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II. The Behaviour of the Hydrodynamic Modes at Large Times

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Барабаненков Ю.Н., Озрин В.Д., Шелест А.В. **E17 - 12301**

Влияние коллективных эффектов на временную асимптотику гидродинамических мод в нейтральных газах. II. Поведение гидродинамических мод при больших временах

Исследуются временные асимптотики гидродинамических нормальных мод газа твердых сфер. Показано, что учет эффектов, обусловленных взаимодействием частиц с коллективными возбуждениями в среде, приводит: (1) к появлению нерегулярных по волновому числу поправок к гидродинамическим частотам и (2) к замене чисто экспоненциальной временной асимптотики нормальной моды на экспоненциально-степенную.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Barabanenkov Yu.N., Ozrin V.D., Shelest A.V. **E17 - 12301**

Collective Effects Influence on the Time Asymptotics of the Hydrodynamic Modes in the Neutral Gases. II. The Behaviour of the Hydrodynamic Modes at Large Times

The time asymptotics of the hydrodynamic normal modes are investigated for the hard-sphere gas. It is shown that consideration of the effects caused by interaction of the particles with collective excitations in medium leads: 1) to the appearance of irregular, with respect to the wave number, corrections to the hydrodynamic frequencies; and 2) to the replacement of the purely exponential time asymptotics of the normal mode by the exponential-power one.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

The character of time dependence of the hydrodynamic normal mode $\chi^j_k(t)$* which may be represented by the integral of type

$$\frac{\chi^j_k(t)}{\chi^j_k(0)} = E^j(t) = \int_{-i\infty+\sigma_0}^{i\infty+\sigma_0} \frac{dz}{2\pi i} e^{zt} D_j^{-1}(k, z) \qquad (1.1)$$

is wholly defined by the analytical singularities of the function $D_j^{-1}(k, z)$ on the z-plane. As is known in the case of Navier-Stokes' hydrodynamics at small k the function $D_j^{-1}(k, z)$ has the simple poles, $D_j(k, z) = z - z_j(k)$, and the frequencies z (k) at low density approximation are computed with the help of the Boltzmann equation and are of the form:

$$z_{\eta_\alpha}(k) = z_\eta(k) = -k^2 D_\eta, \quad \alpha = 1,2; \quad z_T(k) = -k^2 D_T; \qquad (1.2)$$
$$z_\sigma(k) = -i\sigma ck - \frac{1}{2}\Gamma k^2, \quad \sigma = \pm 1;$$

*Here we use the notations adopted in /1/, and in the references to the formulae from this work we add figure 1.

$$D_\eta = \frac{5u}{16\sqrt{\pi}\, a_0^2 n}, \quad D_T = \frac{3}{2} D_\eta ,$$

$$\Gamma = \frac{2}{3} D_T + \frac{4}{3} D_\eta , \quad c = u\sqrt{5/3}, \quad u = \sqrt{\theta/m}.$$

Thus, all the modes damp exponentially.

In recent years the character of the autocorrelation functions damping at $t \to \infty$ was widely discussed. Alder's and Wainwright's computer experiments [2], performed in 1968, showed that at sufficiently large times the velocity correlation function for hard-sphere system behaved as $t^{-3/2}$. Then some authors [3,4] found out that the change of an exponential asymptotics of the autocorrelation function into power one could be obtained on the basis of the kinetic theory and was caused by the interaction of the particles with the long-wave collective excitations in a system described by hydrodynamic equations. In this connection the question arose, how such collective excitations affected the hydrodynamic mode behaviour at $t \to \infty$.

For the first time this problem was examined by Ernst and Dorfman [5]. Taking as a basis the first two equations of hierarchy BBGKY for the hard spheres [5,6], they built the linearized equations of hydrodynamics generalizing Navier-Stokes' equations and being true for the low-density case. The obtained in [5] expression for the hydrodynamic normal mode with wave number $k \ll k_0 \approx \lambda_0^{-1}$, where λ_0 is a free path, may be represented by the integral of type (1.1) with function $D_j(k, z)$ in the form:

$$D_j(k, z) = z - z_j(k) + k^2 U_j(k, z). \tag{1.3}$$

The time asymptotics of $\chi_k^j(t)$ is defined by behaviour of $U_j(k, z)$ at small k and $|z|$ ($|z| \ll ck_0$). In this region the main contribution in the function $U_j(k, z)$ - a sort of "mass operator" - gives the interaction with the intermediate ("virtual") hydrodynamic modes. As is shown in [1] for the calculation of the main terms of function $U_j(k, z)$ asymptotical expansion at $k, |z| \to 0$ it is sufficient to take into account only some combinations of the intermediate modes. Namely (1.3.3.), (1.3.4):

$$U_\eta(k, z) = U_\eta^{(\eta)}(k, z) + U_\eta^{(s)}(k, z); \; U_T(k, z) = U_T^{(\eta T)}(k, z) + U_T^{(s)}(k, z);$$
$$U_\sigma(k, z) = U_\sigma^{(\eta)}(k, z) + U_\sigma^{(\eta T)}(k, z) + U_\sigma^{(s)}(k, z). \tag{1.4}$$

Here $U_j^{(\eta)}, U_j^{(\eta T)}, U_j^{(s)}$ represent the sums of the following quantities

$$U_j^{(r,\ell)}(k,z) = \frac{u^2}{2n} \int_{q<k_0} \frac{d\vec{q}}{(2\pi)^3} \frac{|A_j^{(r,\ell)}(a\vec{k} - \vec{q},\; b\vec{k} + \vec{q})|^2}{z - z_r(a\vec{k} - \vec{q}) - z_\ell(b\vec{k} + \vec{q})} \tag{1.5}$$

over r, ℓ -indices, which take the following values accordingly: $r, \ell = \eta_\alpha, \eta_\beta$; $r, \ell = \eta_\alpha, T$; $r = \sigma$ $\ell = -\sigma$. Integrals (1.5) contain the arbitrary parameters a, b satisfying the condition $a + b = 1$, $|a|, |b| \sim 1$. However, according to [1] this arbitrariness does not affect the asymptotics of the function $U_j^{(r,\ell)}(k,z)$. The coefficients $A_j^{r,\ell}$ are computed in Appendix B.

