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E17 - 11822

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KINETIC EQUATIONS FOR THE ELECTRON-PHONON SYSTEM

1978

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Кинетические уравнения для электрон-фононной системы

Рассматривается взаимодействие электрона с фононным полем и внешним электрическим полем. Получены точные соотношения, которые после применения надлежащей аппроксимационной процедуры переходят в уравнения описывающие электрон-фононной системы.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1978

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Kinetic Equations for the Electron-Phonon System

The interaction of the electron with the phonon field and with the external electric field is considered. The exact equalities which by applying a suitable approximation procedure can be transformed into equations for the study of the kinetic properties of the electron-phonon system are derived.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1978

PART I

1. Introduction

We shall consider here the interaction of the electron with the phonon field and with the external electric field.

In this paper we will derive exact equalities which by applying a suitable approximation procedure can be transformed into equations for the study of kinetic properties.

The hamiltonian for our dynamic system (S, Σ) of interacting electron (S) and phonon field (Σ) may be written in the form:

$$H = \frac{p^2}{2m} + \vec{E}(t)\vec{r} + \sum_{(k)} \hbar\omega_k b_k^+ b_k + \frac{1}{\sqrt{V}} \sum_{(k)} \mathcal{L}(k) \left(\frac{\hbar}{2\omega_k}\right)^{1/2} e^{i(\vec{k}\cdot\vec{r})} (b_k + b_{-k}^+) , \tag{1.1}$$

where $\vec{E}(t)$ is the external electric field, multiplied by the charge e_c ; ω_k, $\mathcal{L}(k)$ are the spherically symmetric real functions of the wave vector $\vec{k}$, e.g. for the Fröhlich polaron

$$\mathcal{L}(k) = \frac{g}{|k|}, \quad \omega_k = \omega \tag{1.2}$$

g being the coupling constant.

In this expression $\vec{r}$ and $\vec{p}$ denote the position and momentum of the electron, V is the volume of the considered dynamical system, while b_k, b_k^+ are the phonon bose amplitudes corresponding to the vector $\vec{k}$. The frequencies ω_k are supposed to be essentially positive.

The summation (k) proceeds over the usual quasi-discrete spectrum of $\vec{k}$, becoming continuous in the limit $V \to \infty$.

In some situations it will be convenient to use the notion of an adiabatic switching off the interactions.

We may then introduce the factor $e^{\epsilon t}$ $(\epsilon > 0)$ to annihilate the interactions at $t \to -\infty$; in the final result we put $\epsilon \to 0$.

Therefore we shall start by considering the hamiltonian:

$$H_t = \frac{p^2}{2m} + e^{\epsilon t} \vec{E}(t) \cdot \vec{r} + \sum_{(k)} \hbar \omega_k b_k^+ b_k + \frac{e^{\epsilon t}}{\sqrt{V}} \sum_{(k)} \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \mathfrak{L}(k) e^{i\vec{k}\cdot\vec{r}} \cdot (b_k + b_{-k}^+). \tag{1.3}$$

Let us now inrtoduce some notation .

So, $A(S)$ will denote a dynamical variable depending only upon dynamical variables $\vec{p}, \vec{r}$ of S and thus commuting with all b_k, b_k^+.

In the same way $A(\Sigma)$ denotes a dynamical variable depending only upon dynamical variables $\ldots b_k \ldots b_k^+ \ldots$ of Σ and thus commuting with $\vec{p}$ and $\vec{r}$.

The general dynamical variable may be denoted by $A(S, \Sigma)$. These conventions being adopted we denote:

$$\Gamma_t(S) = \frac{p^2}{2m} + e^{\epsilon t} \vec{E}(t)\vec{r}$$

$$H(\Sigma) = \sum_{(k)} \hbar \omega_k b_k^+ b_k$$

$$H_t' = H_t'(S,\Sigma) = \frac{e^{\epsilon t}}{\sqrt{V}} \sum_{(k)} \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \mathfrak{L}(k) e^{i\vec{k}\cdot\vec{r}} (b_k + b_{-k}^+). \tag{1.4}$$

Then (1.3) yields:

$$H_t = \Gamma_t(S) + H(\Sigma) + H_t'(S, \Sigma). \tag{1.5}$$

We begin by making use of the Liouville equation for the statistical operator $\mathfrak{D}_t$:

$$i\hbar \frac{\partial \mathfrak{D}_t}{\partial t} = H_t \mathfrak{D}_t - \mathfrak{D}_t H_t \tag{1.6}$$

with the initial condition at some time t_0 far in the past:

$$\mathfrak{D}_{t_0} = \rho(S) \mathfrak{D}(\Sigma), \tag{1.7}$$

where $\rho(S)$ is a statistical operator for the system S

$$\operatorname{Sp}_{(S)} \rho(S) = 1$$

and $\mathfrak{D}(\Sigma)$ describes the statistical equilibrium in the system Σ alone:

$$\mathfrak{D}(\Sigma) = Z^{-1} e^{-\beta H(\Sigma)}; \quad Z = \operatorname{Sp}_{(\Sigma)} e^{-\beta H(\Sigma)}$$

$$\operatorname{Sp}_{(\Sigma)} \mathfrak{D}(\Sigma) = 1. \tag{1.8}$$

Therefore one obtains the conventional normalization for the statistical operator:

$$\operatorname{Sp}_{(S,\Sigma)} \mathfrak{D}_t = \operatorname{Sp}_{(S,\Sigma)} \mathfrak{D}_{t_0} = \operatorname{Sp}_{(S)} \rho(S) \operatorname{Sp}_{(\Sigma)} \mathfrak{D}(\Sigma) = 1 \tag{1.9}$$

