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# CONTINUITY OF STATE FUNCTIONS OF THE BOSE GAS

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Непрерывность функции состояния бозе-газа

В работе показано, что некоторые функции состояния бозе-газа, в особенности энтропия, зависят непрерывно от уровней энергии свободного гамильтониана и от возмущений свободного гамильтониана операторами порядка 0. Использованный здесь метод состоит в том, чтобы ввести подходящую топологию на матрицах.

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Continuity of State Functions of the Bose Gas

In this paper we show that some state functions of the Bose gas, especially the entropy, depend continuously on the energy levels for the free Hamiltonian and on perturbations of the free Hamiltonian by operators of degree 0. The method used here is to introduce an appropriate topology on the density matrices and on the perturbations of the free Hamiltonian.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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## 1. Introduction

The aim of this paper is to show that the entropy and the energy of a Bose gas in equilibrium depend continuously on the Hamiltonian, respectively on perturbations of the Hamiltonian under appropriate assumptions. Especially for the entropy the problem of continuity is very nontrivial /2/. We restrict ourselves to the case where the states can be described by density operators (finite box), but topological methods used for our considerations can be applied also to more general physical systems. In this paper we show, that in the case under consideration the density operators depend continuously on the Hamiltonian with respect to an appropriate topology. The continuity of special thermodynamical state functions is then a simple consequence of this fact.

The algebra of observables $\mathfrak{A}$ of the Bose gas is generated by the creation and annihilation operators $a_i^+, a_i$ satisfying the canonical commutation relations

(1.1) $$[a_i, a_j] = [a_i^+, a_j^+] = 0 \ , \ [a_i, a_j^+] = \delta_{ij} \ ,$$

where i,j=1,2,...,f.For a real gas it is f infinite,but in this paper we shall first regard the case f finite. We suppose $\mathfrak{A}$ to be generated by $a_i^+$, $a_i$ only algebraically,i.e.,any observable $A \in \mathfrak{A}$ is a finite algebraic expression in the creation and annihilation operators.

The operators are in a natural way defined on the Fock space $\mathcal{F}$ with the complete orthonormal base of states $\Phi_{(n)}=n_1,n_2,\ldots$ characterized by the occupation numbers $n_i$ , which for a Bose gas run over $n_i= 0,1,2,\ldots$ .

The annihilation and creation operators act on the $\Phi_{(n)}$ in the following way

$$\begin{aligned} a_i^+ \; |\ldots,n_i,\ldots\rangle &= \sqrt{n_i+1} \; |\ldots,n_i+1,\ldots\rangle \\ a_i \; |\ldots,n_i,\ldots\rangle &= \sqrt{n_i} \; |\ldots,n_i-1,\ldots\rangle \; . \end{aligned} \tag{1.2}$$

The vacuum state is $\Phi_0 = |0,0,\ldots\rangle$ .

The number operator of the $i^{th}$ particle is $N_i = a_i^+a_i$ and the total number operator is

$$N = \sum a_i^+a_i \quad . \tag{1.3}$$

Let us remark that the number operator and also Hamiltonians of the form

$$H = \sum \varepsilon_i a_i^+ a_i \tag{1.4}$$

which we regard below are not elements of the observable algebra $\mathfrak{A}$ .Further, the operators of $\mathfrak{A}$ are first only defined on the algebraicly linear hull $\mathfrak{D}_0$ of the basic vectors $\Phi_{(n)}$. The choice of larger domain $\mathfrak{D}$ of definition for the unbounded operators of $\mathfrak{A}$ is closely related to the physical system under consideration. For our aim of topological considerations we shall choose domains $\mathfrak{D} = \bigcap_{k=0}^{\infty} \mathfrak{D}(T^k)$ , where T is an appropriate self-adjoint operator. In section 3, where we consider the Bose gas of finite degree of freedom, T is simply a polynom of the number operator N. For infinite degree of freedom we must choose T already in a more complicated way, related to the physical problem we have in mind. Further, also in dependence of the closed operator T, we get appropriate 'physical' topologies on the density operators for a Bose system. This problem we regard in the next section.