The main result of [5] is that consideration of the influence of the collective effects leads to the nonanalytical dependence on k of the "renormalized" hydrodynamic frequencies $\bar{z}_j(k)$, e.g., of the solutions of the equations $D_j(k, z) = 0$. The nonanalyticity is of the form $\bar{z}_j(k) - z_j(k) \sim k^{5/2}$ and arises as a result of irregularity of the function $U_j(k, z)$ expansion at $k, |z| \to 0$. Another consequence is the existence of the cut-lines on the z-plane, where $D_j(k, z)$ as well as $U_j(k, z)$ looses its continuity (ref. [1], Appendix A). Thus, as has been noted in [1], integral (1.1) breaks down into two terms, the first of which is of exponential dependence on time, $E_1^j(t) \sim \exp\{t\bar{z}_j(k)\}$, and another one, $E_2^j(t)$, is the integral from the jump of function $D_j^{-1}(k,z)$ along the cut-line. Namely $E_2^j(t)$ determines the main asymptotics of the hydrodynamic mode $\chi_k^j(t)$ at $t \to \infty$.

In the next sections we'll give the concrete calculations for each of the modes and will make sure of the validity of this statement.

2. THE TIME ASYMPTOTICS OF THE VISCOUS MODE

Consider the function $U_\eta(k,z)$ incoming the denominator $D_\eta(k,z)$ of the inverse Laplace transformation (1.1) for the viscous mode $\chi_k^\eta(t)$. According to (1.4), (1.5) the contribution of the intermediate viscous modes in $U_\eta(k,z)$ is represented by the quantity $U_\eta^{(\eta)}(k,z)$ which may be written in the form

$$U_\eta^{(\eta)}(k,z) = \frac{u^2}{8\pi^2 n} \int_0^{k_0} \int_{-1}^{1} \frac{q^2 dq\, d\xi\, M_\eta^{(\eta)}(k^2/q^2, \xi)}{(z + k^2 D_\eta/2) + 2D_\eta q^2}, \qquad (2.1)$$

where $\xi = \cos(\hat{\vec{k}, \vec{q}})$, and the function $M_\eta^{(\eta)}(k^2/q^2, \xi)$ is defined in Appendix (B.5). On deriving of (2.1) we put in formula (1.5) the parameters $a = b = 1/2$.

As is shown in Appendix A /1/, the integral (2.1) is the regular function on the z-plane with a cut along the real axis,

$$y = 0, \quad -2D_\eta k_0^2 \lesssim x \lesssim -D_\eta k^2/2 \quad , \qquad (2.2)$$

crossing which the function's imaginary part changes by jump. Near the real axis the imaginary part of the function $U_\eta^{(\eta)}(k,z)$ is computer with the help of (1.A.7)-type relation and is of the form

$$\operatorname{Im} U_\eta^{(\eta)}(k, x \pm i0) =$$

$$= \mp \frac{u^2}{16\pi n}(2D_\eta)^{-3/2}\sqrt{|x + k^2 D_\eta/2|}\, M_\eta^{(\eta)}\left(\left|\frac{2D_\eta k^2}{x + D_\eta k^2/2}\right|\right)\Theta(-x - D_\eta k^2/2), \qquad (2.3)$$

where

$$M_\eta^{(\eta)}(p^2) = \int_{-1}^{1} d\xi\, M_\eta^{(\eta)}(p^2, \xi). \qquad (2.4)$$

Here $\Theta(x) = 1$ at $x > 0$ and $\Theta(x) = 0$ at $x < 0$. In the right-hand side (2.3) the other Θ-function, $\Theta(x + 2D_\eta k_0^2)$, is dropped out, because only the small values of $|x|$ are of interest.

The nature of the asymptotical expansion of $U_\eta^{(\eta)}(k,z)$ at $k, |z| \to 0$ depends on the parameter k^2/r behaviour, where

$$z + k^2 D_\eta/2 = 2D_\eta r e^{i\phi}, \quad |\phi| < \pi.$$

In the case $r \gg k^2$ we put $q = q'\sqrt{r}$ in the formula (2.1), replace the quantity $M_\eta^{(\eta)}(k^2/rq^2, \xi)$ by $M_\eta^{(\eta)}(0,\xi)$, and using (B.5) obtain

$$U_\eta^{(\eta)}(k,z) - U_\eta^{(\eta)}(0,0) = -\frac{u^2}{16\pi n}(2D_\eta)^{-3/2} M_\eta^{(\eta)}(0)\sqrt{z + k^2 D_\eta/2}, \qquad (2.5)$$

where $\sqrt{z}$ is defined as

$$\sqrt{z} = \sqrt{|z|}\exp\{i\phi/2\}, \quad |\phi| < \pi. \qquad (2.6)$$

With the help of substitution $q = kq'$ it is easy to show that in the region $r \sim k^2$ the quantity $U_\eta^{(\eta)}(k,z)$ at $k \to 0$ behaves as

$$U_\eta^{(\eta)}(k,z) - U_\eta^{(\eta)}(0,0) \sim \frac{u^2 k}{n D_\eta}. \qquad (2.7)$$

This estimation is valid also for the real part of the function $U_\eta^{(\eta)}(k,z)$ in the region $r \ll k^2$.

Consider now the function $U_\eta^{(s)}(k,z)$, which determines the contribution of the "virtual" sound modes in $U_\eta(k,z)$ and may be represented in the form:

$$U_\eta^{(s)}(k, z) =$$

$$= \frac{u^2}{8\pi^2 n} \int_0^{k_0} \int_{-1}^{1} \frac{q^2 M_\eta^{(s)}(k^2/q^2, \xi) dq\, d\xi}{(z + \Gamma k^2/4) + \Gamma q^2 + ic(|\vec{k}/2 - \vec{q}| - |\vec{k}/2 + \vec{q}|)}, \quad (2.8)$$

where $M_\eta^{(s)}(p^2, \xi)$ is the sum of the coefficient squares. This sum is introduced in the Appendix (B.5). In derivation (2.8) it is convenient again to choose in (1.5) $a = b = 1/2$.