It will be convenient to introduce the operator $U(t,t_0)$:

$$i\hbar \frac{\partial U(t,t_0)}{\partial t} = H_t U(t,t_0) \qquad (1.10)$$

$$U(t_0,t_0) = 1$$

As H_t is hermitian we have:

$$-i\hbar \frac{\partial \overset{+}{U}(t,t_0)}{\partial t} = \overset{+}{U}(t,t_0) H_t$$

$$\overset{+}{U}(t_0,t_0) = 1$$

Hence $U(t,t_0)$ is the unitary operator:

$$\overset{+}{U}(t,t_0) = U^{-1}(t,t_0) \qquad (1.11)$$

In terms of these operators the formal solution of (1.6), (1.7) can be written as:

$$\mathfrak{D}_t = U(t,t_0)\, \mathfrak{D}_{t_0}\, U^{-1}(t,t_0). \qquad (1.12)$$

Consider a dynamical variable A. Its average value at the time t is

$$\langle A \rangle_t = \underset{(S,\Sigma)}{Sp}\, A\, \mathfrak{D}_t \qquad (1.13)$$

In virtue of (1.12) one obtains:

$$\langle A \rangle_t = \underset{(S,\Sigma)}{Sp}\, A\, U(t,t_0)\mathfrak{D}_{t_0}\, U^{-1}(t,t_0) = Sp\, A(t)\,\mathfrak{D}_{t_0} \qquad (1.14)$$

where

$$A(t) = \overset{+}{U}(t,t_0) A\, U(t,t_0) = U^{-1}(t,t_0) A\, U(t,t_0) \qquad (1.15)$$

$$A(t_0) = A$$

This leads to:

$$\langle A \rangle_t = \underset{(S,\Sigma)}{Sp}\, A\, \mathfrak{D}_t = \underset{(S,\Sigma)}{Sp}\, A(t)\, \mathfrak{D}_{t_0}. \qquad (1.16)$$

From (1.15) we see that $A(t)$ is nothing else than the Heisenberg representation of A.

Note that if the commutator between A and B is a c-number:

$$[A,B] = AB - BA = c$$

the same relation holds also for their Heisenberg representations:

$$[A(t), B(t)] = U^{-1}(t,t_0)[A,B]\, U(t,t_0) = c.$$

Consider now a dynamical variable of the type $\Phi(S)$ and denote by $\Phi(S_t)$ its Heisenberg representation.

From (1.16) we get:

$$\underset{(S,\Sigma)}{Sp}\, \Phi(S_t)\, \mathfrak{D}_{t_0} = \underset{(S,\Sigma)}{Sp}\, \Phi(S)\, \mathfrak{D}_t =$$

$$= \underset{(S)}{Sp}\, \Phi(S) \{ \underset{(\Sigma)}{Sp}\, \mathfrak{D}_t \}.$$

Therefore in terms of the reduced statistical operator:

$$\rho_t(S) = \underset{(\Sigma)}{Sp}\, \mathfrak{D}_t \; ; \qquad \underset{(S)}{Sp}\, \rho_t(S) = 1 \qquad (1.17)$$

this relation yields:

$$\underset{(S,\Sigma)}{Sp}\, \Phi(S_t)\mathfrak{D}_{t_0} = \underset{(S)}{Sp}\, \Phi(S)\rho_t(S) \qquad (1.18)$$

In the particular case when

$$\Phi(S) = F(\vec{p})$$

we can write:

$$\mathop{Sp}_{(S)} F(\vec{p})\rho_t(S) = \int F(\vec{p}) w_t(\vec{p}) d\vec{p} \tag{1.19}$$

where

$$w_t(\vec{p}_0) = \mathop{Sp}_{(S)} \delta(\vec{p} - \vec{p}_0)\rho_t(S) \tag{1.20}$$

Let $\vec{p}(t)$ denote the Heisenberg operator of the electron momentum. Then (1.18) gives:

$$\mathop{Sp}_{(S,\Sigma)} F(\vec{p}(t))\mathfrak{D}_{t_0} = \int F(\vec{p}) w_t(\vec{p}) d\vec{p}\,. \tag{1.21}$$

From (1.17), (1.20) it follows:

$$\int w_t(\vec{p}) d\vec{p} = 1$$

and it is easy to see that $w_t(\vec{p})$ may be interpreted as the probability density of momentum at the time t.

Let us now turn our attention to the dynamic variable of the type $f(S)$ which does not depend explicitly upon time.

Its Heisenberg operator $f(S_t)$ obeys the equation of motion:

$$ih \frac{\partial f(S_t)}{\partial t} = f(S_t)H(t) - H(t)f(S_t) \tag{1.22}$$

where $H(t)$ is the hamiltonian H_t expressed in terms of the Heisenberg operator:

$$H(t) = \Gamma_t(S_t) + H(\Sigma_t) + H'(t)\,. \tag{1.23}$$

Here:

$$\Gamma_t(S_t) = \frac{p^2(t)}{2m} + e^{\epsilon t}\vec{E}(t)\vec{r}(t)$$

$$H(\Sigma_t) = \sum_{(k)} \hbar\omega_k b_k^+(t) b_k(t)$$

$$H'(t) = \frac{e^{\epsilon t}}{\sqrt{V}} \sum_{(k)} \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \mathfrak{L}(k) e^{i\vec{k}\cdot\vec{r}(t)} \{b_k(t) + b_{-k}^+(t)\} =$$

$$= \frac{e^{\epsilon t}}{\sqrt{V}} \sum_{(k)} \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \mathfrak{L}(k)\{b_k(t) + b_{-k}^+(t)\} e^{i\vec{k}\cdot\vec{r}(t)}\,, \tag{1.24}$$

where $\vec{p}(t)$, $\vec{r}(t)$, $b_k(t)$, $b_k^+(t)$ are the Heisenberg operators obeying the equations of motion:

$$\frac{d\vec{r}(t)}{dt} = \frac{1}{m}\vec{p}(t)$$

$$\frac{d\vec{p}(t)}{dt} = -e^{\epsilon t}\vec{E}(t) - ie^{\epsilon t}\frac{1}{\sqrt{V}} \sum_{(k)} \left(\frac{\hbar}{2\omega_k}\right)^{1/2} \times$$

