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Курбатов А.М. **E17 - 12433**

Асимптотически точное решение задачи классического плотного электронного газа

Целью работы является исследование классического электронного газа. Рассматриваемая система представляет собой модель желе, используемую в физике плазмы. Исследование проводится в рамках аппроксимационной схемы на основе теоремы об аппроксимации систем с положительным сепарабельным взаимодействием. Получено асимптотически точное выражение для функции свободной энергии. Доказано, что равновесное уравнение Власова является точным в пределе высоких плотностей электронного газа.

Работа выполнена в Лаборатории теоретической физики, ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1979

Kurbatov A.M. **E17 - 12433**

Asymptotically Exact Solution of the Problem of Classical Dence Electron Gas

The purpose of the paper is the investigation of classical electron gas. The system under consideration is the jellium model used in plasma physics. The investigation is carried in the frame of an approximation scheme on the basis of the approximation theorem of systems with positive separable interaction. The asymptotically exact expression for the free energy function is found. It is proved that the equilibrium Vlasov equation is exact in the limit of high densities of the electron gas.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1979

In the present paper we shall calculate the free energy of classical electron gas

$$H=\sum_{1\leq i\leq N}\frac{\underline{p}_i^2}{2m}+\sum_{1\leq i<j\leq N}\frac{K}{|\underline{q}_i-\underline{q}_j|} \tag{1}$$

exactly in the limit of high **densities.** The system (1) is known as the jellium model used in plasma physics.

The free energy $F[N,V,\beta;H]$ of the Hamiltonian H is given by

$$F[N,V,\beta;H]=-\frac{1}{\beta}\ln(\rho_H,\mathbb{1})_{\Omega_{\mathbb{V},\underline{N}}}=$$

$$=-\frac{1}{\beta}\ln\int_{\Omega_{\mathbb{V},N}}e^{-\beta\left[\sum_{1\leq i\leq N}\frac{\underline{p}_i^2}{2m}+\sum_{1\leq i<j\leq N}\frac{K}{|\underline{q}_i-\underline{q}_j|}\right]}d\underline{p}_1\ldots d\underline{p}_N\,d\underline{q}_1\ldots d\underline{q}_N= \tag{2}$$

$$=-\frac{1}{\beta}\ln\frac{1}{N!}\left[\left(\frac{2\pi m}{\beta}\right)^{3/2}V\right]^N-\frac{1}{\beta}\ln\left[\frac{1}{V^N}\int_{(\mathbb{V})^N}e^{-\beta\sum_{1\leq i<j\leq N}\frac{K}{|\underline{q}_i-\underline{q}_j|}}d\underline{q}_1\ldots d\underline{q}_N.\right. \tag{3}$$

Let our system be contained in the volume of the sphere of radius R

$$\frac{4\pi}{3} R^3 = V. \tag{4}$$

Making the change of variables

$$\underline{z}_i = R\underline{\xi}_i \tag{5}$$

we obtain

$$F[N,V,\beta;H] = -\frac{1}{\beta}\ln\frac{1}{N!}\left[\left(\frac{2\pi m}{\beta}\right)^{3/2} R^3\right]^N -$$

$$-\frac{1}{\beta}\ln\int_{(E^3)^N} e^{-\beta\frac{K}{R}\sum_{1\le i<j\le N}\frac{1}{|\underline{\xi}_i-\underline{\xi}_j|}}\, d\underline{\xi}_1\ldots d\underline{\xi}_N, \tag{6}$$

or

$$F[N,V,\beta;H] = F[N,E^3,\beta;H_e^{(R)}], \tag{7}$$

where

$$H_e^{(R)} = \sum_{1\le i\le N}\frac{p_i^2}{2mR^2} + \frac{K}{R}U_e^{(N)}(\underline{\xi}_1,\ldots,\underline{\xi}_N),\quad U_e^{(N)} = \sum_{1\le i<j\le N}\frac{1}{|\underline{\xi}_i-\underline{\xi}_j|}. \tag{8}$$

We introduce the notations

$$K(\underline{\xi};\underline{\eta}) = \frac{1}{|\underline{\xi}-\underline{\eta}|}, \tag{9}$$

$$K^{(\tau)}(\underline{\xi},\underline{\eta}) = \begin{cases}\dfrac{1-e^{-\tau|\underline{\xi}-\underline{\eta}|}}{|\underline{\xi}-\underline{\eta}|}, & \underline{\xi}\ne\underline{\eta}\\ \tau, & \underline{\xi}=\underline{\eta}\end{cases},\quad \tau\in(0,\infty). \tag{10}$$

The function $K^{(\tau)}(\underline{\xi},\underline{\eta})$ defined on $E^3\otimes E^3$ for all $\tau\in(0,\infty)$ has the following properties

$$(\forall\langle\underline{\xi},\underline{\eta}\rangle\in E^3\otimes E^3)\quad K^{(\tau)}(\underline{\xi},\underline{\eta}) = K^{(\tau)}(\underline{\eta},\underline{\xi}); \tag{11}$$

$$(\forall\underline{\xi}\in E^3)\quad K^{(\tau)}(\underline{\xi},\cdot)\in L^2[E^3]; \tag{12}$$

$$(\exists M_\tau\in R^1)\quad(\forall\underline{\xi}\in E^3)\quad \|K^{(\tau)}(\underline{\xi},\cdot)\|_2\le M_\tau; \tag{13}$$

