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Динамика модели БКШ-Боголюбова в термодинамическом пределе

Мы покажем существование такой локально-выпуклой топологии ξ_0 на *-алгебре локальных наблюдаемых $\mathfrak{A}_\ell$ модели БКШ-Боголюбова, в которой временное развитие в термодинамическом пределе описывается однопараметрической группой автоморфизмов *-алгебры $\mathfrak{A} = \mathfrak{A}_\ell[\xi_0]$.

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The Dynamics of the BCS-BOGOLUBOV Model in the Thermodynamical Limit

We demonstrate the existence of such a locally convex topology ξ_0 on the *-algebra of local observables $\mathfrak{A}_\ell$ of the BCS-BOGOLJUBOV model in which the time development in the thermodynamical limit can be described by a one-parameter group of automorphisms of the *-algebra $\mathfrak{A} = \mathfrak{A}_\ell[\xi_0]$.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

In last years some progress has been made in understanding the time development of thermodynamical systems in quantum statistical physics. For concrete models the basic object to describe the dynamics is the local Hamiltonian H_Λ, a self-adjoint operator in a Hilbert space $\mathcal{H}_\Lambda$, where Λ is a bounded region of R^r (continuous systems) or Z^r (lattice systems). The local dynamics is then given by

$$(1.1) \qquad \tau_t^\Lambda(A) = e^{iH_\Lambda t} A e^{-iH_\Lambda t}$$

for all A of a certain $*$-algebra $\mathfrak{A}_\Lambda$ of local observables. We suppose $\mathfrak{A}_\Lambda$ to be an Op^*-algebra /12/, i.e.,a $*$-algebra of closable operators A on $\mathcal{H}_\Lambda$ with a common invariant dense domain $\mathcal{D}_\Lambda$, $A, A^* \mathcal{D}_\Lambda \subset \mathcal{D}_\Lambda$ for all $A \in \mathfrak{A}_\Lambda$. By $A^+ = A^*|_{\mathcal{D}_\Lambda}$ we denote the restriction of A^* to $\mathcal{D}_\Lambda$. The $*$-algebra of all such operators on $\mathcal{D}_\Lambda$ is denoted by $\mathcal{L}^+(\mathcal{D}_\Lambda)$.

Further we assume that for $\Lambda \subset \Lambda'$ $\mathfrak{A}_\Lambda$ is (isomorphic) imbedded into $\mathfrak{A}_{\Lambda'}$, $\mathfrak{A}_\Lambda \subset \mathfrak{A}_{\Lambda'}$ and $\mathfrak{A}_\Lambda$ contains the identity operator. The union $\mathfrak{A}_\ell = \bigcup_\Lambda \mathfrak{A}_\Lambda$ is the $*$-algebra of all local observables.

The fundamental question is in what sense the "thermodynamical limit"

(1.2) $$\tau_t(A) = \lim_{\Lambda\to\infty} e^{iH_\Lambda t} A\, e^{-iH_\Lambda t}$$

of the local dynamics exists for every $A \in \mathfrak{A}_\ell$. There are two extreme cases:

<u>Best case:</u> $\mathfrak{A}_\ell$ is a dense $*$-subalgebra of a C^*-algebra $\mathfrak{A}$, and the limit (1.2) exists in $\mathfrak{A}$, and τ_t is a one-parameter group of $*$-automorphism of $\mathfrak{A}$.

<u>Weakest case:</u> The limit dynamics exists for Gibbs states in the sense of Greens functions

(1.3) $$G(A,B;t) = \lim_{\Lambda\to\infty} \omega_\Lambda(A\, \tau_t^\Lambda(B)) ,$$

where $\omega_\Lambda(A) = \mathrm{Tr}\, e^{-H_\Lambda\beta}A/\mathrm{Tr}\, e^{-H_\Lambda\beta}$ and $\omega(A) = G(A,I;t) = \lim_{\Lambda\to\infty}\omega_\Lambda(A)$ is the Gibbs state.

In this case, under some special assumptions /5,6,16,17/ one can describe the dynamics by a one-parameter group τ_t of $*$-automorphisms on the W^*-algebra $\mathfrak{A} = \pi_\omega(\mathfrak{A}_\ell)''$, where π_ω is the GNS representation of the state ω. The essential point is the following one. On the basis of the dynamical system $(\mathfrak{A}, \tau_t)$ one can apply the "KMS-conception" (see e.g./10/), which makes it possible to handle in the algebraic approach on a rigorous level such physical concepts as the entropy /2/ or the stability and the variational principle for equilibrium states.

All these considerations are mainly based on the properties of W^*-algebras. Last years first steps were taken to generalize these considerations to $*$-algebras containing "unbounded" elements, which arise in a very natural way from the physical situations /1,9,11/. One needs an appropriate $*$-algebra $\mathfrak{A}$, the structure of which has to be closely related to the physical model. For the "weakest case" mentioned above the observable algebra $\mathfrak{A} = \pi_\omega(\mathfrak{A}_\ell)''$ is constructed already in dependence on the equilibrium state ω. It is desirable to get the observable algebra $\mathfrak{A}$ independently of a fixed equilibrium state.