According to the reasoning of the Appendix A /1/ the integral (2.8) is the continuous function of $z = x + iy$ regular outside the region

$$\Gamma k_0^2 \lesssim x \leq -\Gamma k^2/4, \quad |y| \leq ck, \quad (2.9)$$

and non-differentiable inside it*.

In order to calculate the asymptotics of $U_\eta^{(s)}(k, z)$ at $k, |z| \to 0$ it is convenient to perform in the integral (2.8) the substitution of the variable $q = q'\sqrt{k}$. Then this integral yields:

$$U_\eta^{(s)}(k, z) - U_\eta^{(s)}(0, 0) =$$

$$= \frac{u^2 \sqrt{k}}{8\pi^2 n \Gamma} \int_0^\infty \int_{-1}^{1} dq\, d\xi \left\{ \frac{q^2 M^{(s)}(k/q^2, \xi)}{\frac{r}{k} e^{i\phi} + q^2 + \frac{ic}{\Gamma}\left(\left|\frac{\vec{k}}{2k} - \frac{\vec{q}}{\sqrt{k}}\right| - \left|\frac{\vec{k}}{2k} + \frac{\vec{q}}{\sqrt{k}}\right|\right)} - M_\eta^{(s)}(0, \xi) \right\}, \quad (2.10)$$

*Note, that the region (2.2) and (2.9) as well as the nature of the nonanalyticity of the functions $U_\eta^{(\eta)}(k, z)$ and $U_\eta^{(s)}(k, z)$ don't depend on the choice of the constants a, b, $a + b = 1$.

where $z + \Gamma k^2/4 = \Gamma r e^{i\phi}$. The integral (2.10) asymptotics depends evidently on the relation of the parameters r/k and k. For the cases $r/k \lesssim k \ll k_0$ we put under the integral sign $r/k = 0$ and pass on to the limit $k \to 0$. After easy calculations we get

$$U_\eta^{(s)}(k, z) - U_\eta^{(s)}(0, 0) = -\frac{u^2}{77\pi n} \sqrt{\frac{ck}{2\Gamma^3}}. \quad (2.11)$$

On condition that $r/k \gg k$ or $|z + \Gamma k^2/4| \gg \Gamma k^2$ under the sign of the integral (2.10) one could pass on to the limit $k \to 0$ at fixed r/k. After that the integral may be computed with the help of (B.5) at any $(z + \Gamma k^2/4)$. The explicit expression for $\{U_\eta^{(s)}(k, r) - U_\eta^{(s)}(0, 0)\}$ which we needn't further has the above-mentioned analytical properties*.

So, from the definition of the functions $U_\eta^{(\eta)}(k, z)$ (2.1) and $U_\eta^{(s)}(k, z)$ (2.8) it follows that the function $D_\eta(k, z)$ with $U_\eta(k, z) = U_\eta^{(\eta)} + U_\eta^{(s)}$ is continuous on the z-plane with the cut (2.2) along the real axis, crossing which $\mathrm{Im}\, U_\eta^{(\eta)}(k, z)$ changes by jump. The solution of the equation $D_\eta(k, z) = 0$ in the low-density approximation can be obtained by the iteration method if one takes into account that the function $U_\eta(k, z)$ as well as $U_\eta(0, 0)$ (1.3.1) is proportional to the small parameter $(na_0^3)^2$, $U_\eta(0, 0) \sim D_\eta (na_0^3)^2$. We find

$$\bar{z}_\eta = \bar{x}_\eta + i\bar{y}_\eta = -k^2 D_\eta - k^2 U_\eta(k, -k^2 D_\eta + i0\, \mathrm{sgn}\, y). \quad (2.12)$$

This equality has a sense of an iteration solution because the imaginary part of the function $U_\eta(k, z)$ satisfies the condition

*Note that the functions $U_\eta^{(s)}(, z)$ and $U_T^{(s)}(k, z)$ calculated in /7/ lose their continuity on the lines $y = \pm ick$, $x \leq -\Gamma k^2/4$. But this fact doesn't correspond to the rigorous calculation.

$$\mathrm{Im}\,U_\eta(k, x+iy) = -\mathrm{sgn}\,y\,|\mathrm{Im}\,U_\eta(k, x+iy)|.$$

Hence with the help of the computed asymptotics (2.7) and (2.11) we get the expression for the real coordinate of the pole,

$$\bar{x}_\eta = -k^2 D_\eta (1-\Delta_\eta),$$
$$\Delta_\eta = \frac{u^2}{77\pi n D_\eta}\sqrt{\frac{ck}{2l^3}}. \qquad (2.13)$$

In (2.13) the quantity $U_\eta(0,0)$ is dropped out, as it gives the nonessential correction into the kinematical viscosity coefficient D_η. This result was obtained in /5/. The contribution into the imaginary coordinate expression is given only by $U_\eta^{(\eta)}(k,z)$, as $\mathrm{Im}\,U_\eta^{(s)}(k,z) \to 0$ at $y \to 0$. From (2.3) it follows

$$\bar{y}_\eta = \pm\frac{2}{3}k^2 D_\eta \gamma_\eta, \quad \gamma_\eta = \frac{u^2 k}{32\pi n D_\eta^2}. \qquad (2.14)$$

Thus, the integral (1.1), representing the function $\chi_k^\eta(t)$, breaks down into two parts

$$\frac{\chi_k^\eta(t)}{\chi_k^\eta(0)} = E^\eta(t) = E_1^\eta(t) + E_2^\eta(t). \qquad (2.15)$$