## 2. Physical topology on density operators

Let $T \geq I$ be a self-adjoint operator, which allows the following estimation of the number operator, $k \geq 0$

$$(2.1) \qquad \| N^k \Phi \| \leq c \; \| T^n \Phi \| \qquad \text{with a certain } n_k$$

on $\mathfrak{D}_0$, where c is a certain constant. Then the domain $\mathfrak{D}(\bar{A})$ of the closure $\bar{A}$ of any operator $A \in \mathfrak{A}$ contains the domain $\mathfrak{D} = \bigcap_{k=0}^{\infty} \mathfrak{D}(T^k)$. Therefore all operators $A \in \mathfrak{A}$ have a natural extension to $\mathfrak{D}$. Further, $A, A^* \mathfrak{D} \subset \mathfrak{D}$. This means that $\mathfrak{A}$ becomes an $Op^*$-algebra on $\mathfrak{D}$ in the sense of /1/, i.e., $\mathfrak{A}$ is an $*$-subalgebra of $\mathcal{L}^+(\mathfrak{D})$, which is the algebra of all (unbounded) operators A defined on $\mathfrak{D}$ so that $A\mathfrak{D} \subset \mathfrak{D}$ and $A^* \mathfrak{D} \subset \mathfrak{D}$. $A^+ = A^* |_{\mathfrak{D}}$ is the restriction of $A^*$ to $\mathfrak{D}$.

By $\| \Phi \|_s = \| T^s \Phi \|$, $\Phi \in \mathfrak{D}$, there is defined a norm on $\mathfrak{D}$ for every real s. We denote by $\mathcal{H}_s$ the completion of $\mathfrak{D}$ with respect to the Hilbert norm $\| \cdot \|_s$. Then $\{\mathcal{H}_s\}$ is a scale of Hilbert spaces with

$$(2.2) \qquad \mathfrak{D} = \mathcal{H}_\infty = \bigcap_{s > -\infty} \mathcal{H}_s \qquad \text{and} \qquad \mathcal{H}_0 = \mathcal{H} .$$

Now let us introduce /3/

(2.3) $\quad \mathcal{S}_1(\mathcal{D}) = \{\rho \in \mathcal{L}^+(\mathcal{D});\ A\,\rho\,B \text{ nuclear for all } A,B \in \mathcal{L}^+(\mathcal{D})\}$.

Every positive and normed operator $\rho \geq 0$, $\operatorname{tr} \rho = 1$ is called density operator. If the density operator $\rho$ is contained in $\mathcal{S}_1(\mathcal{D})$, then $\rho(A) = \operatorname{tr} \rho A$ is a state on $\mathcal{L}^+(\mathcal{D})$ and therefore also on $\mathfrak{A}$.

Now we introduce the strong topology $\beta^*$ on $\mathcal{S}_1(\mathcal{D})$, which we call the physical topology /2/.

The physical topology $\beta^*$ is given by the following system of seminorms

$$\beta^*: \qquad \|\rho\|_{\mathfrak{M}} = \sup_{A \in \mathfrak{M}} |\rho(A)| < \infty \quad ,$$

where $\mathfrak{M}$ runs over all weakly bounded sets in $\mathcal{L}^+(\mathcal{D})$ with respect to the dual pair $(\mathcal{L}^+(\mathcal{D}), \mathcal{S}_1(\mathcal{D}))$.

For the applications we have in mind, the case, where $T^{-1}$ is a nuclear operator, is of importance. Then we have /2/

Theorem 2.1 The physical topology $\beta^*$ is given by the system of seminorms

$$\|\rho\|_k = \|T^k \rho\, T^k\| \quad , \qquad k = 0,1,2,\ldots \quad ,$$

where $\|\cdot\|$ is the usual operator norm. A bounded operator $\rho$ is in $\mathcal{S}_1(\mathcal{D})$ if and only if all norms $\|\rho\|_k < \infty$.

Theorem 2.2 The entropy $S(\rho) = -\operatorname{tr} \rho \ln \rho$ is finite on the positive part $\mathcal{S}_+ = \{\rho \geq 0,\ \rho \in \mathcal{S}_1(\mathcal{D})\}$ of $\mathcal{S}_1(\mathcal{D})$ and continuous with respect to the topology $\beta^*$.

Further, let us recall the following definition /4/

Definition 2.3 An operator $B \in \mathcal{L}^+(\mathcal{D})$ is said to be of degree $r$, and we write $B \in OP_r(\mathcal{D})$ if

$$\|B\phi\|_s \leq c_s \|\phi\|_{s+r}$$

for all real $s$ , where $c_s$ is a certain constant for every $s$ .