$$\times\ \mathfrak{L}(k)\vec{k}\, e^{i\vec{k}\cdot\vec{r}(t)} (b_k(t) + b_{-k}^+(t))$$

$$\frac{db_k(t)}{dt} = -i\omega_k b_k(t) - ie^{\epsilon t}\frac{1}{\sqrt{V}}\mathfrak{L}(k)\left(\frac{1}{2\hbar\omega_k}\right)^{1/2} e^{-i\vec{k}\cdot\vec{r}(t)} \tag{1.25}$$

$$\frac{db_{-k}^+(t)}{dt} = i\omega_k b_{-k}^+(t) + ie^{\epsilon t}\frac{1}{\sqrt{V}}\mathfrak{L}(k)\left(\frac{1}{2\hbar\omega_k}\right)^{1/2} e^{-i\vec{k}\cdot\vec{r}(t)}$$

and the initial conditions:

$$\vec{r}(t_0) = \vec{r}\,,\quad \vec{p}(t_0) = \vec{p}\,,\quad b_k(t_0) = b_k\,,\quad b_{-k}^+(t_0) = b_{-k}^+ \tag{1.26}$$

Substitute (1.23) into (1.22) and remark that

$$[f(S_t), H(\Sigma_t)] = 0$$

because

$$[f(S), H(\Sigma)] = 0 .$$

Therefore (1.22) can be written in the form:

$$\frac{\partial f(S_t)}{\partial t} + \frac{\Gamma_t(S_t) f(S_t) - f(S_t) \Gamma_t(S_t)}{i\hbar} = \frac{f(S_t) H'(t) - H'(t) f(S_t)}{i\hbar} \tag{1.27}$$

from which it follows:

$$\mathop{Sp}_{(S,\Sigma)} \left\{ \frac{\partial f(S_t)}{\partial t} + \frac{\Gamma_t(S_t) f(S_t) - f(S_t) \Gamma_t(S_t)}{i\hbar} \right\} \mathfrak{D}_{t_0} = \frac{1}{i\hbar} \mathop{Sp}_{(S,\Sigma)} \{ f(S_t) H'(t) - H'(t) f(S_t) \} \mathfrak{D}_{t_0} \tag{1.28}$$

Remark that in virtue of (1.18) we have:

$$\mathop{Sp}_{(S,\Sigma)} \left\{ \frac{\partial f(S_t)}{\partial t} + \frac{\Gamma_t(S_t) f(S_t) - f(S_t) \Gamma_t(S_t)}{i\hbar} \right\} \mathfrak{D}_{t_0} =$$

$$= \mathop{Sp}_{(S)} \left\{ f(S) \frac{\partial \rho_t(S)}{\partial t} + \frac{\Gamma_t(S) f(S) - f(S) \Gamma_t(S)}{i\hbar} \rho_t(S) \right\}$$

and thus:

$$\mathop{Sp}_{(S)} \left\{ f(S) \frac{\partial \rho_t(S)}{\partial t} + \frac{\Gamma_t(S) f(S) - f(S) \Gamma_t(S)}{i\hbar} \rho_t(S) \right\} =$$

$$= \frac{1}{i\hbar} \mathop{Sp}_{(S,\Sigma)} \{ f(S_t) H'(t) - H'(t) f(S_t) \}$$

In view of (1.24) this equality yields:

$$\mathop{Sp}_{(S)} \left\{ f(S) \frac{\partial \rho_t(S)}{\partial t} + \frac{\Gamma_t(S) f(S) - f(S) \Gamma_t(S)}{i\hbar} \rho_t(S) \right\} =$$

$$= -ie^{\epsilon t} \frac{1}{\sqrt{V}} \sum_{(k)} \mathfrak{L}(k) \left(\frac{1}{2\hbar\omega_k} \right)^{1/2} \mathop{Sp}_{(S,\Sigma)} f(S_t) e^{i\vec{k}\vec{r}(t)} (b_k(t) + b^+_{-k}(t)) \mathfrak{D}_{t_0} +$$

$$+ ie^{\epsilon t} \frac{1}{\sqrt{V}} \sum_{(k)} \mathfrak{L}(k) \left(\frac{1}{2\hbar\omega_k} \right)^{1/2} \mathop{Sp}_{(S,\Sigma)} (b_k(t) + b^+_{-k}(t)) e^{i\vec{k}\vec{r}(t)} f(S_t) \mathfrak{D}_{t_0} . \tag{1.29}$$

This relation will be studied in the next paragraph.

2. Elimination of the Phonon Field Amplitudes

We now proceed to eliminate the bose amplitudes b_k, b^+_{-k} of the phonon field from the relation (1.29). Our main aim here consists in obtaining the equation in which only positions and momenta

$$\vec{r}(\tau), \ \vec{p}(\tau) \qquad t_0 \leq \tau \leq t \tag{2.1}$$

of the electron enter explicitly.

