(0)[F'_{\beta,0}(0) - f'_{\beta,0}(0)] = 1$, so $f'_{\beta,0}(0) < 1$ whenever $F'_{\beta,0}(0) > 2$, i.e., for $\beta < 1/6 = \beta_c$ (by calculation).

As a consequence, $f_{\beta,H}$ has a unique fixed point, $m_B(\beta,H)$, on $[0,\infty)$, which is seen to be the (largest) root of: $F_{\beta,H}(x)-2x=0$, and the iterations of $f_{\beta,H}$, $m_i(\beta,H,\cdot)$, approach exponentially fast this fixed point when $i\to\infty$. Thus $m_B(\beta,H)$ is the bulk magnetization. The "inverse correlation length" characterizing this approach to m_B is:

$$1-f'_{\beta,H}(m_B(\beta,H))=\sqrt{F'_{\beta,H}(m_B(\beta,H))-2}\cdot\frac{\sqrt{F'_{\beta,H}(m_B)+2}-\sqrt{F'_{\beta,H}(m_B)-2}}{2} \tag{14}$$

We can now easily identify ψ^B and ψ^S in the expansion (1). Let us first remark that ψ_n can be written as

$$\frac{1}{n}\sum_{1\le i\le n}\left[\frac{1}{\beta}G(m_i^{(n)})-\frac{1}{2}Hm_i^{(n)}\right]-\frac{1}{2n}h(m_1^{(n)}+m_n^{(n)}),$$

where

$$G(x)=\frac{x}{2}\mathcal{F}'^{-1}(x)-\mathcal{F}\circ\mathcal{F}'^{-1}(x)$$

appears when solving in Eq. (10) $(J^{(n)}m)_i$ in terms of m_i. We have the necessary bounds to prove:

Proposition 3.2.

The asymptotic expansion (1) holds with:

$$\psi^B(\beta,H)=\frac{1}{\beta}G(m_B(\beta,H))-\frac{1}{2}Hm_B(\beta,H), \tag{15}$$

$$\psi^S(\beta,H,h)=\frac{1}{\beta}\sum_{i=1}^{\infty}\Big\{[G(m_i(\beta,H,h))-G(m_B(\beta,H))]-\frac{\beta H}{2}[m_i(\beta,H,h)-m_B(\beta,H)]\Big\}-\frac{1}{2}hm_1(\beta,H,h). \tag{16}$$

4. THE CRITICAL BEHAVIOUR

We start by rescaling $F_{\beta,H}$ and $f_{\beta,H}$ so that the fixed point $m_B(\beta,H)$ remains at finite distance from other fixed points at the critical point $t=H=0$. Namely, let us define:

$$\tilde{F}_{t,H}(x)=|t|^{-1/2}F_{\beta,|t|^{3/2}H}(|t|^{1/2}x);\ \tilde{f}_{t,H}(x)=|t|^{-1/2}f_{\beta,|t|^{3/2}H}(|t|^{1/2}x). \tag{17}$$

These functions have the same convexity properties as the original ones and satisfy:

$$x + \tilde{f}\circ\tilde{f}(x) = \tilde{F}_{t,H}\circ\tilde{f}(x). \tag{18}$$

The fixed point of $\tilde{f}_{t,H}$ is:

$$\mathfrak{M}_B(t,H) = |t|^{-1/2} m_B(\beta, |t|^{3/2}H). \tag{19}$$

L e m m a 1. The functions $|t|^{-1}[\tilde{F}_{t,H}(x) - 2x]$ converge uniformly on compacts (together with all derivatives) for $t \to \pm 0$ to $\mathfrak{F}^{\pm}(x) - H \equiv 2x^3 \mp 6x - H$ (and its corresponding derivatives).

This follows easily by inspection. It implies:

Proposition 4.1.

(i) $\lim_{t\to\pm 0} \mathfrak{M}_B(t,H) = \mathfrak{M}^{\pm}_B(H)$; ; here $\mathfrak{M}^{\pm}_B(H)$ is the positive solution of $\mathfrak{F}^{\pm}(x) = H$ for $H > 0$, and $\lim_{H\to+0} \mathfrak{M}^{\pm}_B(H)$ for $H = 0$;

(ii) $\lim_{t\to\pm 0} |t|^{-2}\psi^B(\beta, |t|^{3/2}H) = -\frac{1}{2}\mathfrak{M}^{\pm}_B(H)^4 - \frac{H}{2}\mathfrak{M}^{\pm}_B(H) \equiv \phi^{\pm}_B(H)$.