$$K^{\tau}(\cdot,\cdot)\in L^2[E^3\otimes E^3]; \tag{14}$$

$$K^{(\tau)}(\cdot,\cdot)>0. \tag{15}$$

Therefore (see, e.g., ref.1) for all $\tau\in(0,\infty)$ the integral operator $K^{(\tau)}: L^2[E^3]\longrightarrow L^2[E^3]$

$$(K^{(\tau)}\varphi)(\underline{\xi}) = (K^{(\tau)}(\underline{\xi},\cdot),\varphi(\cdot))_{L^2[E^3]},\quad \varphi(\cdot)\in L^2[E^3] \tag{16}$$

has a countable orthonormal in $L^2[E^3]$ set of eigenfunctions $\{\varphi_\alpha^{(\tau)}(\cdot)\}$ with corresponding eigenvalues $\{\lambda_\alpha^{(\tau)}\}$ which is complete in $L^2[E^3]$.

We designate

$$K_S^{(z)}(\underline{\xi},\underline{\eta})=\sum_{\alpha=1}^{S}\lambda_\alpha^{(z)}\varphi_\alpha^{(z)}(\underline{\xi})\varphi_\alpha^{(z)}(\underline{\eta}), \tag{17}$$

$$\Delta K_S^{(z)}(\underline{\xi},\underline{\eta})=K_S^{(z)}(\underline{\xi},\underline{\eta})-K^{(z)}(\underline{\xi},\underline{\eta}), \tag{18}$$

and note that also

$$K(\cdot,\cdot)\in C[\mathbb{E}^3\otimes\mathbb{E}^3]. \tag{19}$$

Then (see, e.g., ref.1)

$$(\forall z)\ (\forall\alpha)\quad \lambda_\alpha^{(z)}>0, \tag{20}$$

$$(\forall z)\quad \|\Delta K_S^{(z)}\|_\infty=\varepsilon_z(S)\xrightarrow[S\to\infty]{}0. \tag{21}$$

From the other hand, the potential energy $U_e^{(N)}(\underline{\xi}_1,\dots,\underline{\xi}_N)$ may be represented in the form

$$U_e^{(N)}(\underline{\xi}_1,\dots,\underline{\xi}_N)=U_{zS}^{(N)}(\underline{\xi}_1,\dots,\underline{\xi}_N)+U_z^{(N)}(\underline{\xi}_1,\dots,\underline{\xi}_N)-$$
$$-\frac{N}{2}z+U_a^{(N)}(\underline{\xi}_1,\dots,\underline{\xi}_N), \tag{22}$$

where

$$U_{zS}^{(N)}(\underline{\xi}_1,\dots,\underline{\xi}_N)=\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha=1}^{S}\lambda_S^{(z)}\varphi_\alpha^{(z)}(\underline{\xi}_i)\varphi_\alpha^{(z)}(\underline{\xi}_j), \tag{23}$$

$$U_z^{(N)}(\underline{\xi}_1,\dots,\underline{\xi}_N)=\frac{1}{2}\sum_{i,j=1}^{N}\Delta K_S^{(z)}(\underline{\xi}_i,\underline{\xi}_j), \tag{24}$$

$$U_a^{(N)}(\underline{\xi}_1,\dots,\underline{\xi}_N)=\sum_{1\le i<j\le N}\frac{e^{-z|\underline{\xi}_i-\underline{\xi}_j|}}{|\underline{\xi}_i-\underline{\xi}_j|}. \tag{25}$$

Now we estimate the difference of the free energies $F[N,V,\beta;H]$ and $F[N,\mathbb{E}^3,\beta;H_{zS}^{(R)}]$, where

$$H_{zS}^{(R)}=\sum_{1\le i\le N}\frac{p_i^2}{2mR^2}+\frac{K}{R}U_{zS}^{(N)}(\underline{\xi}_1,\dots,\underline{\xi}_N). \tag{26}$$

By virtue of the Bogolubov inequality for classical systems[2] we obtain

$$\frac{K}{R}\left\langle -U_z^{(N)}-U_a^{(N)}+\frac{N}{2}z\right\rangle_{H_{zS}}\le$$
$$\le F[N,\mathbb{E}^3,\beta;H_{zS}^{(R)}]-F[N,\mathbb{E}^3,\beta;H_e^{(R)}]\le \tag{27}$$
$$\le\frac{K}{R}\left\langle -U_z^{(N)}-U_a^{(N)}+\frac{N}{2}z\right\rangle_{H_e}.$$

According to (21) for any Hamiltonian H we have the inequalities

$$|\langle U_z\rangle_H|\le\|U_z^{(N)}\|_\infty\le\frac{1}{2}N^2\|\Delta K_S^{(z)}\|_\infty=\frac{1}{2}N^2\varepsilon_z(S), \tag{28}$$

moreover, since $U_a^{(N)}>0$,

$$\langle -U_a^{(N)} \rangle_{He} \leq 0 . \tag{29}$$

Let us consider the average $\langle U_a^{(N)} \rangle_{H_{zs}}$

$$\langle U_a^{(N)} \rangle_{H_{zs}} =$$

$$= \frac{\int\limits_{(\mathbb{E}^3)^N} d\underline{\xi}_1 \dots d\underline{\xi}_N \sum\limits_{1\leq i<j\leq N} \frac{e^{-z|\underline{\xi}_i - \underline{\xi}_j|}}{|\underline{\xi}_i - \underline{\xi}_j|} e^{-\beta \frac{K}{R} \sum\limits_{1\leq i<j\leq N} K_s^{(2)}(\underline{\xi}_i, \underline{\xi}_j)}}{\int\limits_{(\mathbb{E}^3)^N} d\underline{\xi}_1 \dots d\underline{\xi}_N \, e^{-\beta \frac{K}{R} \sum\limits_{1\leq i<j\leq N} K_s^{(2)}(\underline{\xi}_i, \underline{\xi}_j)}} =$$