The equations (1.25) lead to:

$$b_k(t) = -iB_k(t) + \tilde{b}_k(t); \quad b^+_{-k}(t) = iB^+_{-k}(t) + \tilde{b}^+_{-k}(t) \tag{2.2}$$

where

$$B_k(t) = \frac{1}{\sqrt{V}} \mathfrak{L}(k)\left(\frac{1}{2\hbar\omega_k}\right)^{1/2} \int_{t_0}^{t} d\tau\, e^{-i\omega_k(t-\tau)+\epsilon\tau} e^{-i\vec{k}\vec{r}(\tau)}$$

$$B^{+}_{-k}(t) = \frac{1}{\sqrt{V}} \mathfrak{L}(k)\left(\frac{1}{2\hbar\omega_k}\right)^{1/2} \int_{t_0}^{t} d\tau\, e^{+i\omega_k(t-\tau)+\epsilon\tau} e^{-i\vec{k}\vec{r}(\tau)} \qquad (2.3)$$

and

$$\tilde{b}_k(t) = e^{-i\omega_k(t-t_0)} b_k, \quad \tilde{b}^{+}_{-k}(t) = e^{i\omega_k(t-t_0)} b^{+}_{-k}. \qquad (2.4)$$

Therefore (1.29) can be put in the form:

$$\underset{(S)}{Sp}\left\{ f(S)\frac{\partial\rho_t(S)}{\partial t} + \frac{\Gamma_t(S)f(S) - f(S)\Gamma_t(S)}{i\hbar}\rho_t(S)\right\} =$$

$$= -ie^{\epsilon t}\frac{1}{\sqrt{V}}\sum_{(k)} \mathfrak{L}(k)\left(\frac{1}{2\hbar\omega_k}\right)^{1/2} \underset{(S,\Sigma)}{Sp} f(S_t)\, e^{i\vec{k}\vec{r}(t)} \times$$

$$\times\{\tilde{b}_k(t) + \tilde{b}^{+}_{-k}(t) - iB_k(t) + iB^{+}_{-k}(t)\}\mathfrak{D}_{t_0} + \qquad (2.5)$$

$$+ie^{\epsilon t}\frac{1}{\sqrt{V}}\sum_{(k)} \mathfrak{L}(k)\left(\frac{1}{2\hbar\omega_k}\right)^{1/2} \underset{(S,\Sigma)}{Sp}\{\tilde{b}_k(t) + \tilde{b}^{+}_{-k}(t) -$$

$$- iB_k(t) + iB^{+}_{-k}(t)\}e^{i\vec{k}\vec{r}(t)} f(S_t)\mathfrak{D}_{t_0}.$$

Here $B_k(t), B^{+}_{-k}(t)$ in fact depend explicitly only upon $\vec{r}(\tau)$ $(t_0 \leq \tau \leq t)$ but the "free" bose amplitudes $\tilde{b}_k(t), \tilde{b}^{+}_{-k}(t)$ are still present.

In order to get rid of these $\tilde{b}_k, \tilde{b}^{+}_{-k}$ we now propose to prove the following <u>lemma</u>:

$$\underset{(S,\Sigma)}{Sp}\, \tilde{b}_k(t)\, \mathfrak{U}(S,\Sigma)\, \mathfrak{D}_{t_0} =$$

$$= \frac{1}{1 - e^{-\beta\hbar\omega_k}} \underset{(S,\Sigma)}{Sp}\{\tilde{b}_k(t)\mathfrak{U}(S,\Sigma) - \mathfrak{U}(S,\Sigma)\tilde{b}_k(t)\}\mathfrak{D}_{t_0}$$

<u>Proof:</u>

By noticing (1.7) and observing that b_k commutes with any operator of the type $\Phi(S)$, we have:

$$\underset{(S,\Sigma)}{Sp}\, \tilde{b}_k(t)\, \mathfrak{U}(S,\Sigma)\mathfrak{D}_{t_0} = \underset{(S,\Sigma)}{Sp}\, \tilde{b}_k(t)\, \mathfrak{U}(S,\Sigma)\rho(S)\mathfrak{D}(\Sigma) =$$

$$= \underset{(\Sigma)}{Sp}\, \tilde{b}_k(t)\{\underset{(S)}{Sp}\, \mathfrak{U}(S,\Sigma)\rho(S)\}\mathfrak{D}(\Sigma)$$

$$\underset{(S,\Sigma)}{Sp}\, \mathfrak{U}(S,\Sigma)\tilde{b}_k(t)\mathfrak{D}_{t_0} = \underset{(S,\Sigma)}{Sp}\, \mathfrak{U}(S,\Sigma)\tilde{b}_k(t)\rho(S)\mathfrak{D}(\Sigma) =$$

$$= \underset{(\Sigma)}{Sp}\{\underset{(S)}{Sp}\, \mathfrak{U}(S,\Sigma)\rho(S)\}\tilde{b}_k(t)\mathfrak{D}(\Sigma)$$

Denote

$$\underset{(S)}{Sp}\, \mathfrak{U}(S,\Sigma)\rho(S) = B(\Sigma).$$

Then:

$$\underset{(S,\Sigma)}{Sp}\, \tilde{b}_k(t)\, \mathfrak{U}(S,\Sigma)\mathfrak{D}_{t_0} = \underset{(\Sigma)}{Sp}\, \tilde{b}_k(t) B(\Sigma)\mathfrak{D}(\Sigma)$$

$$\underset{(S,\Sigma)}{Sp}\, \mathfrak{U}(S,\Sigma)\tilde{b}_k(t)\mathfrak{D}_{t_0} = \underset{(\Sigma)}{Sp}\, B(\Sigma)\tilde{b}_k(t)\mathfrak{D}(\Sigma). \qquad (2.6)$$

Let us recall here an important property of the equilibrium averages in the statistical mechanics.

Consider an isolated dynamical system, characterized by some time independent hamiltonian H and two dynamical variables A ,B , corresponding to this system, which do not depend explicitly upon t.