In this paper we need only some properties of operators of degree 0. Since for $r = 0$ (2.5) is equivalent to $\|T^s B T^{-s}\Phi\| \leq c_s \|\Phi\|$ for all $s$ , the operators of order 0 are characterized by the property that

(2.6) $$T^s B T^{-s} \text{ is bounded in } \mathcal{H} \text{ for every real } s .$$

Note that $OP_o$ is an $Op^*$-algebra.

In section 4 we need the following lemma

Lemma 2.4

i) If $B$ is an operator of degree 0 then also $e^B$ is of degree 0 .

ii) $\sup_{p \leq s \leq q} \|T^s B T^{-s}\| = \alpha < \infty$ .

Proof: i) Every operator of degree 0 is bounded. Therefore

$$e^B = \sum \frac{1}{n!} B^n$$

from which we get

$$\| T^s e^B T^{-s} \| \leq \sum \frac{1}{n!} \| T^s B T^{-s}\|^n.$$

ii) is a consequence of the fact that $T^s B T^{-s}$ is a continuous function of $s$ .

Definition 2.5 We define on the $Op^*$-algebra $OP_o$ of all operators of degree 0 the locally convex topology $\eta$ by the following system of seminorms

(2.7) $\eta$ : $p_k(B) = \sup_{-k\leq s\leq k} \| T^s B T^{-s} \|$ , $k=0,1,2,\ldots$ .

In fact, the topology $\eta$ is already defined by the (semi-)norms $\| T^s B T^{-s} \| = q_s(B)$ , $s \in R^1$ .

Now we want to apply these general topological considerations to the Bose gas. If we have finite degree of freedom (f=finite), then the operator $T = (1+N)^{f+1}$ has the desired properties.

It is a straightforward consequence of the structure of the number operator $N$ (1.3) that $T$ satisfies the conditions (2.1). The main property of $T$ which we need in the following is the nuclearity of $T^{-1}$.

<u>Lemma 2.6</u> For finite degree of freedom, f=finite, the operator $T^{-1} = (1+N)^{-f-1}$ is nuclear.

<u>Proof</u>: To prove the nuclearity of $T^{-1}$ we first remark that $T\, |n_1,\ldots,n_f\rangle = (1 + \sum n_i)^{f+1} \, | n_1,\ldots,n_f \rangle$ . Therefore the eigenvalues of $T$ are $t_{(n)} = (1 + \sum n_i)^{f+1}$, where $(n) = (n_1,\ldots,n_f)$ runs over all n-tupels of nonnegative integers. We have to show the convergence of the series

$$\sum_{n_1,\ldots,n_f} \left(\frac{1}{1 + \sum_{i=1}^{f} n_i}\right)^{f+1} < \infty \quad (2.8)$$

To do this we apply the following integral criterion, which we need also in the following section

<u>Criterion 2.7</u> Let $F(x)$ be a function in $f$ variables, which is positive, depending only on $r = |x|$ and monotonically decreasing for $c < r \to \infty$. The series $\sum_{(n)} F(n)$ converges if and only if $\int_{r>0}^{\infty} F(x)\, dx < \infty$ .

The members of the series (2.7) can be estimated in the form $(1 + \sum n_i)^{-f-1} \leq F(n)$ with $F(x) = r^{-f-1}$. Since

$\int_{r>c} F(x)\, dx = \int_{r>c} r^{-2}\, dr\, d\omega = \frac{\Omega_f}{c}$ , where $\Omega_f$ is the area of f-dimensional unit sphere, from the criterion we get the convergence of the series (2.8).

## 3. The Gibbs' states of the Bose gas

We shall consider the Gibbs states of the Bose gas for finite number of energy levels, more precisely for f = finite.

For T we take the operator $(1+N)^{f+1}$ . Than, as outlined in the foregoing section, the observable algebra $\mathfrak{A}$ is defined on $\mathfrak{D} = \bigcap_{n=0}^{\infty} \mathfrak{D}(T^n)$ .