$$= \frac{N(N-1)}{2} \Bigg[\int\limits_{(\mathbb{E}^3)^{N-1}} d\underline{\xi}_2 \dots d\underline{\xi}_N \, e^{-\beta \frac{K}{R} \sum\limits_{2\leq i<j\leq N} K_s^{(2)}(\underline{\xi}_i, \underline{\xi}_j)} \int\limits_{\substack{\mathbb{E}^3 \\ \varepsilon \leq |\underline{\xi}_1 - \underline{\xi}_2|}} d\underline{\xi}_1 \frac{e^{-z|\underline{\xi}_1 - \underline{\xi}_2|}}{|\underline{\xi}_1 - \underline{\xi}_2|} e^{-\beta \frac{K}{R} \sum\limits_{2\leq j\leq N} K_s^{(2)}(\underline{\xi}_1, \underline{\xi}_j)} +$$

$$+ \int\limits_{(\mathbb{E}^3)^{N-1}} d\underline{\xi}_2 \dots d\underline{\xi}_N \, e^{-\beta \frac{K}{R} \sum\limits_{2\leq i<j\leq N} K_s^{(2)}(\underline{\xi}_i, \underline{\xi}_j)} \int\limits_{\mathbb{E}^3} d\underline{\xi}_1 \frac{e^{-z|\underline{\xi}_1 - \underline{\xi}_2|}}{|\underline{\xi}_1 - \underline{\xi}_2|} e^{-\beta \frac{K}{R} \sum\limits_{2\leq j\leq N} K_s^{(2)}(\underline{\xi}_1, \underline{\xi}_j)} \Bigg] :$$

$$: \int\limits_{(\mathbb{E}^3)^N} d\underline{\xi}_1 \dots d\underline{\xi}_N \, e^{-\beta \frac{K}{R} \sum\limits_{1\leq i<j\leq N} K_s^{(2)}(\underline{\xi}_i, \underline{\xi}_j)} , \tag{30}$$

where ε is a parameter which may depend on z , β , κ , K , N . For the integral $\int\limits_{\mathbb{E}^3} d\underline{\xi}_1 \, |\underline{\xi}_1 - \underline{\xi}_2|^{-1}$ converges,

$$\int\limits_{\substack{\mathbb{E}^3 \\ |\underline{\xi}_1 - \underline{\xi}_2| \leq \varepsilon}} d\underline{\xi}_1 \frac{1}{|\underline{\xi}_1 - \underline{\xi}_2|} \leq \rho(\varepsilon), \tag{31}$$

where $\rho(\varepsilon)$ vanishes monotonously, as $\varepsilon \to 0$. Therefore the estimation holds

$$\int\limits_{\substack{\mathbb{E}^3 \\ |\underline{\xi}_1 - \underline{\xi}_2| \leq \varepsilon}} d\underline{\xi}_1 \frac{e^{-z|\underline{\xi}_1 - \underline{\xi}_2|}}{|\underline{\xi}_1 - \underline{\xi}_2|} < \int\limits_{\substack{\mathbb{E}^3 \\ |\underline{\xi}_1 - \underline{\xi}_2| \leq \varepsilon}} d\underline{\xi}_1 \frac{1}{|\underline{\xi}_1 - \underline{\xi}_2|} . \tag{32}$$

Taking into account that $e^{-z\xi}/\xi$ is a monotonously decreasing function of ξ and introducing the notation

$$Y(z, \beta, \kappa, K, N) = \frac{\int\limits_{(\mathbb{E}^3)^{N-1}} d\underline{\xi}_2 \dots d\underline{\xi}_N \, e^{-\beta \frac{K}{R} \sum\limits_{2\leq i<j\leq N} K_s^{(2)}(\underline{\xi}_i, \underline{\xi}_j)}}{\int\limits_{(\mathbb{E}^3)^N} d\underline{\xi}_1 \dots d\underline{\xi}_N \, e^{-\beta \frac{K}{R} \sum\limits_{1\leq i<j\leq N} K_s^{(2)}(\underline{\xi}_i, \underline{\xi}_j)}} \tag{33}$$

for sufficiently small δ we obtain

$$\langle U_a^{(N)} \rangle_{H_{zs}} \leq \frac{N(N-1)}{2} \left[\frac{e^{-z\varepsilon}}{\varepsilon} + \rho(\varepsilon) Y(z, \beta, \kappa, N) \right] . \tag{34}$$

Since $\rho(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0$,

$$(\forall z)\,(\exists \varepsilon_1)\,(\forall \varepsilon > \varepsilon_1)\quad \sqrt{\rho(\varepsilon)}\; Y(z,\beta,K,N) \leq 1 . \tag{35}$$