Then, for the equilibrium averages:

$$\langle A(t)B\rangle_{eq} = Sp\, A(t)B\,\mathfrak{D}_{eq}$$
$$\langle BA(t)\rangle_{eq} = Sp\, B A(t)\,\mathfrak{D}_{eq}$$

in which

$$A(t) = e^{\frac{i}{\hbar}Ht} A e^{-\frac{i}{\hbar}Ht}$$

we have

$$\langle A(t)B\rangle_{eq} = \int_{-\infty}^{\infty} J(\omega) e^{-i\omega t} d\omega$$
$$\langle BA(t)\rangle_{eq} = \int_{-\infty}^{\infty} e^{-\beta\omega\hbar} J(\omega) e^{-i\omega t} d\omega .$$

We write these relations in the form:

$$Sp(e^{\frac{i}{\hbar}H(t-t_0)} A e^{-\frac{i}{\hbar}H(t-t_0)} B\mathfrak{D}_{eq}) = \int_{-\infty}^{\infty} J(\omega) e^{-i\omega(t-t_0)} d\omega$$

$$Sp(B e^{\frac{i}{\hbar}H(t-t_0)} A e^{-\frac{i}{\hbar}H(t-t_0)} \mathfrak{D}_{eq}) = \tag{2.7}$$

$$= \int_{-\infty}^{\infty} e^{-\beta\omega\hbar} J(\omega) e^{-i\omega(t-t_0)} d\omega .$$

Take now as this system our system Σ and put:

$$H = H(\Sigma), \quad \mathfrak{D}_{eq} = \mathfrak{D}(\Sigma)$$
$$A = b_k, \quad B = B(\Sigma).$$

Note also that in this case:

$$\tilde{b}_k(t) = e^{-i\omega_k(t-t_0)} b_k = e^{\frac{i}{\hbar}(t-t_0)H} b_k e^{-\frac{i}{\hbar}(t-t_0)H}$$

Thus (2.7) lead to:

$$\underset{(\Sigma)}{Sp}\, \tilde{b}_k(t) B(\Sigma)\mathfrak{D}(\Sigma) = e^{-i\omega_k(t-t_0)} \underset{(\Sigma)}{Sp}\, b_k B(\Sigma)\mathfrak{D}(\Sigma) = \int_{-\infty}^{\infty} J_k(\omega) e^{-i\omega(t-t_0)} d\omega$$

$$\underset{(\Sigma)}{Sp}\, B(\Sigma)\tilde{b}_k(t)\mathfrak{D}(\Sigma) = e^{-i\omega_k(t-t_0)} \underset{(\Sigma)}{Sp}\, B(\Sigma) b_k \mathfrak{D}(\Sigma) = \int_{-\infty}^{\infty} e^{-\beta\hbar\omega} J_k(\omega) e^{-i\omega(t-t_0)} d\omega . \tag{2.8}$$

These relations show that $J_k(\omega)$ is proportional to $\delta(\omega-\omega_k)$:

$$J_k(\omega) = I_k\, \delta(\omega-\omega_k)$$

and hence:

$$e^{-\beta\hbar\omega} J_k(\omega) = e^{-\beta\hbar\omega_k} J_k(\omega)$$

Therefore from (2.8) we also obtain:

$$\underset{(\Sigma)}{Sp}\, B(\Sigma)\tilde{b}_k(t)\mathfrak{D}(\Sigma) = e^{-\beta\hbar\omega_k} \underset{(\Sigma)}{Sp}\, \tilde{b}_k(t) B(\Sigma)\mathfrak{D}(\Sigma)$$

or, in virtue of (2.6):

$$\underset{(S,\Sigma)}{Sp}\, \mathfrak{U}(S,\Sigma)\,\tilde{b}_k(t)\,\mathfrak{D}_{t_0} = e^{-\beta\hbar\omega_k}\, \underset{(S,\Sigma)}{Sp}\, \tilde{b}_k(t)\,\mathfrak{U}(S,\Sigma)\,\mathfrak{D}_{t_0}$$

which yields

$$\underset{(S,\Sigma)}{Sp}\, \{\tilde{b}_k(t)\,\mathfrak{U}(S,\Sigma) - \mathfrak{U}(S,\Sigma)\,\tilde{b}_k(t)\}\,\mathfrak{D}_{t_0} =$$

$$= (1 - e^{-\beta\hbar\omega_k})\, \underset{(S,\Sigma)}{Sp}\, \tilde{b}_k(t)\,\mathfrak{U}(S,\Sigma)\,\mathfrak{D}_{t_0} .$$

We now see that:

$$\underset{(S,\Sigma)}{Sp}\, \tilde{b}_k(t)\,\mathfrak{U}(S,\Sigma)\,\mathfrak{D}_{t_0} = \frac{1}{1-e^{-\beta\hbar\omega_k}} \times$$

$$\times\, \underset{(S,\Sigma)}{Sp}\, \{\tilde{b}_k(t)\,\mathfrak{U}(S,\Sigma) - \mathfrak{U}(S,\Sigma)\,\tilde{b}_k(t)\}\,\mathfrak{D}_{t_0}$$

$$\underset{(S,\Sigma)}{Sp}\, \mathfrak{U}(S,\Sigma)\,\tilde{b}_k(t)\,\mathfrak{D}_{t_0} = \frac{e^{-\beta\hbar\omega_k}}{1-e^{-\beta\hbar\omega_k}} \times \qquad (2.9)$$

$$\times\, \underset{(S,\Sigma)}{Sp}\, \{\tilde{b}_k(t)\,\mathfrak{U}(S,\Sigma) - \mathfrak{U}(S,\Sigma)\,\tilde{b}_k(t)\}\,\mathfrak{D}_{t_0}$$

and our lemma is proved.