In the following section we outline the construction of an appropriate observable algebra $\mathfrak{A}$ for the BCS-Bogolubov model of superconductivity. This algebra is the completion $\mathfrak{A} = \widetilde{\mathfrak{A}_\ell[\xi_0]}$ of the algebra $\mathfrak{A}_\ell$ of local observables with respect to an appropriate chosen locally convex topology ξ_0. The dynamics of the model can be described by a one-parameter group of automorphisms on $\mathfrak{A}$. Let us call the method to construct this topology ξ_0 the "method of N-operators". The sense of this notion will be clear in what follows.

2. Dynamics of the BCS-BOGOLUBOV Model

The treatment of the BCS-Bogolubov model /3,4/ in the algebraic approach had been done for the first time in /8/. Our considerations and notions are founded on the papers /18,19/.

The BCS Hamiltonian in the quasi-spin formulation looks

$$H_\Lambda = \varepsilon \sum_{p\in\Lambda} (1 - \sigma_p^z) - 2gV(\Lambda)^{-1} \sum_{p,p'\in\Lambda} \sigma_p^- \sigma_{p'}^+ \quad . \tag{2.1}$$

The approximating Bogolubov Hamiltonians are

$$H_{B,\Lambda} = -\alpha \sum (\sigma_p \cdot n) \quad , \tag{2.2}$$

where $\alpha^2 = \varepsilon^2 + g^2 \eta^2 (n_1^2 + n_2^2)$. The direction $n = (n_1, n_2, n_3)$ and the number $-1 \leq \eta \leq 1$ depend on the temperature and the phase of the superconductor.

The algebra $\mathfrak{A}_\Lambda$ is generated by all Pauli operators σ_p^i with $p \in \Lambda$. $\sigma_p^\pm = 1/2(\sigma_p^x \pm \sigma_p^y)$.

Let $\{\varepsilon\} = \{\varepsilon_p\}$ be a sequence of signs ± 1 and n an arbitrary direction. Following /18/ we define

$$(2.3) \qquad M^{\Lambda}_{\{\varepsilon\},n} = 1 + 1/2 \sum_{p\in\Lambda} (1 - \varepsilon_p(\sigma_p \cdot n)) \quad .$$

Now we regard only such sequences $\{\varepsilon\}$ of signs for which $\eta = \lim_{\Lambda\to\infty} V(\Lambda)^{-1} \sum_{p\in\Lambda} \varepsilon_p$ exists for a fixed increasing sequence $\Lambda_\nu \to Z^r$ of boundeddomains. Further, Γ_o denotes the set of multi-indices $\gamma = (\{\varepsilon\}, n)$, n a direction, for which $n_3 = \frac{\varepsilon}{g\eta}$. $g > \varepsilon$ is assumed.

Lemma 2.1

Let F be the set of all continuous functions $f(x)$ on R^1_+, which decrease faster than any inverse powers, i.e., $\sup (1+x)^k |f| < \infty$ for all positive integers. Then

$$(2.4) \qquad \| A \|_\alpha = \lim_{\Lambda\to\infty} \| (M^\Lambda_\gamma)^k A \, f(M^\Lambda_\gamma) \|$$

exists for every $A \in \mathfrak{A}_\ell$ and defines a seminorm on $\mathfrak{A}_\ell$. α is the multi-index $\alpha = (f,k,\gamma)$, $\gamma = (\{\varepsilon\}, n) \in \Gamma_o$.

We have proved this Lemma in /15, Theorem 3.2/. On $\mathfrak{A}_\ell$ we define now a locally convex topology ξ_o by the following system of seminorms

$$(2.5) \quad \xi_o : \qquad p_\alpha(A) = \max (\| A \|_\alpha , \| A^+ \|_\alpha) ,$$

where α runs over all the multi-indices $\alpha = (f,k,\gamma)$ with $f \in F$, $k = 0,1,2,\ldots$, $\gamma = (\{\varepsilon\}, n) \in \Gamma_o$.

Further we have the following lemma /15, Lemma 3.1/.

Lemma 2.2

$\mathfrak{A}_\ell[\xi_o]$ is a locally convex topological $*$-algebra. The algebraic operations can be continuously extended to the completion $\mathfrak{A} = \widetilde{\mathfrak{A}_\ell[\xi_o]}$, which then is a complete locally convex $*$-algebra.

Let us remark that the second part of this lemma is nontrivial, since the multiplication in $\mathfrak{A}_\ell[\xi_0]$ is not jointly continuous.

On the basis of these lemmas we can formulate the main result on the dynamics of the BCS-Bogolubov model /15, Theorem 4.4/.