If we set

$$\varepsilon = \varepsilon_2 = \min\left\{\frac{1}{\sqrt{2}}, \varepsilon_1\right\}, \tag{36}$$

we find

$$\langle U_a^{(N)} \rangle_{H_{zs}} \leq \frac{N(N-1)}{2}\left[\sqrt{2}\, e^{-\sqrt{2}} + \sqrt{\rho\left(\frac{1}{\sqrt{2}}\right)}\right] = \frac{N(N-1)}{2}\, \zeta(z), \tag{37}$$

$$\zeta(z) \xrightarrow[z \to \infty]{} 0 . \tag{38}$$

Thus, according to (28),(29),(37) the inequality (27) can be rewritten in the form

$$-\frac{1}{2}\frac{N^2}{R} K \varepsilon z(s) - \frac{1}{2}\frac{N(N-1)}{R} K \zeta(z) \leq F[N, \mathbb{E}^3, \beta; H_{zs}^{(R)}] -$$

$$F[N, \mathbb{E}^3, \beta; H_e^{(R)}] \leq \frac{1}{2}\frac{N^2}{R} K \varepsilon z(s) + \frac{1}{2}\frac{N}{R} K . \tag{39}$$

Let us transform now H_{zs} to

$$H_{zs}^{(R)} = \sum_{1 \leq i \leq N} \frac{p_i^2}{2mR^2} + \frac{K}{R}\frac{1}{2}\sum_{i,j=1}^{N}\sum_{\alpha=1}^{s} \lambda_\alpha^{(z)} \varphi_\alpha^{(z)}(\underline{\xi}_i)\, \varphi_\alpha^{(z)}(\underline{\xi}_j) =$$

$$= N\, T(\underline{p}_1, \dots, \underline{p}_N) + N \sum_{\alpha=1}^{s} A_\alpha^{(z)}(\underline{\xi}_1, \dots, \underline{\xi}_N)\, A_\alpha^{(z)}(\underline{\xi}_1, \dots, \underline{\xi}_N), \tag{40}$$

where

$$T(\underline{p}_1, \dots, \underline{p}_N) = \frac{1}{N}\sum_{i=1}^{N} \frac{p_i^2}{2mR^2} = \frac{1}{N}\sum_{i=1}^{N} \mathcal{T}(\underline{p}_i), \tag{41}$$

$$A_\alpha^{(z)}(\underline{\xi}_1, \dots, \underline{\xi}_N) = \frac{1}{N}\sum_{i=1}^{N} \sqrt{\frac{1}{2}\frac{N}{R} K \lambda_\alpha^{(z)}}\; \varphi_\alpha^{(z)}(\underline{\xi}_i) = \frac{1}{N}\sum_{\alpha=1}^{N} \mathcal{A}_\alpha^{(z)}(\underline{\xi}_i). \tag{42}$$

Note that by virtue of the Cauchy-Buniakowsky inequality and in view of (13)

$$|\lambda_\alpha^{(z)} \varphi_\alpha^{(z)}(\underline{\xi})| = |(K^{(z)}(\underline{\xi}, \cdot), \varphi_\alpha^{(z)}(\cdot))_{L^2[\mathbb{E}^3]}| \leq$$

$$\leq \|K^{(z)}(\underline{\xi}, \cdot)\|_2\, \|\varphi_\alpha(\cdot)\|_2 \leq M_z , \tag{43}$$

therefore

$$\|\mathcal{A}_\alpha^{(z)}\|_\infty = \sqrt{\frac{1}{2}\frac{K}{R}}\, \frac{M_z}{\sqrt{\lambda_\alpha^{(z)}}}\, \sqrt{N} = M_z(\alpha, N) \tag{44}$$

and

$$\sum_{\alpha=1}^{s} \|\mathcal{A}_\alpha^{(z)}\|_\infty = \sum_{\alpha=1}^{s} M_z(\alpha, N) = \sqrt{\frac{1}{2}\frac{K}{R}}\, M_z \sum_{\alpha=1}^{s} \frac{1}{\sqrt{\lambda_\alpha^{(z)}}} \sqrt{N} = M_z(N). \tag{45}$$

It is easy to see that for $H_{zs}^{(R)}$ the conditions of Theorem 2 of ref.3 are satisfied at any fixed z and s , therefore

$$-\frac{1}{R}\kappa M_2\Big(\sum_{\alpha=1}^{S}\frac{1}{\sqrt{\lambda_\alpha^{(z)}}}\Big)^2 \leq \frac{1}{N}F[N,\mathbb{E}^3,\beta;H_{zso}^{(R)}(\bar{a}_1^{(z,s,N)},\dots,\bar{a}_S^{(z,s,N)})]-$$

$$-\frac{1}{N}F[N,\mathbb{E}^3,\beta,H_{zs}^{(R)}]\leq 0, \qquad (46)$$

where

$$H_{zso}^{(R)}(a_1,\dots,a_S)=NT(\underline{p}_1,\dots,\underline{p}_N)+N\sum_{\alpha=1}^{S}\big[2a_\alpha A_\alpha^{(z)}(\xi_1,\dots,\xi_N)-a_\alpha^2\big]= \qquad (47)$$

$$=NT(\underline{p}_1,\dots,\underline{p}_N)+N\,U_{zso}^{(N)}(a_1,\dots,a_S;\underline{\xi}_1,\dots,\underline{\xi}_N), \qquad (48)$$