The first term contains the contribution from the poles of the function $D_\eta^{-1}(k,z)$ and is of the form

$$E_1^\eta(t) = 2\cos\left(\frac{2}{3}\gamma_\eta\tau\right)\exp\{-\tau(1-\Delta_\eta)\}, \qquad (2.16)$$

where the dimensionless time $\tau = k^2 D_\eta t$ is introduced. The second term is the integral around the cut (2.2). Using (2.3) we obtained after some transformations the expression for $E_2^\eta(t)$ as the function of τ,

$$E_2^\eta(t) = -e^{-\tau/2}\int_0^\infty \frac{dx}{\pi}\,\frac{\gamma_\eta\sqrt{x/2}\,M_\eta^{(\eta)}(2/x)\,e^{-\tau x}}{[x-1/2+\Delta_\eta]^2 + [\gamma_\eta\sqrt{x/2}\,M_\eta^{(\eta)}(2/x)]^2} \qquad (2.17)$$

The integral (2.17) depends on two parameters: the dimensionless time τ and quantity γ_η (2.14) estimated as

$$\gamma_\eta = \frac{u^2 k_0}{32\pi n D_\eta^2}\left(\frac{k}{k_0}\right) \sim (na_0^3)^2\left(\frac{k}{k_0}\right). \qquad (2.18)$$

We can see that γ_η is small in the low-density approximation, $na_0^3 \ll 1$, for the wave numbers $k \ll k_0$. For finite τ we pass on in the integral (2.17) and in (2.16) to the limit $\gamma_\eta \to 0$ and get

$$E^\eta(t) = \exp\{-\tau(1-\Delta_\eta)\}. \qquad (2.19)$$

Note that in the exponent index we retain the quantity Δ_η (2.13), as its simple estimation gives $\gamma_\eta/\Delta_\eta \sim \sqrt{k/k_0}$.

At $\tau \to \infty$ and γ_η fixed the main asymptotics of the integrals of the (2.17) type is calculated, as is known, by the integrand expansion into the asymptotical series near $x=0$. According to (2.4) and (B.5) the function $M_\eta^{(\eta)}(p^2)$ at $p^2 \to \infty$ behaves as

$$M_\eta^{(\eta)}(p^2) \approx \frac{16}{3p^2}. \qquad (2.20)$$

Hence for the integral (2.17) we obtain

$$E^\eta(t) \approx E^\eta(t) = \frac{8}{\sqrt{2\pi}}\gamma_\eta \tau^{-5/2} e^{-\tau/2}. \qquad (2.21)$$

In this limit the function $E_1^{\eta}(t)$ (2.16) gives the exponentially small correction to $E^{\eta}(t)$.

In the case when both parameters change, $\tau \to \infty$ and $\gamma_\eta \to 0$, the contribution into the integral is given on the whole by the regions $x \leq 1/\tau$ and $|x - 1/2| \leq \gamma_\eta$. So the asymptotics (2.17) depends on the relation between τ and γ_η. It is possible to show that for the times

$$\tau = k^2 D_\eta t >> 2\ln(1/\gamma_\eta) \sim -2\ln\{(n a_0^3)^2 \frac{k}{k_0}\} \qquad (2.22)$$

the main contribution into (2.17) is given by the integration over the first of the mentioned regions, and the power part of the function $E^{\eta}(t)$ asymptotics begins to play the dominating role.

Thus, at sufficiently small (in the $1/k^2 D_\eta$ scale) times the viscous mode damps exponentially (2.19) and taking into account of the influence of the collective motions in gas leads to the appearance of the nonanalytical dependence of the viscosity effective coefficient on the wave number, $\tilde{D}_\eta = D_\eta + a k^{1/2}$. For large times the asymptotics of $\chi_k^{\eta}(t)$ changes and acquires the exponentially-power character (2.21).

3. THE TIME ASYMPTOTICS OF THE HEAT MODE

Let us consider the analytical properties of the function $U_T(k, z) = U_T^{(s)} + U_T^{(\eta T)}$ (1.4). On deriving $U_T^{(s)}(k, z)$ put in (1.5) $a = b = 1/2$ and get

$$U_T^{(s)}(k, z) = \frac{u^2}{8\pi^2 n} \int_0^{k_0}\int_{-1}^{1} \frac{q^2 M_T^{(s)}(k^2/q^2, \xi)\,dq\,d\xi}{z + \Gamma k^2/4 + \Gamma q^2 + ic\{|\vec{k}/2 - \vec{q}| - |\vec{k}/2 + \vec{q}|\}}, \qquad (3.1)$$

where the sum of the coefficient $A_T^{\sigma,-\sigma}(\vec{k}/2 - \vec{q}, \vec{k}/2 + \vec{q})$ squares is designated as $M_T^{(s)}(k^2/q^2, \xi)$ (B.5).

To simplify the investigation of the function $U_T^{(\eta T)}(k, z)$ it is convenient to choose the parameters a and b, corresponding to the arguments of the hydrodynamic frequencies $z_\eta(a\vec{k} - \vec{q})$ and $z_T(b\vec{k} + \vec{q})$ in (1.5), in the form:

$$a = \frac{D_T}{2D'}, \quad b = \frac{D_\eta}{2D'}, \quad D' = \frac{D_\eta + D_T}{2}. \qquad (3.2)$$

Then the function $U_T^{(\eta T)}(k, z)$ is written down

$$U_T^{(\eta T)}(k, z) = \frac{u^2}{8\pi^2 n} \int_0^{k_0}\int_{-1}^{1} \frac{q^2 M_T^{(\eta T)}(a^2k^2/q^2, \xi)\,dq\,d\xi}{(z + 2abD'k^2) + 2D'q^2}, \qquad (3.3)$$

where the quantity $M_T^{(\eta T)}(p^2, \xi)$ is defined in the Appendix (B.5).