The Hamiltonian

$$H_o = \sum \varepsilon_i a_i^+ a_i \tag{3.1}$$

is an element of $\mathcal{L}^+(\mathfrak{D})$, since the sum on the right-hand side of (3.1) is finite. We suppose $\varepsilon_i \geqslant 0$. Let $\mu < 0$ be the chemical potential, then the operator $e^{-\beta(H_o - \mu N)}$ is nuclear. Its trace is the partition function Z,

$$Z = \mathrm{tr}\, e^{-\beta(H_o-\mu N)} = \prod_{i=1}^{f} [1 - e^{-\beta(\varepsilon_i - \mu)}]^{-1} . \tag{3.2}$$

The Gibbs state on $\mathfrak{A}$ is given by the density operator

$$\varrho_G = \frac{1}{Z} e^{-\beta(H_o-\mu N)} \quad . \tag{3.3}$$

The fact that $\mathrm{tr}\, A\, \varrho_G$ is well-defined and finite for all $A \in \mathfrak{A}$ is a consequence of the following lemma, which describes the structure of $\varrho_G$ more precisely.

<u>Lemma 3.1</u> The Gibbs density operator $\varrho_G = \frac{1}{Z} e^{-\beta(H_o - \mu N)}$ belongs to $\mathfrak{S}_1(\mathfrak{D})$ .

<u>Proof</u>: It is sufficient to show that $T^k \varrho T^k$ is nuclear for every integer k (see also /2/). The eigenvalues of this operator are $e^{-\sum \beta(\varepsilon_i - \mu) n_i} (1 + \sum n_i)^{(2f+2)k}$. Since $(\varepsilon_i - \mu) > 0$

by assumption, the infinite sum of all these eigenvalues is finite. This is already the complete proof. |–|

Now we can prove

<u>Theorem 3.2</u> The Gibbs density operator $\rho_G \in \mathfrak{S}_1(\mathcal{D})[\beta^*]$ depends continuously on $(\varepsilon_1, \dots, \varepsilon_f)$, $\mu$ and $\beta$ with respect to the topology $\beta^*$.

<u>Proof:</u> Since the partition function $Z = Z(\varepsilon_i, \mu, \beta)$ depends continuously on the parameters $\varepsilon_i \geq 0$, $\mu < 0$, $\beta > 0$ we have only to estimate the exponential operator $e^{-A}$, where $A = \beta(H_0 - \mu N)$. Let $A'$ be the corresponding operator for the parameters $\varepsilon_i'$, $\mu'$, $\beta'$. Since $A$ and $A'$ commute, we can apply the operator inequality

$$(3.4) \qquad \| e^{-A} - e^{-A'} \| \leq \| ( A' - A )\| \|( e^{-A} + e^{-A'} )\| .$$

To prove $e^{-A'} \longrightarrow e^{-A}$ with respect to the topology $\beta^*$, we have to estimate $\| e^{-A'} - e^{-A} \|_k$ ( see Theorem 2.1) . Applying (3.4) we get

$$\| e^{-A} - e^{-A'} \|_k \leq \| T^k (e^{-A} - e^{-A'}) T^k \| \leq$$

$$(3.5) \qquad \leq \| T^{-1} (A - A') \| \, \| T^{2k+1} (e^{-A} + e^{-A'}) \| .$$

The second factor in (3.5) is bounded for $A' \longrightarrow A$, and for the first factor we get the estimation

$$(3.6) \qquad \| T^{-1} (A - A') \| \leq \sum_i \| T^{-1} a_i^+ a_i \| \, | (\varepsilon_i - \mu)\beta - \beta'(\varepsilon_i' - \mu') | .$$

This completes the proof. |–|

Let us remark that all estimations in this proof have such a simple form, since all operators in the formulae of the proof commute. In the case, where the perturbation of the Hamiltonian does not commute with it, the estimations are much more complicated. That case will be handled in the next section. Nevertheless, the simple perturbation Theorem 3.2 is of own interest. First, it gives a clear impression on the topological questions which must be solved. Secondly, as a consequence of it we get the continuous dependence of all expectation values in the Gibbs equilibrium in the sense of the following corollary.

<u>Corollary 3.3</u> Every expectation value $\varrho_G = \mathrm{tr}\ \varrho_G A$, $A \in \mathcal{L}^+(\mathcal{D})$, depends continuously on $\varepsilon_i$, $\mu$, and $\beta = \frac{1}{kT}$.

This follows immediately from Theorem 3.2, since the trace of $(\varrho_G \cdot A)$ depends continuously on $\varrho_G$ with respect to the topology $\beta^*$.