$$U_{zso}^{(N)}(a_1,\dots,a_S;\underline{\xi}_1,\dots,\underline{\xi}_N)=\sum_{\alpha=1}^{S}\Big[\frac{1}{N}\sum_{i=1}^{N}2a_\alpha\sqrt{\frac{1}{2}\frac{N}{R}\cdot\kappa\,\lambda_\alpha^{(z)}}\,\varphi_\alpha^{(z)}(\underline{\xi}_i)-a_\alpha^2\Big], \qquad (49)$$

and $\langle\bar{a}_1^{(S,z,N)},\dots,\bar{a}_S^{(S,z,N)}\rangle$ is a solution of the following absolute maximum problem

$$\frac{1}{N}F[N,\mathbb{E}^3,\beta;H_{zso}^{(R)}(\bar{a}_1^{(z,S,N)},\dots,\bar{a}^{(z,S,N)})]=\max_{\langle a_1,\dots,a_S\rangle}\frac{1}{N}F[N,\mathbb{E}^3,\beta;H_{zso}^{(R)}(a_1,\dots,a_S)], \qquad (50)$$

by virtue of the same theorem this solution always existing and

$$|\bar{a}_\alpha^{(z,S,N)}|\leq M_2(\alpha,\Lambda). \qquad (51)$$

Thus taking into account (39) and (46) we have

$$-\frac{1}{2}\frac{N^2}{R^{1+3z}}\kappa\varepsilon z(S)-\frac{1}{2}\frac{N(N-1)}{R^{1+3z}}\kappa\,\zeta(z)-\frac{N}{R^{1+3z}}\kappa M_2\Big(\sum_{\alpha=1}^{S}\frac{1}{\sqrt{\lambda_\alpha^{(z)}}}\Big)^2\leq$$

$$\leq\frac{1}{R^{3z}}F[N,\mathbb{E}^3,\beta;H_{zso}^{(R)}(\bar{a}_1^{(z,S,N)},\dots,\bar{a}_S^{(z,S,N)})]-\frac{1}{R^{3z}}F[N,V,\beta;H]\leq$$

$$\leq\frac{1}{2}\frac{N^2}{R^{1+3z}}\kappa\varepsilon z(S)+\frac{1}{2}\frac{N}{R}\kappa, \qquad (52)$$

where z is a positive parameter. Finally, taking in (51) subsequently the limits $\frac{N}{R^{1+3z}}\kappa\to 0$: $\frac{N^2}{R^{1+3z}}\kappa=O(1)$, $S\to\infty$, $z\to\infty$ we obtain

$$\lim_{z\to\infty}\ \lim_{S\to\infty}\ \lim_{\substack{\frac{N}{R^{1+3z}}\kappa\to 0\\ \frac{N^2}{R^{1+3z}}\kappa=O(1)}}\Big\{\frac{1}{R^{3z}}F[N,\mathbb{E}^3,\beta;H_{zso}^{(R)}(\bar{a}_1^{(z,S,N)},\dots,\bar{a}_S^{(z,S,N)})]-$$

$$-\frac{1}{R^{3z}}F[N,V,\beta;H]\Big\}=0. \qquad (53)$$

We write $U_{zso}^{(N)}(a_1,\dots,a_S;\underline{\xi}_1,\dots,\underline{\xi}_N)$ in the form

$$U_{zso}^{(N)}(a_1,\dots,a_S;\underline{\xi}_1,\dots,\underline{\xi}_N)=-\sum_{\alpha=1}^{S}a_\alpha^2+N^x\sum_{i=1}^{N}\sum_{\alpha=1}^{S}2a_\alpha\sqrt{\frac{1}{2}\frac{\kappa}{R}\frac{\lambda_\alpha^{(z)}}{N^{1+2x}}}\,\varphi_\alpha^{(z)}(\underline{\xi}_i)=$$

$$=-\sum_{\alpha=1}^{S}a_\alpha^2+N^x\frac{\kappa}{RN^{1+2x}}\sum_{i=1}^{N}\psi_{zs}(\underline{\xi}_i), \qquad (54)$$

where

$$\Psi_{zs}(\underline{\xi}) = \sum_{\alpha=1}^{s} 2a_\alpha \sqrt{\frac{1}{2}\frac{RN^{1+2x}}{K}} \sqrt{\lambda_\alpha^{(z)}}\, \varphi_\alpha^{(z)}(\underline{\xi}), \tag{55}$$

then

$$\frac{1}{R^{3z}} F[N, \mathbb{E}^3, \beta; H_{zs0}(a_1,\dots,a_s)] = -\frac{1}{\beta R^{3z}} \ln \frac{1}{N!}\left[\left(\frac{2\pi m}{\beta}\right)^{3/2} R^3\right]^N -$$

$$-\frac{1}{\beta R^{3z}} \ln \int e^{-\beta N\left[-\sum_{\alpha=1}^{s} a_\alpha^2 + N^x \frac{K}{RN^{1+2x}} \sum_{i=1}^{N} \Psi_{zs}(\underline{\xi}_i)\right]} d\underline{\xi}_1 \dots d\underline{\xi}_N =$$

$$= -\frac{1}{\beta R^{3z}} \ln \frac{1}{N!}\left[\left(\frac{2\pi m}{\beta}\right)^{3/2} R^3\right] - \frac{N}{R^{3z}} \sum_{\alpha=1}^{s} a_\alpha^2 -$$

$$-\frac{N}{\beta R^{3z}} \ln \int_{\mathbb{E}^3} e^{-\beta \frac{K}{RN^x} \Psi_{zs}(\underline{\xi})} d\underline{\xi}\,. \tag{56}$$