Theorem 2.3
For $A \in \mathfrak{A}_\ell$ the limit $\tau_t(A) = \lim_{\Lambda\to\infty} e^{iH_\Lambda t} A e^{-iH_\Lambda t}$ exists in $\mathfrak{A} = \widetilde{\mathfrak{A}_\ell[\xi_0]}$. $\tau_t(A)$ is continuous with respect to the topology ξ_σ, and it can be extended to a one-parameter group of $*$-automorphisms of the complete locally convex $*$-algebra $\mathfrak{A}$.

The crucial point for the proof of the theorem is the existence of the mean quasi-spin $s = \lim 1/2\, V(\Lambda)^{-1} \sum_{p\in\Lambda} \sigma_p$ in $\mathfrak{A}$, where the limit is taken with respect to the topology ξ_σ (/15/ Theorem 3.3, /14/ lemma 3.2). This limit fails to exist with respect to the C^*-norm, which is given on $\mathfrak{A}_\ell$. This is also the reason, why the dynamics of the BCS-Bogolubov model cannot be handled in a C^*-theory.

Let us, in the next section, yet explain the connection of the topology ξ_σ with the Op^*-topologies introduced in /7,12,13/.

3. Op^*-Topologies

Let M_γ^Λ be the operators of the foregoing section (N-operators, see below). Let us fix the index $\gamma = (\{\varepsilon\},n)$. Then there exists a faithful representation of $\mathfrak{A}_\ell$ on a certain Hilbert space $\mathcal{H}_\gamma$, so that $\lim_{\Lambda\to\infty} M_\gamma^\Lambda = M$ exists on a certain dense domain with respect to the weak topology /18, Lemma 2/. M is a self-adjoint operator. $\mathcal{D} = \bigcap_{k=0}^\infty \mathcal{D}(M^k)$ is invariant for all operators $A \in \mathfrak{A}_\ell$. We define on $\mathcal{D}$ a locally convex topology t by the system of seminorms

$$(3.1) \qquad t: \quad \|\phi\|_k = \|M^k \phi\| , \quad \phi \in \mathcal{D} , \quad k = 0,1,2,\dots .$$

$\mathcal{D}[t]$ is a Frechet space. Let $\mathcal{L}^+(\mathcal{D})$ be the maximal Op*-algebra on $\mathcal{D}$ and $\mathcal{L}(\mathcal{D})$ the algebra of all continuous operators of $\mathcal{D}[t]$ into itself. The topology of uniformly bounded convergence on $\mathcal{L}(\mathcal{D})$ will be denoted by $\mathfrak{T}^{\mathcal{D}}$. $\mathcal{L}(\mathcal{D})[\mathfrak{T}^{\mathcal{D}}]$ is a complete space.

The topology $\mathfrak{T}^{\mathcal{D}}$ is given by the seminorms

$$(3.2) \qquad \mathfrak{T}^{\mathcal{D}}: \quad \|A\|^{\mathcal{M},k} = \sup_{\phi \in \mathcal{M}} \|M^k A \phi\|$$

where $\mathcal{M}$ runs over all bounded sets in $\mathcal{D}[t]$ and $k = 0,1,2,\dots$. On $\mathcal{L}^+(\mathcal{D})$ we define a more strong topology $\mathfrak{T}^{\mathcal{D}}_*$ by the following system of seminorms

$$(3.3) \qquad \mathfrak{T}^{\mathcal{D}}_*: \quad \|A\|^{\mathcal{M},k} , \quad \|A\|^{\mathcal{M},k}_+ = \|A^+\|^{\mathcal{M},k} .$$

Then also the involution $A \to A^+$ is a continuous mapping with respect to the topology $\mathfrak{T}^{\mathcal{D}}_*$.

<u>Lemma 3.1</u> /13,14/

$\mathcal{L}^+(\mathcal{D})[\mathfrak{T}^{\mathcal{D}}_*]$ is a complete locally convex *-algebra.

Since $\mathfrak{A}_e \subset \mathcal{L}^+(\mathcal{D})$, the completion $\widetilde{\mathfrak{A}_e[\mathfrak{T}^{\mathcal{D}}_*]} = \mathfrak{A}_{\mathcal{D}}$ is also a *-algebra. Now let us fix the index γ in (2.5). Then we get a weaker topology ξ_γ and $\xi_\circ = \sup_\gamma \xi_\gamma$.

<u>Lemma 3.2</u> /15, Theorem 3.2/

The topologies ξ_γ and $\mathfrak{T}^{\mathcal{D}}_*$ coincide on $\mathfrak{A}_e$.

The topology $\mathfrak{T}^{\mathcal{D}}_*$ depends only on the dense domain $\mathcal{D}$ and is a natural generalization of the C^*-topology on a *-algebra of bounded operators. Therefore ξ_γ and $\xi_o = \sup_\gamma \xi_\gamma$ are "C^*-like" topologies /13/. The relations (2.4) and (2.5) show that ξ_o can be defined by an explicit use of the operators $\hat{M}_\gamma$. This is an interesting fact, since the operators M_{t}^{Λ} are of the type of the operators N_{Λ} in /5/, which we call "N-operators". They estimate the unboundedness of H_{Λ} (see /5/, N2-N3) in the limit $\Lambda \to \infty$.

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