Since $H_o \in \mathcal{L}^+(\mathcal{D})$, we have also proved the continuous dependence of the total energy $E = \varrho_G(H_o)$ on the physical parameters $\varepsilon_i$, $\mu$, $\beta$.

Finally, by combining Theorem 3.2 with Theorem 2.2 we get yet the following result

<u>Corollary 3.4</u> The entropy $S = -\mathrm{tr}\ \varrho_G \ln \varrho_G$ is a continuous function of the physical parameters $\varepsilon_i$, $\mu$ and $\beta$.

Especially in the case of entropy the desired result is by no means trivial, since the entropy will be mostly infinity and is only finite on a set of density matrices of category I as investigated by Wehrl /6/.

## 4. Perturbations of the Hamiltonian

In this section we investigate the continuous dependence of the Gibbs states on some perturbations $h$ of the Hamiltonian $H_o$ which in general do not commute with $H_o$.

First we prove the following lemma

<u>Lemma 4.1</u> Let be $A = H_o - \mu N$, $\mu < 0$, then
$\| T^k e^{-tA} \| \leq \left(\frac{k(f+1)}{t}\right)^{k(f+1)} e^{-k(f+1)-t\mu}$ ,for $t > 0$ .

<u>Proof:</u> Since $H_o$ is positive and commutes with $N$ we have

$$\| T^k e^{-tA} \| \leq \| T^k e^{t\mu N} \| = \| (1+N)^{(f+1)} e^{t\mu N} \| .$$

Therefore

$$\| T^k e^{-tA} \| \leq \sup_{x \geq 0} ( 1+x )^{k(f+1)} e^{t\mu x} \leq$$

$$\left(\frac{k(f+1)}{t}\right)^{k(f+1)} e^{-k(f+1) - t\mu} .$$

<u>Lemma 4.2</u> Let $H_o$, $N$ , $\mathcal{E}$ , $\mu$ satisfying the same assumptions as in the foregoing section. If $h$ is a self-adjoint operator of degree zero (see (2.6) ) then $e^{-\beta(H_o+h-\mu N)} \in \in \mathfrak{S}_1(\mathfrak{D})$.

<u>Proof:</u> We put $A = H_o - \mu N$. Then $e^{-\beta A} \in \mathfrak{S}_1(\mathfrak{D})$ (Lemma 3.1). Now we apply the Trotter formulae (/5/, Theorem VIII.31)

$$\text{(4.1)} \qquad \underset{n \to \infty}{\text{s-lim}} ( e^{-\frac{\beta}{n} A} e^{-\frac{\beta}{n} h} )^n = e^{-\beta( A+h)} .$$

Every operator $s_n = ( e^{-\frac{\beta}{n} A} e^{-\frac{\beta}{n} h} )^n$ is contained in $\mathfrak{S}_1(\mathfrak{D})$ since $e^{-\frac{\beta}{n} A} \in \mathfrak{S}_1(\mathfrak{D})$ and $e^{-\frac{\beta}{n} h}$ is of degree zero (Lemma 2.4). Now we estimate the norm $\| s_n \|_{p,q}$ of $s_n$ as an operator of $\mathcal{H}_p$ into $\mathcal{H}_q$. Essential is only the case $p < q$ . We decompose this interval in $n$ parts $p=p_o < p_1 < \dots < p_n=q$ , $p_k = p + \frac{k}{n} (q-p)$. Then

$$\text{(4.2)} \quad \| s_n \|_{p,q} \leq \prod_{i=1}^{n} \| e^{-\frac{\beta}{n} A} e^{-\frac{\beta}{n} h} \|_{p_{i-1}, p_i} .$$

Further we get for every $i$

$$\text{(4.3)} \quad \| e^{-\frac{\beta}{n} A} e^{-\frac{\beta}{n} h} \|_{p_{i-1},p_i} = \| T^{p_i} e^{-\frac{\beta}{n} A} T^{-p_{i-1}} T^{p_{i-1}} e^{-\frac{\beta}{n} h} T^{-p_{i-1}} \|$$

$$\leq \| T^{\frac{q-p}{n}} e^{-\frac{\beta}{n}A} \| \; \| T^{p_{i-1}} e^{-\frac{\beta}{n}h} T^{-p_{i-1}} \| .$$

Let us estimate the norms on the right-hand side. For the first norm we get by Lemma 4.1

(4.4) $$\| T^{\frac{q-p}{n}} e^{-\frac{\beta}{n}A} \| \leq \left( \frac{(q-p)(f+1)}{\beta} \right)^{\frac{q-p}{n}(f+1)} e^{-\frac{1}{n}(q-p)(f+1) - \frac{\beta\mu}{n}} = \gamma^{\frac{1}{n}},$$

where the number $\gamma$ is independent of $n$ and $i$.