It is easy to see that the integrals (3.1) and (3.3) transform into the expressions (2.8) and (2.1) respectively, if one performs the following substitutions:

$U_T^{(\eta T)} \Rightarrow U_\eta^{(\eta)}$ when $M_T^{(\eta T)}(a^2k^2/q^2, \xi) \Rightarrow M_\eta^{(\eta)}(k^2/q^2, \xi)$, $(z + 2abD'k^2) \Rightarrow (z + D_\eta k^2/2)$, $D' \Rightarrow D_\eta$;

$U_T^{(s)} \Rightarrow U_\eta^{(s)}$ when $M_T^{(s)}(p^2, \xi) \Rightarrow M_\eta^{(s)}(p^2, \xi)$. (3.4)

This symmetry allows us to use the main part of the previous chapter results directly for the considered case.

So, using (3.4), we can affirm at once that the function $U_T(k, z)$ (and so $D_T(k, z)$) is continuous on the z-plane with cut along the real axis,

$$y=0,\quad -2D'k_0^2 \lesssim x \le -2abD'k^2 = -\frac{D_\eta D_T}{D_\eta + D_T} k^2, \qquad (3.5)$$

crossing which the imaginary part of $U_T^{(\eta T)}(k, z)$ changes by jump. (It is easy to show that the analytical properties of the functions $U_T^{(s)}(k, z)$ and $U_T^{(\eta T)}(k, z)$, their nonanalyticity regions, in particular, don't depend on the choice of the parameters a, b).

The imaginary part of the function $U_T^{(\eta T)}(k,z)$ near the real axis accordingly to (3.4) and (2.3) is of the form

$$\mathrm{Jm}\, U_T^{(\eta T)}(k, x \pm i0) =$$

$$(3.6)$$

$$= \mp \frac{u^2 a k}{32\pi n D'} \sqrt{\left|\frac{x+2abD'k^2}{2D'a^2k^2}\right|}\, M_T^{(\eta T)}\left(\left|\frac{2D'a^2k^2}{x+2abD'k^2}\right|\right)\Theta(-x-2abD'k^2),$$

where the function $M_T^{(\eta T)}(p^2)$ is computed with the help of (B.5),

$$M_T^{(\eta T)}(p^2) = \int_{-1}^{1} d\xi\, M_T^{(\eta T)}(p^2, \xi). \qquad (3.7)$$

As all the estimations of the behaviour of the functions $U_\eta^{(\eta)}$ and $U_\eta^{(s)}$ at small k and $|z|$ are valid, owing to (3.4), also for $U_T^{(\eta T)}(k, z)$, $U_T^{(s)}(k, z)$, one can immediately get the expression for the roofs of the equation $D_T(k, z) = 0$ in the form similar to (2.12). For the real coordinate of the pole one gets /5/

$$\bar{x}_T = -k^2 D_T (1 - \Delta_T), \qquad (3.8)$$

$$\Delta_T = \frac{u^2}{21\pi n D_T}\sqrt{\frac{ck}{2\Gamma^3}}.$$

The expression for the pole imaginary coordinate can be found from (3.6) at $x = -k^2 D_T = -2aD'k^2$,

$$\bar{y}_T = \pm k^2 D_T \gamma_T, \qquad \gamma_T = \frac{u^2 k}{32\pi n D'^2}. \qquad (3.9)$$

Because of the equivalence of the analytic properties of the functions $D_\eta(k,z)$ and $D_T(k,z)$ the behaviour of the heat $\chi_k^T(t)$ and viscous $\chi_k^\eta(t)$ modes at $t \to \infty$ is greatly similar. Really the function $\chi_k^T(t)$ breaks into two parts

$$\frac{\chi_k^T(t)}{\chi_k^T(0)} = E^T(t) = E_1^T(t) + E_2^T(t). \qquad (3.10)$$

The first of these parts contains the contribution from the integrand $D_T^{-1}(k,z)$ poles of the inverse Laplace transformation (1.1) and is of the form

$$E_1^T(t) = 2\cos(\gamma_T \tau)\exp\{-\tau(1-\Delta_T)\}. \qquad (3.11)$$

Here we introduce the dimensionless time $\tau = k^2 D_T t$. The second term in the right-hand side of (3.10) represents the integral around the cut (3.5) and is written down with the help of (3.6) as

$$E_2^T(t) =$$

$$= -a e^{-\tau b} \int_0^\infty \frac{dx}{\pi} e^{-\tau a x} \frac{\frac{1}{2}\gamma_T \sqrt{x}\, M_T^{(\eta T)}(1/x)}{[ax - a + \Delta_T]^2 + [\frac{1}{2}\gamma_T \sqrt{x}\, M_T^{(\eta T)}(1/x)]^2}, \qquad (3.12)$$

where the constants a and b are defined in (3.2).

As in the case of the viscous mode (2.18), the quantity γ_T is the small parameter. Therefore at finite τ (and $\gamma_T \to 0$) the main contribution into the integral (3.12) gives the region $x \sim 1$ and the time dependence of the heat mode is exponential,

$$E^T(t) \approx \exp\{-\tau(1-\Delta_T)\}. \quad (3.13)$$

In large time limit $\tau \to \infty$ the expression (3.11) may be neglected, and the integral (3.12) is calculated with the help of the function $M_T^{(\eta T)}(1/x)$ expansion at $x \to 0$, which according to (3.7) and (B.5) is of the form

$$M_T^{(\eta T)}(p^2 \to \infty) \approx \frac{8}{3p^2}. \quad (3.14)$$

Put this expression in (3.12) and get

$$E^T(t) \approx E_2^T(t) \approx -\frac{1}{a\sqrt{\pi}}\gamma_T(a\tau)^{-5/2}e^{-b\tau}. \quad (3.15)$$

This power asymptotics becomes the leading one at the times estimated by inequality (2.22).