Denote

$$\frac{e^{-\beta\hbar\omega_k}}{1-e^{-\beta\hbar\omega_k}} = N_k \ . \qquad (2.10)$$

Then the relations (2.9) can be written in terms of these equilibrium averages of the occupation numbers $b_k^+ b_k$:

$$\underset{(S,\Sigma)}{Sp}\, \tilde{b}_k(t)\,\mathfrak{U}(S,\Sigma)\,\mathfrak{D}_{t_0} =$$

$$= (1+N_k)\, \underset{(S,\Sigma)}{Sp}\, \{\tilde{b}_k(t)\,\mathfrak{U}(S,\Sigma) - \mathfrak{U}(S,\Sigma)\,\tilde{b}_k(t)\}\,\mathfrak{D}_{t_0}$$

$$(2.11)$$

$$\underset{(S,\Sigma)}{Sp}\, \mathfrak{U}(S,\Sigma)\,\tilde{b}_k(t)\,\mathfrak{D}_{t_0} =$$

$$= N_k\, \underset{(S,\Sigma)}{Sp}\, \{\tilde{b}_k(t)\,\mathfrak{U}(S,\Sigma) - \mathfrak{U}(S,\Sigma)\,\tilde{b}_k(t)\}\,\mathfrak{D}_{t_0} .$$

By replacing here:

$$\mathfrak{U}(S,\Sigma) \to \overset{+}{\mathfrak{U}}(S,\Sigma), \qquad k \to -k$$

and performing the complex conjugation procedure we immediately obtain:

$$\underset{(S,\Sigma)}{Sp}\, \mathfrak{U}(S,\Sigma)\,\tilde{b}^+_{-k}(t)\,\mathfrak{D}_{t_0} =$$

$$= (1+N_k)\, \underset{(S,\Sigma)}{Sp}\, \{\mathfrak{U}(S,\Sigma)\,\tilde{b}^+_{-k}(t) - \tilde{b}^+_{-k}(t)\,\mathfrak{U}(S,\Sigma)\}\,\mathfrak{D}_{t_0}$$

$$(2.12)$$

$$\underset{(S,\Sigma)}{Sp}\, \tilde{b}^+_{-k}(t)\,\mathfrak{U}(S,\Sigma)\,\mathfrak{D}_{t_0} =$$

$$= N_k\, \underset{(S,\Sigma)}{Sp}\, \{\mathfrak{U}(S,\Sigma)\,\tilde{b}^+_{-k}(t) - \tilde{b}^+_{-k}(t)\,\mathfrak{U}(S,\Sigma)\}\,\mathfrak{D}_{t_0}$$

We now proceed to use (2.11), (2.12) for the case when:

$$\mathfrak{U}(S,\Sigma) = \Phi(S_t) \tag{2.13}$$

Here as always $\Phi(S_t)$ denotes the Heisenberg operator corresponding to the dynamical variable $\Phi(S)$.

Therefore, because b_k, b^+_{-k} commute with $\Phi(S)$ the operators $b_k(t), b^+_{-k}(t)$ must also commute with $\Phi(S_t)$

$$b_k(t)\Phi(S_t) - \Phi(S_t)b_k(t) = 0$$
$$\Phi(S_t)b^+_{-k}(t) - b^+_{-k}(t)\Phi(S_t) = 0. \tag{2.14}$$

By taking into account (2.2) we obtain:

$$\tilde{b}_k(t)\Phi(S_t) - \Phi(S_t)\tilde{b}_k(t) = i\{B_k(t)\Phi(S_t) - \Phi(S_t)B_k(t)\}$$
$$\Phi(S_t)\tilde{b}^+_{-k}(t) - \tilde{b}^+_{-k}(t)\Phi(S_t) = i\{B^+_{-k}(t)\Phi(S_t) - \Phi(S_t)B^+_{-k}(t)\}. \tag{2.15}$$

In this situation (2.11), (2.12) lead to:

$$\underset{(S,\Sigma)}{Sp}\{\tilde{b}_k(t)+\tilde{b}^+_{-k}(t)\}\Phi(S_t)\mathfrak{D}_{t_o} = i(1+N_k)\underset{(S,\Sigma)}{Sp}\{B_k(t)\Phi(S_t) - \Phi(S_t)B_k(t)\}\mathfrak{D}_{t_o} + iN_k\underset{(S,\Sigma)}{Sp}\{B^+_{-k}(t)\Phi(S_t) - \Phi(S_t)B^+_{-k}(t)\}\mathfrak{D}_{t_o}$$

$$\underset{(S,\Sigma)}{Sp}\,\Phi(S_t)\{\tilde{b}_k(t)+\tilde{b}^+_{-k}(t)\}\mathfrak{D}_{t_o} = iN_k\underset{(S,\Sigma)}{Sp}\{B_k(t)\Phi(S_t) - \Phi(S_t)B_k(t)\}\mathfrak{D}_{t_o} + i(1+N_k)\underset{(S,\Sigma)}{Sp}\{B^+_{-k}(t)\Phi(S_t) - \Phi(S_t)B^+_{-k}(t)\}\mathfrak{D}_{t_o}$$

from which it follows:

$$\underset{(S,\Sigma)}{Sp}\,\Phi(S_t)\{\tilde{b}_k(t)+\tilde{b}^+_{-k}(t) - iB_k(t) + iB^+_{-k}(t)\}\mathfrak{D}_{t_o} = i\underset{(S,\Sigma)}{Sp}\{N_kB_k(t)+(1+N_k)B^+_{-k}(t)\}\Phi(S_t)\mathfrak{D}_{t_o} - i\underset{(S,\Sigma)}{Sp}\{(1+N_k)\Phi(S_t)B_k(t)+N_k\Phi(S_t)B^+_{-k}(t)\}\mathfrak{D}_{t_o} \tag{2.16}$$

$$\underset{(S,\Sigma)}{Sp}\{\tilde{b}_k(t)+\tilde{b}^+_{-k}(t) - iB_k(t) + iB^+_{-k}(t)\}\Phi(S_t)\mathfrak{D}_{t_o} = i\underset{(S,\Sigma)}{Sp}\{N_kB_k(t)+(1+N_k)B^+_{-k}(t)\}\Phi(S_t)\mathfrak{D}_{t_o} - i\underset{(S,\Sigma)}{Sp}\{(1+N_k)\Phi(S_t)B_k(t)+N_k\Phi(S_t)B^+_{-k}(t)\}\mathfrak{D}_{t_o} \tag{2.17}$$