The quantities $\bar{a}_1^{(z,s,N)}, \dots, \bar{a}_s^{(z,s,N)}$ which are solutions of the absolute maximum problem (45) are to satisfy the set of equations

$$\partial \frac{1}{R^{3z}} F[N, \mathbb{E}^3, \beta; H_{zs0}(a_1,\dots,a_s)] \big/ \partial a_\alpha = 0. \tag{57}$$

or

$$-2a_\alpha + \frac{K}{RN^x} \frac{\int_{\mathbb{E}^3} \frac{\partial \Psi_{zs}(\underline{\xi})}{\partial a_\alpha} e^{-\beta \frac{K}{RN^x} \Psi_{zs}(\underline{\xi})} d\underline{\xi}}{\int_{\mathbb{E}^3} e^{-\beta \frac{K}{RN^x} \Psi_{zs}(\underline{\xi})} d\underline{\xi}}\,. \tag{58}$$

Since

$$\frac{\partial \Psi_{zs}(\underline{\xi})}{\partial a_\alpha} = 2\sqrt{\frac{1}{2}\frac{RN^{1+2x}}{K}} \sqrt{\lambda_\alpha^{(z)}}\, \varphi_\alpha^{(z)}(\underline{\xi}), \tag{59}$$

we have

$$-a_\alpha + \sqrt{\frac{1}{2}\frac{K}{RN^x}} \frac{\int_{\mathbb{E}^3} \sqrt{\lambda_\alpha^{(z)}}\, \varphi_\alpha^{(z)}(\underline{\eta})\, e^{-\beta \frac{K}{RN^x} \Psi_{zs}(\underline{\eta})} d\underline{\eta}}{\int_{\mathbb{E}^3} e^{-\beta \frac{K}{RN^x} \Psi_{zs}(\underline{\eta})} d\underline{\eta}}\,. \tag{60}$$

Multiplying these equations by $2\sqrt{\frac{1}{2}\frac{RN^{1+2x}}{K}} \sqrt{\lambda_\alpha^{(z)}}\, \varphi_\alpha^{(z)}(\underline{\xi})$ and adding them together we obtain that the function $\Psi_{zs}(\underline{\xi})$ is to satisfy the following integral equation

$$\Psi_{zs}(\underline{\xi}) = \frac{\int_{\mathbb{E}^3} K_s^{(z)}(\underline{\xi}, \underline{\eta})\, e^{-\beta \frac{K}{RN^x} \Psi_{zs}(\underline{\eta})} d\underline{\eta}}{\int_{\mathbb{E}^3} e^{-\beta \frac{K}{RN^x} \Psi_{zs}(\underline{\eta})} d\underline{\eta}}\,. \tag{61}$$

Then, note that the sum $\sum_{\alpha=1}^{s}[\bar{a}_\alpha^{(z,s,N)}]^2$ may be expressed directly in terms of the function $\bar{\Psi}_{2s}(\underline{\xi})$, satisfying the equation (61). In fact, by virtue of (60)

$$-\sum_{\alpha=1}^{s}\left[\bar{a}_\alpha^{(z,s,N)}\right]^2 = -\sum_{\alpha=1}^{s}{}' \frac{K}{2RN^x}\,\frac{\int\limits_{E^3} 2\bar{a}_\alpha^{(z,s,N)}\sqrt{\frac{1}{2}\frac{RN^x}{K}}\sqrt{\lambda_\alpha^{(z)}}\,\varphi_\alpha^{(z)}(\underline{\xi})\, e^{-\beta\frac{K}{RN^x}\Psi_{2s}(\underline{\xi})}\,d\underline{\xi}}{\int\limits_{E^3} e^{-\beta\frac{K}{RN^x}\Psi_{2s}(\underline{\xi})}\,d\underline{\xi}} =$$

$$= \frac{K}{2RN^x}\,\frac{\int\limits_{E^3}\bar{\Psi}_{2s}(\underline{\xi})\, e^{-\beta\frac{K}{RN^x}\bar{\Psi}_{2s}(\underline{\xi})}\,d\underline{\xi}}{\int\limits_{E^3} e^{-\beta\frac{K}{RN^x}\bar{\Psi}_{2s}(\underline{\xi})}\,d\underline{\xi}}\,. \qquad (62)$$

Thus

$$\frac{1}{R^{3z}}F\left[N,E^3,\beta;H_{2s0}^{(R)}\left(\bar{a}_1^{(z,s,N)},\dots,\bar{a}_s^{(z,s,N)}\right)\right] =$$

$$= -\frac{1}{\beta R^{3z}}\ln\frac{1}{N!}\left[\left(\frac{2\pi m}{\beta}\right)^{3/2}R^3\right]^N -$$

$$-\frac{1}{2}\frac{N}{R^{3z}}\frac{K}{RN^x}\,\frac{\int\limits_{E^3}\bar{\Psi}_{2s}(\underline{\xi})\, e^{-\beta\frac{K}{RN^x}\bar{\Psi}_{2s}(\underline{\xi})}\,d\underline{\xi}}{\int\limits_{E^3} e^{-\beta\frac{K}{RN^x}\bar{\Psi}_{2s}(\underline{\xi})}\,d\underline{\xi}} -$$

$$-\frac{N}{R^{3z}}\frac{1}{\beta}\ln\int\limits_{E^3} e^{-\beta\frac{K}{RN^x}\bar{\Psi}_{2s}(\underline{\xi})}\,d\underline{\xi}\,, \qquad (63)$$

where $\bar{\Psi}_{2s}(\underline{\xi})$ satisfies the equation (61).