4. TIME ASYMPTOTICS OF THE SOUND MODE

After we have considered the shear and heat modes, the investigation of the sound mode $\chi_k^\sigma(t)$ time dependence is not difficult. The function $U_\sigma(k,z)$ included in the denominator

$$D_\sigma(k,z) = z + i\sigma ck + \Gamma k^2/2 + k^2 U_\sigma(k,z)$$

of the inverse transformation for $\chi_k^\sigma(t)$ is defined by the relations (1.4). Note that each of the terms in the sum $U_\sigma = U_\sigma^{(\eta)} + U_\sigma^{(s)} + U_\sigma^{(\eta T)}$, which are defined by the integrals of (1.5) type, differs from $U_\eta^{(\eta)}(k,z)$, $U_\eta^{(s)}(k,z)$ and $U_T^{(\eta T)}(k,z)$, respectively, only by the coefficient $A_j^{r,\ell}(\vec{q},\vec{q}\,')$. The simple transformation shows that:

$U_\eta^{(\eta)}$ turns into $U_\sigma^{(\eta)}$ at the replacement $M_\eta^{(\eta)}(p^2,\xi)$ by $M_\sigma^{(\eta)}(p^2,\xi)$;
$U_\eta^{(s)}$ turns into $U_\sigma^{(s)}$ at the replacement $M_\eta^{(s)}(p^2,\xi)$ by $M_\sigma^{(s)}(p^2,\xi)$;

$$U_T^{(\eta T)}(k,z) = 3U_\sigma^{(\eta T)}(k,z). \quad (4.1)$$

Here the quantities $M_\sigma^{(\eta)}(p^2,\xi)$ and $M_\sigma^{(s)}(p_\ell^2,\xi)$ are the sums of the square coefficients $A_\sigma^{r,\ell}(\vec{q},\vec{q}\,')$ calculated in the Appendix B. The last equality (4.1) follows from (B.5).

The analytical properties of the functions $U_\eta^{(\eta)}(k,z)$ (2.1), $U_\eta^{(s)}(k,z)$ (2.8) and $U_T^{(\eta T)}(k,z)$ (3.3) are defined by the form of the denominators in the integrands. Therefore from the results of the previous sections it follows that $U_\sigma(k,z)$ and $D_\sigma(k,z)$ are continuous functions on the z-plane with the cut along the real axis $y=0$,

$$x \le -k^2 \min\{D_\eta/2;\ \frac{D_\eta D_T}{D_\eta + D_T}\} = -k^2 D_\eta/2. \quad (4.2)$$

Here we have used the relations between the coefficients in the low-density approximation (1.2).

The equation for the function $D_\sigma^{-1}(k,z)$ poles is solved by the iteration method,

$$\bar{z}_\sigma(k) = -i\sigma ck - k^2\Gamma/2 - k^2\{\delta U_\sigma^{(\eta)}(k,-i\sigma ck) +$$

$$+\delta U_\sigma^{(\eta T)}(k,-i\sigma ck) + \delta U_\sigma^{(s)}(k,-i\sigma ck)\},$$

where $\delta U(k,z) = U(k,z) - U(0,0)$. Here, as in sections 2 and 3 the quantity $U_\sigma(0,0)$ is dropped out. The expressions for $\delta U_\sigma^{(\eta)}(k,-i\sigma ck)$ and $\delta U_\sigma^{(\eta T)}(k,-i\sigma ck)$ follow from (2.5) if we use the conditions (4.1) and (3.4). The quantity $\delta U_\sigma^{(s)}(k,-i\sigma ck)$ is found with the help of (2.10), where it is necessary to put $r = ck$, $\phi = -\pi/2$, pass onto the limit $k \to 0$ under the integral sign and, finally, to perform the replacement (4.1). As a result, we get

$$\bar{z}_\sigma(k) = -i\sigma ck - k^2\Gamma/2 +$$

$$+\frac{u^2\sqrt{ck}}{16\pi n}k^2 e^{-i\pi\sigma/4}\left\{\frac{M_\sigma^{(\eta)}}{(2D_\eta)^{3/2}}+\frac{M_\sigma^{(\eta T)}}{(D_\eta+D_T)^{3/2}}+\frac{M_\sigma^{(s)}}{\Gamma^{3/2}}\right\}, \qquad (4.3)$$

where the constants are computed with the help of (B.5) and have the following values:

$$M_\sigma^{(\eta)} = \int_{-1}^{1} d\xi\, M_\sigma^{(\eta)}(0,\xi) = \frac{4\cdot 14}{45},$$

$$M_\sigma^{(\eta T)} = \frac{1}{3}\int_{-1}^{1} d\xi\, M_T^{(\eta T)}(0,\xi) = \frac{8}{9}, \qquad (4.4)$$

$$M_\sigma^{(s)} = \int_{0}^{2} d\xi\sqrt{\xi}\, M_\sigma^{(s)}(0,\xi-1) = \frac{2^{11}\sqrt{2}}{3\cdot 5\cdot 7\cdot 9\cdot 11}.$$

These results coincide with [5].

Thus, in this case also the integral of (1.1) type, representing the function $\chi_k^\sigma(t)$, splits into two parts

$$\frac{\chi_k^\sigma(t)}{\chi_k^\sigma(0)} = E^\sigma(t) = E_1^\sigma(t) + E_2^\sigma(t), \qquad (4.5)$$

where

$$E_1^\sigma(t) = \exp\{\bar{z}_\sigma(k)t\}. \qquad (4.6)$$

The second term on the right-hand side of (4.5) is the integral around the cut (4.2) and can be written in the form

$$E_2^\sigma(t) =$$

$$= -\int_{-\infty}^{-k^2D_\eta/2}\frac{dx}{\pi}\,\frac{k^2|\,\mathrm{Im}U_\sigma(k,x\pm i0)|\,e^{xt}}{[x+i\sigma ck+\Gamma k^2/2+k^2\mathrm{Re}U_\sigma(k,x)]^2+[k^2\mathrm{Im}U_\sigma(k,x)]^2}. \qquad (4.7)$$

Here the imaginary part of the function $U_\sigma(k,z)$ near the real axis can be defined by the formulae (2.3), (3.6) and by relation (4.1),

$$\mathrm{Im}U_\sigma(k, x\pm i0) =$$

$$= \mp D_\eta\gamma_\eta\sqrt{\left|\frac{x+D_\eta k^2/2}{2D_\eta k^2}\right|}\,M_\sigma^{(\eta)}\left(\left|\frac{2D_\eta k^2}{x+D_\eta k^2/2}\right|\right)\Theta(-x-D_\eta k^2/2)\mp \qquad (4.8)$$

$$\mp\frac{1}{3}D'\gamma_T a\sqrt{\left|\frac{x+2abD'k^2}{2D'a^2k^2}\right|}\,M_\sigma^{(\eta T)}\left(\left|\frac{2D'a^2k^2}{x+2abD'k^2}\right|\right)\Theta(-x-2abD'k^2),$$

where the constants a, b, D' are given in (3.2), and the quantities γ_η (2.14) and γ_T (3.9) are proportional to the small parameter of the problem.