Put in (2.16)

$$\Phi(S_t) = f(S_t)e^{i\vec{k}\vec{r}(t)} \tag{2.16a}$$

and in (2.17)

$$\Phi(S_t) = e^{i\vec{k}\vec{r}(t)}f(S_t). \tag{2.17a}$$

Then by using the expressions for $B_k(t), B^+_{-k}(t)$ from (2.3) one obtains:

$$\underset{(S,\Sigma)}{Sp}\,f(S_t)e^{i\vec{k}\vec{r}(t)}\{\tilde{b}_k(t)+\tilde{b}^+_{-k}(t) - iB_k(t) + iB^+_{-k}(t)\}\mathfrak{D}_{t_o} =$$

$$= ie^{\epsilon t}\frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\omega_k}\right)^{1/2}\mathfrak{L}(k)\int_{t_0}^{t} d\tau e^{-\epsilon(t-\tau)}\{N_k e^{-i\omega_k(t-\tau)} +$$

$$+(1+N_k)e^{i\omega_k(t-\tau)}\}\underset{(S,\Sigma)}{Sp}\, e^{-i\vec{k}\vec{r}(\tau)} f(S_t)e^{i\vec{k}\vec{r}(t)}\mathfrak{D}_{t_0} -$$

$$-ie^{\epsilon t}\frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\omega_k}\right)^{1/2}\mathfrak{L}(k)\int_{t_0}^{t} d\tau e^{-\epsilon(t-\tau)}\{(1+N_k)e^{-i\omega(t-\tau)} +$$

$$+N_k e^{i\omega_k(t-\tau)}\}\underset{(S,\Sigma)}{Sp} f(S_t)e^{i\vec{k}\vec{r}(t)}e^{-i\vec{k}\vec{r}(\tau)}\mathfrak{D}_{t_0} \qquad (2.18)$$

$$\underset{(S,\Sigma)}{Sp}\{\tilde{b}_k(t)+\tilde{b}^+_{-k}(t)-iB_k(t)+iB^+_{-k}(t)\}e^{i\vec{k}\vec{r}(t)} f(S_t)\mathfrak{D}_{t_0} =$$

$$= ie^{\epsilon t}\frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\omega_k}\right)^{1/2}\mathfrak{L}(k)\int_{t_0}^{t} d\tau e^{-\epsilon(t-\tau)}\{N_k e^{-i\omega_k(t-\tau)} +$$

$$+(1+N_k)e^{i\omega_k(t-\tau)}\}\underset{(S,\Sigma)}{Sp}\, e^{-i\vec{k}\vec{r}(\tau)}e^{i\vec{k}\vec{r}(t)} f(S_t)\mathfrak{D}_{t_0} -$$

$$-ie^{\epsilon t}\frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\omega_k}\right)^{1/2}\mathfrak{L}(k)\int_{t_0}^{t} d\tau e^{-\epsilon(t-\tau)}\{(1+N_k)e^{-i\omega_k(t-\tau)} +$$

$$+N_k e^{i\omega_k(t-\tau)}\}\underset{(S,\Sigma)}{Sp}\, e^{i\vec{k}\vec{r}(t)} f(S_t)e^{-i\vec{k}\vec{r}(\tau)}\mathfrak{D}_{t_0} .$$

Substituting these results into (2.5) we find that:

$$\underset{(S)}{Sp}\{f(S)\frac{\partial\rho_t(S)}{\partial t}+$$

$$+\frac{e^{\epsilon t}\bar{E}(t)(\vec{r}f(S)-f(S)\vec{r})+\frac{p^2}{2m}f(S)-f(S)\frac{p^2}{2m}}{i\hbar}\rho_t(S)\}=$$

$$=\frac{1}{V}e^{2\epsilon t}\sum_{(k)}\frac{\mathfrak{L}^2(k)}{2\hbar\omega_k}\int_{t_0}^{t} d\tau e^{-\epsilon(t-\tau)}\{N_k e^{-i\omega_k(t-\tau)}+$$

$$+(1+N_k)e^{i\omega_k(t-\tau)}\}\times$$

$$\times\underset{(S,\Sigma)}{Sp}\{e^{-i\vec{k}\vec{r}(\tau)}f(S_t)e^{i\vec{k}\vec{r}(t)}-e^{-i\vec{k}\vec{r}(\tau)}e^{i\vec{k}\vec{r}(t)}f(S_t)\}\mathfrak{D}_{t_0}+ \qquad (2.19)$$

$$+\frac{1}{V}e^{2\epsilon t}\sum_{(k)}\frac{\mathfrak{L}^2(k)}{2\hbar\omega_k}\int_{t_0}^{t} d\tau e^{-\epsilon(t-\tau)}\{(1+N_k)e^{-i\omega_k(t-\tau)}+N_k e^{i\omega_k(t-\tau)}\}\times$$

$$\times\underset{(S,\Sigma)}{Sp}\{e^{i\vec{k}\vec{r}(t)}f(S_t)e^{-i\vec{k}\vec{r}(\tau)}-f(S_t)e^{i\vec{k}\vec{r}(t)}e^{-i\vec{k}\vec{r}(\tau)}\}\mathfrak{D}_{t_0} .$$

In this equation the phonon field amplitudes do not enter explicitly. Indeed here the right-hand side depends only upon "trajectory" of the electron i.e. upon (2.1).

It must be stressed however that $\vec{r}(\tau)$, $\vec{p}(\tau)$ depend in a very complicated way upon the initial values $\vec{r}, \vec{p}, \ldots b_k, \ldots b_k^+ \ldots$.

Therefore in order to obtain from (2.19) an explicit equation we must rely on a suitable approximation procedure.