These are the scaling properties of the bulk magnetizations and free energy, respectively. Comparing to (2), we conclude:

$$\alpha = 0, \quad \Delta = 3/2, \quad \beta = 1/2 = 2 - \alpha - \Delta. \tag{20}$$

We now turn to the behaviour of the row magnetizations. The essential point is the following:

L e m m a 2. The functions $\tilde{g}_{t,H}(x) \equiv |t|^{-1/2}[\tilde{f}_{t,H}(x) - x]$ converge uniformly on compacts for $t \to \pm 0$ to:

$$\mathfrak{M}^{\pm}_1(H,x) \equiv [2\int_{\mathfrak{M}^{\pm}_B(H)}^{x} (\mathfrak{F}^{\pm}(x') - H)\,dx']^{1/2}. \tag{21}$$

Instead of a formal proof we give a plausibility argument. Subtracting 2 $\tilde{f}(x)$ from both sides of (18), we obtain:

$$|t|^{-1/2}[\tilde{g}_{t,H}\circ\tilde{f}_{t,H}(x) - \tilde{g}_{t,H}(x)] = |t|^{-1}[\tilde{F}_{t,H}\circ\tilde{f}_{t,H}(x) - 2\tilde{f}_{t,H}(x)].$$

Approximating the l.h.s. with the 1st Taylor term and using Lemma 1:

$$\tilde{g}'_{t,H}(x)\tilde{g}_{t,H}(x) + \text{remainder} \to \mathfrak{L}^{\pm}(x) - H \quad (t \to \pm 0)$$

wherefrom the result follows by integration (if one manages to control the remainder).

Otherwise stated, Lemma 2 tells us that the continuous families of contractions $|t|^{1/2} \to f_{t,H}$ $(t \gtrless 0)$ have derivatives at 0. If so, the nonlinear analog of the Trotter-Chernoff formula/14/ ensures that (uniformly on compacts of $[0, \infty)$) the sequences of nonlinear contractions $k \to (\tilde{f}_{\pm y^2/k^2, H})^{\circ k}(\cdot)$ converge for $k \to \infty$ to the (nonlinear) semigroups generated by $\mathfrak{M}_1^{\pm}(H, \cdot)$ respectively, i.e., to the solutions $q^{\pm}(\cdot, y)$ of the following differential equations:

$$\frac{dq}{dy} = \mathfrak{M}_1^{\pm}(H, q), \tag{22}$$

where the first argument of q stands for the initial datum ($q(h, 0) = h$). These statements have immediate translation into "physical" terms and represent our main results:

<u>Proposion 4.2.</u>

(i) (Scaling property of the row magnetizations).

$$\forall H, h \geq 0, \quad \lim_{t \to \pm 0} |t|^{-1}[m_k(\beta, |t|^{3/2}H, |t|^{1/2}h) - |t|^{1/2}h] = k\,\mathfrak{M}_1^{\pm}(H, h).$$

(ii) (Scaling property of the magnetization profile). Let us define the scaled magnetization profile to be the step functions:

$$q_{t,H}(h, y) = |t|^{-1/2} m_k(\beta, |t|^{3/2}H, |t|^{1/2}h) \quad \text{for} \quad y \in E_{t,k}, \; k = 1, 2, \ldots \tag{23}$$

where $E_{t,k} = \{y \in \mathbb{R} \mid (k-1)|t|^{1/2} \leq y < k|t|^{1/2}\}$. Then, uniformly on compacts:

$$\lim_{t \to \pm 0} q_{t,H}(h, y) = q_H^{\pm}(h, y) \qquad H, h \geq 0. \tag{24}$$

(iii) (Scaling property of the surface free-energy):

$$\begin{aligned} \lim_{t \to \pm 0} |t|^{-3/2}[\psi^S(\beta, |t|^{3/2}H, |t|^{1/2}h) + \frac{1}{2}|t|h^2] &= \\ = -\frac{1}{2}\int_0^{\infty} [q_H^{\pm}(h, y)^4 - \mathfrak{M}_B^{\pm}(H)^4]\,dy &- \\ -\frac{H}{2}\int_0^{\infty} [q_H^{\pm}(h, y) - \mathfrak{M}_B^{\pm}(H)]\,dy + \frac{1}{2}h\,\mathfrak{M}_1^{\pm}(H, h). \end{aligned} \tag{25}$$

The last assertion can be obtained by writting (16) as the integral of an appropriate step function, and by exploiting (ii).