In contradistinction to (2.17) or (3.12) in the integrand denominator (4.7) stands the complex quantity. Therefore, at the finite dimentionless time $\tau \sim k^2D_\eta t$ and $\gamma_\eta, \gamma_T \to 0$ the integral (4.7) disappears and time dependence of the sound mode becomes of the exponential character (4.6). In the limit $\tau\to\infty$ the integral (4.7) asymptotics is computed by the expansion of the fraction under the integral sign around $x=-k^2D_\eta/2$. In the vicinity of this real axis point, due to (4.2), only the first terms on the right-hand side of (4.8) differ from zero. According to (B.5) at $p^2 \to \infty$ the function $M_\sigma^{(\eta)}(p^2)$ behaves as

$$M_\sigma^{(\eta)}(p^2) = \int_{-1}^{1} d\xi M_\sigma^{(\eta)}(p^2, \xi) = \frac{4}{9} + O(1/p^2). \qquad (4.9)$$

Use this expression. Conserve in the denominator (4.7) only the most "powerfull" at $k \to 0$ terms, and get

$$E_2^\sigma(t) \approx \frac{1}{72}\left(\frac{k^3u^2}{c^2n}\right)(2\pi k^2 D_\eta t)^{-3/2} \exp\{-D_\eta k^2 t/2\}. \qquad (4.10)$$

One can show that this asymptotics plays the main part and $E^\sigma \approx E_2^\sigma$ if

$$(\Gamma - D_\eta)k^2 t \gg -2\ln(k^3u^2/nc^2). \qquad (4.11)$$

Here the times are much larger than in the case of the viscous or heat modes (2.22) as the parameter

$$\frac{u^2k^3}{nc^2} \sim (na_0^3)^2\left(\frac{k}{k_0}\right)^3 \sim \gamma_\eta\left(\frac{k}{k_0}\right)^2 .$$

Mention the essential distinction of the asymptotics (4.10) from the corresponding results for the viscous and heat modes: at $t \to \infty$ the function $E_2^\sigma(t)$ is proportional to $t^{-3/2}$, but not to $t^{-5/2}$. The index t in the asymptotics is defined by the function $M_\sigma^{(\eta)}(p^2)$ behaviour which at $p^2 \to \infty$ doesn't disappear (4.9) in contradiction to $M_\eta^{(\eta)}(p^2)$ (2.20) or $M_T^{(\eta T)}(p^2)$ (3.14). This circumstance is connected with the kind of "forbiddance rules" due to which the viscous mode with the wave vector $\vec{k}$ cannot split into two "virtual" modes with the wave vectors $\vec{k}/2$ and $\vec{k}/2$ and the coefficient $A_\eta^{\eta a, \eta\beta}(\vec{k}, \vec{k}) = 0$ (B.3). As to sound mode, such a split is permitted and $A_\sigma^{\eta a, \eta\beta}(\vec{k}, \vec{k}) \neq 0$.

6. CONCLUDING REMARKS

Thus, we have shown that each of the hydrodynamic normal modes time dependence at sufficiently large times changes and turns from purely exponential into exponential-power one. This effect is conditioned by existence of the cuts (jumps) in the kinetic equation collision integral, considered as a function of the complex variable z. In turn the cuts appear as a result of the integration over the wave vectors of the functions, which have the poles in the z-plane and represent (in the quantum theory language) the "free" hydrodynamic modes propagators. So the change of the time asymptotics behaviour at large t, as well as the nonregularity of the corrections in wave number **k** to the hydrodynamic frequencies ($\sim k^{5/2}$), is connected directly with the consideration of the collective motions in the gas.

The leading principle, which helped us to obtained the main results of the work, was to pick out only the main terms of real and imaginary parts of the functions $D_j(k, z)$ expansion in the region of small k and $|z|$ with the coefficients, computed in the low-density limit. In this connection it should be noted that the specifying of the approximation, in the frame of which from the generalized kinetic equation (1.1.5) the expressions for the hydrodynamic modes (1.3), (1.4), (1.5) were obtained apparently, will not affect the location of the cut-line of $D_j(k,z)$ but will give only density corrections to the value of $D_j(k,z)$ jump on the cut. So there is a hope that the effect of the change of the damping character of hydrodynamic modes at large times conserves also for moderately dense gas.

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APPENDIX B

The coefficient functions $A_j^{r,\ell}(\vec{q},\vec{q}')$ defined in (1.2.13), represent the integrals of the form

$$A_j^{r,\ell}(\vec{q},\vec{q}')=(ku)^{-1}\int d\vec{v}\,\phi(v)\,a_{\vec{k}}^{j}(\vec{v})\,a_{\vec{q}}^{r}(\vec{v})\,a_{\vec{q}'}^{\ell}(\vec{v})\,\{\vec{k}\vec{v}-k\,\omega_j\},\qquad (B.1)$$

where $\phi(v)=(2\pi u^2)^{-3/2}\exp\{-v^2/2u^2\}$ is the Maxwell distribution, and ω_j takes the values

$$\omega_T=\omega_{\eta_1}=\omega_{\eta_2}=0,\qquad \omega_\sigma=\sigma c\,.$$

The quantities $a_{\vec{q}}^{j}(\vec{v})$ are the eigenfunctions of the linearized Boltzmann equation, computed in the low-wave-number approximation, $k\to 0$,

$$a_{\vec{q}}^{\eta_a}(\vec{v})=\vec{g}^{(a)}\vec{v}/u,\quad a=1,2;$$

$$a_{\vec{q}}^{T}(\vec{v})=\frac{1}{\sqrt{10}}(v^2/u^2-5);\qquad (B.2)$$

$$a_{\vec{q}}^{\sigma}(\vec{v})=\frac{v^2}{u^2\sqrt{30}}+\sigma\vec{g}^{(3)}\vec{v}/u\sqrt{2},\quad \sigma=\pm1.$$