We remind that our task is to estimate the free energy $F[H]$ at given values of N_0, R_0, β_0. To this end in view we make use of the relation (53) for

$$z=-\frac{7}{3},\ N=nN_0,\ R=n^{-1/3}R_0,\ \beta=n^{-4/3}\beta_0,\ x=-1, \qquad (64)$$

and consider the expression

$$\frac{1}{(n^{-1/3}R_0)^{3/2}}F\left[nN_0,E^3,n^{-4/3}\beta_0;H_{2s0}^{(n^{-1/3}R_0)}\left(\bar{a}_1^{(z,s,nN_0)},\dots,\bar{a}_s^{(z,s,nN_0)}\right)\right].$$

Its limit, as $n\to\infty$, exists and is equal to

$$\lim_{n\to\infty}\frac{1}{(n^{-1/3}R_0)^{3z}}F\left[nN_0,E^3,n^{-4/3}\beta_0;H_{2s0}^{(n^{-1/3}R_0)}\left(\bar{a}_1^{(s,z,nN_0)},\dots,\bar{a}_s^{(s,z,nN_0)}\right)\right]=$$

$$= -\frac{1}{\beta_0 R_0^{3z}}\ln\frac{1}{N_0!}\left[\left(\frac{2\pi m}{\beta_0}\right)^{3/2}R_0^3\right]^{N_0} -$$

$$-\frac{1}{2}\frac{N_0}{R_0^{3z}}\frac{KN_0}{R_0}\,\frac{\int\limits_{E^3}\bar{\Psi}_{2s}(\underline{\xi})\, e^{-\beta_0\frac{KN_0}{R_0}\bar{\Psi}_{2s}(\underline{\xi})}\,d\underline{\xi}}{\int\limits_{E^3} e^{-\beta_0\frac{KN_0}{R_0}\bar{\Psi}_{2s}(\underline{\xi})}\,d\underline{\xi}} -$$

$$-\frac{N_0}{R_0^{3z}}\frac{1}{\beta_0}\ln\int_{E^3} e^{-\beta_0\frac{KN_0}{R_0}\bar{\Psi}_{zs}(\underline{\xi})}\,d\underline{\xi}\,, \tag{65}$$

where $\bar{\Psi}_{zs}(\underline{\xi})$ satisfies the equation

$$\Psi_{zs}(\underline{\xi}) = \frac{\int_{E^3} K_S^{(z)}(\underline{\xi},\underline{\eta})\, e^{-\beta_0\frac{KN_0}{R_0}\Psi_{zs}(\underline{\eta})}\,d\underline{\eta}}{\int_{E^3} e^{-\beta_0\frac{KN_0}{R_0}\Psi_{zs}(\underline{\eta})}\,d\underline{\eta}}\,. \tag{66}$$

It is possible to show (see, e.g., ref.4) that since

$$K_S^{(z)}(\underline{\xi},\underline{\eta}) \xrightarrow[z\to\infty]{S\to\infty} \frac{1}{|\underline{\xi}-\underline{\eta}|}\,,$$

the function $K_S^{(z)}(\underline{\xi},\underline{\eta})$ converges to $\frac{1}{|\underline{\xi}-\underline{\eta}|}$, as $S\to\infty$ and $z\to\infty$:

$$\bar{\Psi}_{zs}(\underline{\xi}) \xrightarrow[z\to\infty]{S\to\infty} \bar{\Psi}(\underline{\xi}) \tag{67}$$

where $\bar{\Psi}(\underline{\xi})$ satisfies the equation

$$\Psi(\underline{\xi}) = \frac{\int_{E^3} \frac{1}{|\underline{\xi}-\underline{\eta}|}\, e^{-\beta_0\frac{KN_0}{R_0}\Psi(\underline{\eta})}\,d\underline{\eta}}{\int_{E^3} e^{-\beta_0\frac{KN_0}{R_0}\Psi(\underline{\eta})}\,d\underline{\eta}}\,. \tag{68}$$

Therefore the limit

$$\lim_{z\to\infty}\lim_{S\to\infty}\lim_{n\to\infty}\frac{1}{(n^{-1/3}R_0)^{3z}}F\Big[nN_0, E^3, n^{-1/3}\beta_0; H_{zs0}^{(n^{-1/3}R_0)}\big(\bar{a}_1^{(z,S,nN_0)},\ldots,\bar{a}_S^{(z,S,nN_0)}\big)\Big]$$

exists and is equal to

$$\lim_{z\to\infty}\lim_{S\to\infty}\lim_{n\to\infty}\frac{1}{(n^{-1/3}R_0)^{3z}}F\Big[nN_0, E^3, n^{-1/3}\beta_0; H_{zs0}^{(n^{-1/3}R_0)}\big(\bar{a}_1^{(z,S,nN_0)},\ldots,\bar{a}_S^{(z,S,nN_0)}\big)\Big] =$$

$$= -\frac{1}{\beta_0 R_0^{3z}}\ln\frac{1}{N_0!}\Big[\Big(\frac{2\pi m}{\beta_0}\Big)^{3/2}R_0^3\Big]^{N_0} -$$

$$-\frac{1}{2}\frac{N_0}{R_0^{3z}}\frac{KN_0}{R_0}\frac{\int_{E^3}\bar{\Psi}(\underline{\xi})\,e^{-\beta_0\frac{KN_0}{R_0}\bar{\Psi}(\underline{\xi})}\,d\underline{\xi}}{\int_{E^3} e^{-\beta_0\frac{KN_0}{R_0}\bar{\Psi}(\underline{\xi})}\,d\underline{\xi}} -$$

$$-\frac{N_0}{R_0^{3z}}\frac{1}{\beta_0}\ln\int_{E^3} e^{-\beta_0\frac{KN_0}{R_0}\bar{\Psi}(\underline{\xi})}\,d\underline{\xi} = f[N_0,R_0,\beta_0]\,. \tag{69}$$