Thus, we succeeded in calculating the critical exponents:

$$\alpha_S = 1/2, \quad \Delta_1 = 1/2, \quad \beta_1 = 1 = 2 - \alpha_S - \Delta_1. \tag{26}$$

We consider now the "two point correlations", Eq. (8). For the finite slab we can calculate them as derivatives of $m_i^{(n)}$ using the minimum condition. The result is that the matrix $\chi^{(n)}$ is the inverse of $X^{(n)} = D^{(n)} - A^{(n)}$, where $D^{(n)}$ is diagonal and $D_{ii}^{(n)} = 1/\mathcal{F}''\circ\mathcal{F}'^{-1}(m_i^{(n)})$, while $A_{ij}^{(n)} = \delta_{|i-j|,1}$. Let us again suppose a homogeneous magnetic field $H \geq 0$ plus a field $h \geq 0$ on the boundary row. Then the results of sec. III ensure the existence of $\lim_{n\to\infty} X_{ij}^{(n)}$, i,j. If we look at $X^{(n)}$ as an operator in the ℓ^2-space of square-summable sequences $\xi = (\xi_i)_{i=1,2,\ldots}$ (acting as the unit operator on components $i > n$), then it converges strongly to the bounded operator $X = D - A$, $D_{ij} = \delta_{ij} / \mathcal{F}''\circ\mathcal{F}'^{-1}(m_i)$. We can show that for $(\beta, H) \neq (\beta_c, 0)$, $X^{(n)}$ are strictly positive uniformly in n. Thus $\chi^{(n)}$ itself converges strongly to X^{-1}. In particular, we have defined the "local susceptibilities" $\chi_{ij}(\beta, H, h)$ of the semi-infinite system. Besides, we consider the response of a row to a uniform field: $\sum_{j=1}^{\infty} \chi_{ij}$, and its bulk $(i \to \infty)$ limit.

<u>Proposition 4.3.</u>

(Scaling properties of the susceptibilities).

(i) $\lim_{t\to\pm 0} \chi_{ij}(\beta, |t|^{3/2} H, |t|^{1/2} h) = \min(i,j)$, $i,j = 1,2,\ldots$

(ii) $\lim_{t\to\pm 0} |t|^{1/2} \sum_{j=1}^{\infty} \chi_{ij}(\beta, |t|^{3/2} H, |t|^{1/2} h) = i\,k_1^{\pm}(\beta, h)$, $i = 1,2,\ldots$

(the functions $k_1^{\pm}$ will be described below).

(iii) $\lim_{t\to\pm 0} |t| \lim_{i\to\infty} \sum_{j=1}^{\infty} \chi_{ij}(\beta, |t|^{3/2} H, |t|^{1/2} h) = k^{\pm}(H) \equiv 1/\mathfrak{D}^{\pm\prime}(\mathfrak{M}_B^{\pm}(H))$.

Outline of proof. Define X°, χ° by replacing D through $D^\circ = 1/\mathcal{F}''\circ\mathcal{F}'^{-1}(m_B)$. For χ°, statements (i) and (ii) are true as such, while (ii) holds with $k^{\pm}(H)$ instead of $k_1^{\pm}(H,h)$. This is checked by calculation using the explicit form:

$$\chi_{ij}^\circ = (z^{|i-j|} - z^{i+j})/(z^{-1} - z) \text{ with } z = f'_{\beta,H}(m_B(\beta, H)) \tag{27}$$

and Eq. (14) combined with Lemma 1. For χ, use is made of the perturbation formula $(V = D^\circ - D)$:

$$\chi = \chi^\circ + \chi^\circ V^{1/2}[1 - V^{1/2}\chi^\circ V^{1/2}]^{-1} V^{1/2}\chi^\circ \tag{28}$$

showing that $\chi_{ij} - \chi^\circ_{ij} = (\xi^{(i)}, [1 - V^{1/2}\chi^\circ V^{1/2}]^{-1}\ \xi^{(j)})_{\ell^2}$, where $\xi^{(i)}_k = \chi^\circ_{ik} V^{1/2}_{kk}$, $k = 1,2,\ldots$. To control this difference, we imbed (inspired from the scaling of the magnetization profile) ℓ^2 into $L_2(\boldsymbol{R}_+)$ through the isometry: $(U_t\xi)(y) = |t|^{-1/4}\xi_i$ for $y \in E_{t,i}$ $(i = 1,2,\ldots)$. Then, again by calculation and using Prop. 4.2, one can convince itself that:

a) $|t|U_t\chi^\circ U_t^*$ converges to the resolvent of $-\frac{d^2}{dy^2}$ (with zero boundary condition at $y = 0$) at the point $\kappa_\pm = -\mathfrak{D}^{\pm\prime}(\mathfrak{M}^\pm_B(H))$.

b) $|t|^{-1/2}U_tVU_t^*$ converges to the multiplication with $W^\pm_{H,h}(y) = \mathfrak{D}^{\pm\prime}(\mathfrak{M}^\pm_B(H)) - \mathfrak{D}^{\pm\prime}(q^\pm_H(h,y))$.

c) $|t|^{-1/4}U_t\,\xi^{(i)}$ converges (in norm) to $iW^\pm_{H,h}(y)^{1/2}\,e^{\kappa_\pm y}$.

d) $|t|^{3/4}U_t\sum_{i=1}^{\infty}\xi^{(i)}$ conveges (in norm) to $(1-e^{\kappa_\pm y})W_{H,h}(y)^{1/2}\,k^\pm(H)^2$.