Here we introduce the unit orthonormalized reper, connected with the wave number $\vec{q}$,

$$\vec{q}=\vec{g}^{(3)}q;\quad \sum_{i=1}^{3}g_j^{(i)}g_\ell^{(i)}=\delta_{j\ell}\ ;\quad \sum_{i=1}^{3}g_i^{(j)}g_i^{(\ell)}=\delta_{j\ell}\ .$$

The lower indices number the vector projections onto the coordinate system $\{x_1,x_2,x_3\}$.
We introduce another orthoreper $\vec{f}^{(1)},\vec{f}^{(2)},\vec{f}^{(3)}$ connected similar to (B.2) with the wave vector $\vec{q}'$, $\vec{q}'=\vec{f}^{(3)}q'$, and will suppose everywhere that the vector $\vec{k}$ is parallel to the axis x_3, $k=k_3$. Then all the integrals (B.1) can be easily computed. We give here the expressions for some of them:

$$A_{\eta_a}^{\eta_\beta,\eta_\gamma}(\vec{q},\vec{q}')=g_a^{(\beta)}f_3^{(\gamma)}+g_3^{(\beta)}f_a^{(\gamma)};$$

$$A_{\eta_a}^{\sigma,\sigma'}(\vec{q},\vec{q}')=\frac{1}{2}\sigma\sigma'\{g_a^{(3)}f_3^{(3)}+g_3^{(3)}f_a^{(3)}\};$$

$$A_{T}^{\sigma,\sigma'}(\vec{q},\vec{q}')=\frac{1}{\sqrt{6}}\{\sigma g_3^{(3)}+\sigma' f_3^{(3)}\};$$

$$A_{T}^{\eta_a,T}(\vec{q},\vec{q}')=g_3^{(a)};$$

$$A_{\sigma}^{\eta_a,\eta_\beta}(\vec{q},\vec{q}')=\sigma\sqrt{2}\{g_3^{(a)}f_3^{(\beta)}-\frac{1}{3}\vec{g}^{(a)}\vec{f}^{(\beta)}\};\qquad (B.3)$$

$$A_{\sigma}^{\sigma',\sigma''}(\vec{q},\vec{q}')=\frac{\sigma' g_3^{(3)}+\sigma'' f_3^{(3)}}{3\sqrt{2}}+\frac{\sigma\sigma'\sigma''}{3\sqrt{2}}\times$$

$$\times\{3g_3^{(3)}f_3^{(3)}-\vec{g}^{(3)}\vec{f}^{(3)}\};$$

$$A_{\sigma}^{\eta_a,T}(\vec{q},\vec{q}')=\frac{1}{\sqrt{3}}A_{T}^{\eta_a,T}(\vec{q},\vec{q}').$$

With the help of these relations and the condition of orthonormality of the vectors $\vec{g}^{(i)}$ and $\vec{f}^{(i)}$ it is easy to get the expressions for the sums of square coefficients $A_j^{r,\ell}(\vec{q},\vec{q}')$ which appear at calculation of asymptotics. Denote

$$\xi=\cos(\widehat{\vec{k},\vec{q}}),\quad p=k/q\qquad (B.4)$$

and get

$$M_j^{(\eta)}(p^2,\xi)=\sum_{\alpha,\beta=1,2}|A_j^{\eta_\alpha,\eta_\beta}(\vec{k}/2-\vec{q},\vec{k}/2+\vec{q})|^2,$$

$$M_\eta^{(\eta)}(p^2,\xi)=\frac{(1-\xi^2)(p^2/4+1+2\xi^2)}{(p^2/4+1)^2-p^2\xi^2}=\sum_{\alpha=1,2}M_{\eta_\alpha}^{(\eta)}(p^2,\xi),$$

$$M_\sigma^{(\eta)}(p^2,\xi)=\frac{1}{18}\;\frac{p^4/2+(6-10\xi^2)p^2+4(5-12\xi^2+9\xi^4)}{(p^2/4+1)^2-p^2\xi^2},$$

$$M_j^{(s)}(p^2,\xi)=2|A_j^{+1,-1}(\vec{k}/2-\vec{q},\vec{k}/2+\vec{q})|^2,$$

$$M_\eta^{(s)}(p^2,\xi)=\frac{\xi^2(1-\xi^2)}{(p^2/4+1)^2-p^2\xi^2}=\sum_{\alpha=1,2}M_{\eta_\alpha}^{(s)}(p^2,\xi),$$

$$M_T^{(s)}(p^2,\xi)=\frac{2}{3}\left\{\frac{(p^2/4+\xi^2)(p^2/4+1)-p^2\xi^2}{(p^2/4+1)^2-p^2\xi^2}-\frac{p^2/4-\xi^2}{\sqrt{(p^2/4+1)^2-p^2\xi^2}}\right\},\qquad (B.5)$$

$$M_\sigma^{(s)}(0,\xi)=(\xi-\sigma)^2(\xi+\sigma/3)^2,$$

$$M_T^{(\eta T)}(a^2p^2,\xi)=2\sum_{\alpha=1,2}|A^{\eta_\alpha,T}(a\vec{k}-\vec{q},b\vec{k}+\vec{q})|^2=$$

$$=3M_\sigma^{(\eta T)}(a^2p^2,\xi)=\frac{2(1-\xi^2)}{a^2p^2+1-2ap\xi}.$$

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