Since

$$\frac{N}{R^{1+3z}}K = n^{-1}N_0R_0^6K \xrightarrow[n\to\infty]{} 0\,, \tag{70}$$

$$\frac{N^2}{R^{1+3z}}K = N_0R_0^6K = O(1)\,, \tag{71}$$

according to (53) we have

$$\frac{1}{R^{3z}} F[N,V,\beta;H] \xrightarrow[h\to\infty]{} f[N_0,R_0,\beta_0]. \qquad (72)$$

The smaller the quantity $\frac{N}{R^{1+3z}} = n^{-1} N_0 R_0^6$, the smaller the difference

$$\frac{1}{R^{3z}} F[N,V,\beta;H] - f[N_0,R_0,\beta_0].$$

Therefore at sufficiently small $\frac{N}{R^{1+3z}}$ we can set approximately

$$\frac{1}{R^{3z}} F[N,V,\beta;H] = f[N_0,R_0,\beta_0] \qquad (73)$$

and, if the quantity $N_0 R_0^6$ itself is small enough, we can take

$$\frac{1}{R_0^{3z}} F[N_0,V_0,\beta_0;H] = f[N_0,R_0,\beta_0]. \qquad (74)$$

Thus, we have finally

$$F[N_0,V_0,\beta_0;H] =$$

$$= -\frac{1}{\beta_0} \ln \frac{1}{N_0!}\left[\left(\frac{2\pi m}{\beta_0}\right)^{3/2} V_0\right]^{N_0} -$$

$$-\frac{1}{2} N_0 \frac{K N_0}{\left(\frac{3}{4\pi}V_0\right)^{1/3}} \frac{\int_{E^3} \bar{\Psi}(\underline{\xi})\, e^{-\beta_0 \frac{K N_0}{\left(\frac{3}{4\pi}V_0\right)^{1/3}} \bar{\Psi}(\underline{\xi})}\, d\underline{\xi}}{\int_{E^3} e^{-\beta_0 \frac{K N_0}{\left(\frac{3}{4\pi}V_0\right)^{1/3}} \bar{\Psi}(\underline{\xi})}\, d\underline{\xi}} -$$

$$-N_0 \frac{1}{\beta_0} \ln \frac{3}{4\pi} \int_{E^3} e^{-\beta_0 \frac{K N_0}{\left(\frac{3}{4\pi}V_0\right)^{1/3}} \bar{\Psi}(\underline{\xi})}\, d\underline{\xi}\ , \qquad (75)$$

where $\bar{\Psi}(\underline{\xi})$ satisfies the equation

$$\Psi(\underline{\xi}) = \frac{\int_{E^3} \frac{1}{|\underline{\xi}-\underline{\eta}|}\, e^{-\beta_0 \frac{K N_0}{\left(\frac{3}{4\pi}V_0\right)^{1/3}} \Psi(\underline{\eta})}\, d\underline{\eta}}{\int_{E^3} e^{-\beta_0 \frac{K N_0}{\left(\frac{3}{4\pi}V_0\right)^{1/3}} \Psi(\underline{\eta})}\, d\underline{\eta}}\ . \qquad (76)$$

The smaller is the quantity $N_0 V_0^2 = N_0^3 \frac{1}{n_0^2}$, i.e., the higher is the dencity of the electron gas n_0, the more exact is the expression (75) for the free energy of the system (1). The expression (75) is asymptotically exact at high densities.

It should be noted that to keep the gas classical one needs sufficiently high temperatures $\frac{1}{\beta_0}$. In this connection it is necessary to emphasize, that in taking the limit $N = n N_0$, $V = n^{-1} V_0$, $\beta = n^{-1/3}\beta_0$, $n \to \infty$, we are always in the range of the thermodynamic parameters, where our system is classical. In fact, if

$$\frac{2m}{\hbar} \gg \beta_0 \left(\frac{N_0}{V_0}\right)^{2/3}, \qquad (77)$$

we have also

$$\frac{2m}{\hbar} \gg \beta\left(\frac{N}{V}\right)^{2/3} = \beta_0\left(\frac{N_c}{V_0}\right)^{2/3}. \tag{78}$$

It is just the condition that is realized in hot plasma. Thus, for the system (1) is the jellium model, the obtained result (75),(76) is asymptotically exact value for the free energy of this model of plasma.

R e f e r e n c e s

1. И.Г.Петровский. Лекции по теории интегральных уравнений, Москва, Наука, 1965.
2. A.M.Kurbatov. JINR E5-12431, Dubna, 1979.
3. A.M.Kurbatov. JINR E5-12432, Dubna, 1979.
4. P.Hertel, W.Thirring. CERN TH. 1338, 1971.

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