There is one more delicate point: in proving the convergence of $U_t[1 - V^{1/2}\chi^\circ V^{1/2}]^{-1}U_t^*$ we have to convince ourselves that $U_t[1 - V^{1/2}\chi^\circ V^{1/2}]U_t^*$ is invertible. We are lucky that in the worst case $(H = h = 0)$ this is related to estimating the lowest eigenvalue of a Schrödinger operator, which is one of the few solvable ones (potential $-1/\cosh^2 y$; see /15/ , Problem 1.14).

5. CONCLUDING REMARK

I shall outline here a geometric interpretation of the functional equation (12). Remark that the infinite system (11) allows the translation of one step away the boundary: it gives (m_i, m_{i+1}) in terms of (m_{i-1}, m_i). It is natural to define:

$$T_{\beta,H}(x,y) = (y, F_{\beta,H}(y) - x) \tag{29}$$

which is a diffeomorphism of the strip $\{-1 < y < 1\}$ onto the strip $\{-1 < x < 1\}$ of the (x,y)-plane. The fixed points of $T_{\beta,H}$ satisfy $x = y$, $F_{\beta,H}(x) = 2x$. For $\beta \le \beta_c$, it has one fixed point (in $\boldsymbol{R}^2_+$), while for $\beta > \beta_c$, there are three fixed points, out of which one is in $\boldsymbol{R}^2_+$. Thus, always, the fixed point in $\boldsymbol{R}^2_+$ is (m_B, m_B). The tangent map of $T_{\beta,H}$ is $\begin{pmatrix} 0 & 1 \\ -1 & F'_{\beta,H}(y) \end{pmatrix}$. It is easily seen that, at (m_B, m_B)

its eigenvalues are z and z^{-1} (where $z = f'_{\beta,H}(m_B)$).). Thus (m_B, m_B) is hyperbolic. Eq. (12) is nothing but the equation for the stable manifold of (m_B, m_B), which appears to be the graph of $f_{\beta,H}$. We enter thus a general scheme (see, e.g., ref. [16]). The information we get out of it is the uniqueness and differentiability of $f_{\beta,H}$ near m_B. In our case, the existence of a solution of Eq. (12) defined on $[0,\infty)$ allows one to derive more than local properties of the stable manifold; Lemma 2 is one such example.

REFERENCES

1. Angelescu N., Costache G., Nenciu G. phys. stat.sol.(b), 1972, 51, p.205.
2. Angelescu N., Costache G., Nenciu G. Rev.Roum.Phys., 1972, 17, p.811.
3. Costache G. Phys. Lett. 1975, 54A, p.128.
4. Angelescu N. et al. In: "Topics in Theoretical Physics 2", a volume dedicated to Prof. V.Novacu. ICEFIZ, Bucharest, 1979, p.69.
5. Angelescu N. et al."The Critical Behaviour of Kac-Type Models for Semi.Infinite System" (to be published).
6. Pearce P.A. J.Phys. A. Math.Gen., 1977, 10, p.1009.
7. Bray A., Moore M. J.Phys.A: Math.Gen., 1977, 10, p.1927.
8. Binder K., Hohenberg P. Phys.Rev., 1974, B9, p.2194.
9. Burkhardt T., Eisenriegler E. Phys.Rev., 1977, B16, p.3213.
10. McCoy B.M., Wu T.T.Phys.Rev., 1967, 162, p.436.
11. Bariev R.Z. Theor. Mat.Fizika, 1979, 40, p.95 (in Russian).
12. Angelescu N., Bundaru M., Costache G. Preprint ICEFIZ, FT-160-1978, Bucharest (to appear in J.Phys.A:Math.Gen).
13. Griffiths R.B., Hurst C.A., Sherman C.A. J.Math.Phys., 1970, 11, p.790.
14. Brezis H., Pazy A. J.Funct.Anal., 1970, 6, p.237.
15. Ter Haar D. In:"Problems in Quantum Mechanics", London, Pion Ltd. 1975.
16. Hirsch M., Pugh C. In: "Global Analysis", Providence. Amer.Math.Soc., 1970, p.133.

Received by Publishing Department on November 16 1979.