For the second norm we get

(4.5) $$\| T^{p_{i-1}} e^{-\frac{\beta}{n}h} T^{-p_{i-1}} \| \leq \sum \frac{\left(\frac{\beta}{n}\right)^k}{k!} \| T^{p_{i-1}} h T^{-p_{i-1}} \|^k \leq$$
$$\leq e^{\frac{\beta}{n}\alpha} ,$$

where $\alpha$ is the constant of Lemma 2.4 ii).

Combining the estimations (4.2) - (4.5) we get

(4.6) $$\| s_n \|_{p,q} \leq \gamma e^{\beta\alpha} = c .$$

The important point is that the constant $c$ on the right-hand side of (4.6) is independent of $n$.

Now we are going to prove

(4.7) $$\| e^{-\beta(A+h)} \|_{p,q} \leq c < \infty .$$

Condition (4.6) is equivalent to $\| T^q s_n T^{-p} \| \leq c$.

Now for $\phi \in \mathcal{D}$ we get

$$T^q s_n T^{-p} \phi = T^q ( e^{-\frac{\beta}{n}A} e^{-\frac{\beta}{n}h} )^n T^{-p} \phi$$
$$= ( e^{-\frac{\beta}{n}A} e^{-\frac{\beta}{n}(T^q h T^{-q})} )^n T^{q-p} \phi \longrightarrow e^{-\beta(A+T^q h T^{-q})} T^{q-p} .$$

Therefore s-lim $( T^q s_n T^{-p} ) = T^q e^{-\beta(A+h)} T^{-p}$ and consequently $\| T^q e^{-\beta(A+h)} T^{-p} \| = \| e^{-\beta(A+h)} \|_{p,q} \leq c$.

Especially we have for the norms $\| \cdot \|_k$, (2.4),

$\| e^{-\beta(A+h)} \|_k = \| e^{-\beta(A+h)} \|_{-k,k} < \infty$. Therefore $e^{-\beta(A+h)} \in \mathcal{S}_1(\mathcal{D})$ and the Lemma is completely proved. $\dashv$

Lemma 4.3 Let $h, h' \in OP_o$ and $p_k(h), p_k(h') \leq \alpha$ (see (2.7) ). For $s_n(h) = ( e^{-\frac{\beta}{n}A} e^{-\frac{\beta}{n}h} )^n$ and $-k \leq p, q \leq k$ we get

(4.8) $\| s_n(h) - s_n(h') \|_{p,q} \leq \gamma e^{2\beta\alpha} p_k( h - h' )$,

where $\gamma$ and $\alpha$ are the constants of (4.6) which are independent of $n$.

Proof: It is

(4.9) $$s(h) - s(h') = \sum_{i=1}^{n} \left(e^{-\frac{\beta}{n}A} e^{-\frac{\beta}{n}h'}\right)^{i-1} \left[e^{-\frac{\beta}{n}A}\left(e^{-\frac{\beta}{n}h} - e^{-\frac{\beta}{n}h'}\right)\right]\left(e^{-\frac{\beta}{n}A} e^{-\frac{\beta}{n}h}\right)^{n-i}.$$

Quite analoguos as in the foregoing proof we get

(4.10) $$\|s(h) - s(h')\| \leq \sum_{i=1}^{n} \gamma^{\frac{i-1}{n}} e^{\frac{\beta}{n}\alpha(i-1)} \left\| e^{-\frac{\beta}{n}A}\left(e^{-\frac{\beta}{n}h} - e^{-\frac{\beta}{n}h'}\right)\right\|_{p_{i-1},p_i} \cdot \gamma^{\frac{n-i}{n}} \cdot e^{\frac{\beta}{n}\alpha(n-i)}$$

Further

(4.11) $$\begin{aligned} \| e^{-\frac{\beta}{n}A} (e^{-\frac{\beta}{n}h} - e^{-\frac{\beta}{n}h'}) \|_{p_{i-1},p_i} &\leq \gamma^{\frac{1}{n}} \| T^{p_{i-1}} (\tfrac{h}{n} - \tfrac{h'}{n}) T^{-p_{i-1}} \| e^{\frac{\beta}{n}\alpha \cdot 2} \\ &\leq \gamma^{\frac{1}{n}} e^{\frac{\beta}{n}\alpha \cdot 2} \frac{1}{n} p_k(h - h'). \end{aligned}$$

Substituting (4.11) into (4.10) yields (4.8) $\dashv$ .

Now we prove the main theorem of this paper.

Theorem 4.4 Let $h = h^* \in OP_o$ be a perturbation of the Hamiltonian $H_o$ and

(4.12) $$\rho(h) = e^{-\beta(H_o+h-\mu N)} / \operatorname{tr} e^{-\beta(H_o+h-\mu N)}$$

the corresponding Gibbs state. Then $h \longrightarrow \rho(h)$ is a continuous mapping of $OP_o[\eta]$ (see (2.7)) into $\mathfrak{S}_1(\mathfrak{D})[\beta^*]$.

Proof: $B(h) = e^{-\beta(H_o+h-\mu N)}$ is in $\mathfrak{S}_1(\mathfrak{D})$ ( Lemma 4.2). It remains to prove that $B(h) \in \mathfrak{S}_1(\mathfrak{D})[\beta^*]$ continuously depends on $h$ with respect to the topology $\eta$ . Let $h, h' \in OP_o$

and $p_k(h)$, $p_k(h') \leq \alpha$. Further, let $\psi$ be an arbitrary element of $\mathfrak{D}$ with $\|\psi\| \leq 1$. Let $s_n(h)$, $s_n(h')$ be the same operators as in the foregoing Lemma, then

$$(4.13) \quad \| T^k ( B(h) - B(h') ) T^k \psi \| \leq \| T^k( B(h) - s_n(h))T^k \psi \| +$$

$$+ \| T^k (B(h') - s_n(h')) T^k \psi \| + \| T^k(s_n(h) - s_n(h'))T^k \psi \| .$$

Now by applying (4.8) and taking $n \longrightarrow \infty$ we get

$$(4.14) \quad \| T^k (B(h) - B(h')) T^k \psi \| \leq \gamma e^{2\beta\alpha} p_k (h - h') .$$

Since $\psi \in \mathfrak{D}, \|\psi\| = 1$, was arbitrary taken, we get

$$(4.15) \quad \| B(h) - B(h') \|_k = \| T^k(B(h) - B(h') T^k \| \leq$$

$$\leq \gamma e^{2\beta\alpha} p_k(h - h') .$$

This completes the proof of the theorem. |.|

With that it was shown if the Gibbs state of the Bose gas is perturbed by an operator of degree 0 than this perturbed Gibbs state belongs yet to the set $\mathfrak{S}_1(\mathfrak{D})$ and is continuous with respect to the perturbation h. By combining of Theorem 4.4 and Theorem 2.2 we yet get

<u>Corollary 4.5</u> The entropy $S = -\mathrm{tr}\, \varrho \ln \varrho$ is a continuous function of the perturbation h with respect to the topology $\eta$.

Note that also the free energy $E = \varrho(H_o + h)$ continuously depends on the perturbation h .

Finally let us remark that for all foregoing considerations it was essentially that $T^{-1}$ is a nuclear operator, what follows from the property of the number operator in the case of finite degree of freedom.

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References

1. G. Lassner. Rep.Math.Phys. 3,279 (1972)
2. G. Lassner, G.A. Lassner. On the Continuity of the Entropy , Preprint, Dubna 1977
3. G. Lassner, W. Timmermann. Rep. Math. Phys. 9, 205 (1976)
4. Ra Palais. Seminar on the Atiyah-Singer Index Theorem, Princeton University Press, Princeton New Jersey 1965 (Russ. Trans. Moscow 1970)
5. M. Reed, B.Simon. Methods of Modern Mathematical Physics, I. Functional Analysis,Academic Press, New York London 1972
6. A. Wehrl. Rep.Math. Phys. 10, 159 (1976)

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