Consider now the case when we take

$$f(S) = f(\vec{p})$$

and hence

$$f(S_t) = f(\vec{p}(t)).$$

We have

$$\vec{r} f(\vec{p}) - f(\vec{p})\vec{r} = i\hbar \frac{\partial f(\vec{p})}{\partial \vec{p}}$$

Thus from (2.19) it follows:

$$\underset{(S)}{Sp}\{ f(\vec{p}) \frac{\partial \rho_t(S)}{\partial t} + e^{\epsilon t} \vec{E}(t) \frac{\partial f(\vec{p})}{\partial \vec{p}} \rho_t(S)\} =$$

$$= \int d\vec{p} \{ f(\vec{p}) \frac{\partial w_t(\vec{p})}{\partial t} + e^{\epsilon t} \vec{E}(t) \frac{\partial f(\vec{p})}{\partial \vec{p}} w_t(\vec{p})\}.$$

It is easy to see that:

$$e^{i\vec{k}\vec{r}} f(\vec{p}) = f(\vec{p} - \hbar\vec{k}) e^{i\vec{k}\vec{r}}$$

$$f(\vec{p}) e^{i\vec{k}\vec{r}} = e^{i\vec{k}\vec{r}} f(\vec{p} + \hbar\vec{k})$$

which yields:

$$f\{\vec{p}(t)\} e^{i\vec{k}\vec{r}(t)} = e^{i\vec{k}\vec{r}(t)} f\{\vec{p}(t) + \hbar\vec{k}\}$$

$$e^{i\vec{k}\vec{r}(t)} f\{\vec{p}(t)\} = f\{\vec{p}(t) - \hbar\vec{k}\} e^{i\vec{k}\vec{r}(t)}.$$

Because both sums over (k) in the right hand side of (2.19) are invariant with respect to the transformation:

$$\vec{k} \to -\vec{k}$$

this transformation will be performed in the first of these sums.

With all such remarks we find:

$$\int d\vec{p}\, f(\vec{p}) \{ \frac{\partial w_t(\vec{p})}{\partial t} - e^{\epsilon t} \vec{E}(t) \frac{\partial w_t(\vec{p})}{\partial \vec{p}} \} =$$

$$= \int d\vec{p} \{ f(\vec{p}) \frac{\partial w_t(\vec{p})}{\partial t} + e^{\epsilon t} \vec{E}(t) \frac{\partial f(\vec{p})}{\partial \vec{p}} w_t(\vec{p}) \} =$$

$$= e^{2\epsilon t} \frac{1}{V} \sum_{(k)} \frac{\mathcal{L}^2(k)}{2\hbar\omega_k} \int_{t_0}^{t} d\tau\, e^{-\epsilon(t-\tau)} \{ N_k e^{-i\omega_k(t-\tau)} +$$

$$+ (1+N_k) e^{i\omega_k(t-\tau)} \} \underset{(S,\Sigma)}{Sp} [e^{i\vec{k}\vec{r}(t)} e^{-i\vec{k}\vec{r}(\tau)} \{ f(\vec{p}(t) - \hbar\vec{k}) - f(\vec{p}(t)) \} \mathfrak{D}_{t_0}] \tag{2.20}$$

$$+ e^{2\epsilon t} \frac{1}{V} \sum_{(k)} \frac{\mathcal{L}^2(k)}{2\hbar\omega_k} \int_{t_0}^{t} d\tau\, e^{-\epsilon(t-\tau)} \{ (1+N_k) e^{-i\omega_k(t-\tau)} +$$

$$+ N_k e^{i\omega_k(t-\tau)} \} \underset{(S,\Sigma)}{Sp} [\{ f(\vec{p}(t) - \hbar\vec{k}) - f(\vec{p}(t)) \} e^{i\vec{k}\vec{r}(t)} e^{-i\vec{k}\vec{r}(\tau)} \mathfrak{D}_{t_0}]$$

where

$$\mathfrak{D}_{t_0} = \rho(S)\, \mathfrak{D}(\Sigma) \tag{2.21}$$

This exact relation will be considered in the next paragraphs as a source for obtaining approximate equations.

3. Kinetic Equation in the First Approximation for the Case of Small Interaction

We will examine here the case when the interactions are small. It is convenient to characterize the coupling constant by a small parameter which will be denoted by α assuming that $\mathfrak{L}^2(k)$ is proportional to α.

For example, in the framework of the Frohlich model of the polaron the standard dimensionless parameter characterizing the intensity of the electron-phonon interaction is:

$$\alpha = \frac{g^2}{4\pi\hbar\omega^2}\sqrt{\frac{m}{2\hbar\omega}} \tag{3.1}$$

in our notation.

We will also treat external force $\vec{E}$ as being formally proportional to the small parameter.

Then, in zero order approximation, when the interactions are completely neglected, we may write:

$$\vec{r}(\tau) = \vec{r}(t) - \frac{\vec{p}(t)}{m}(t-\tau) \tag{3.2}$$

Such an approximation will be used in (2.20) only for terms proportional to α.

Specifically we put (3.2) under the sign of Spur, using the "zero order approximation" in the following way:

$$\mathcal{E}_{app} = \{ \underset{(s,\Sigma)}{Sp}\, e^{i\vec{k}\vec{r}(\tau)} e^{-i\vec{k}\vec{r}(t)} [f(\vec{p}(t)-\hbar\vec{k}) - f(\vec{p}(t))] \mathfrak{D}_{t_0} \}_{app}$$

$$= \underset{(s,\Sigma)}{Sp}\, e^{i\vec{k}\vec{r}_0(t,\tau)} e^{-i\vec{k}\vec{r}(t)} [f(\vec{p}(t)-\hbar\vec{k}) - f(\vec{p}(t))] \mathfrak{D}_{t_0},$$

$$\mathcal{E}^*_{app} = \{ \underset{(s,\Sigma)}{Sp}\, [f(\vec{p}(t)-\hbar\vec{k}) - f(\vec{p}(t))] e^{i\vec{k}\vec{r}(t)} e^{-i\vec{k}\vec{r}(\tau)} \mathfrak{L}_{t_0} \}_{app} =$$

$$= \underset{(s,\Sigma)}{Sp}\, [f(\vec{p}(t)-\hbar\vec{k}) - f(\vec{p}(t))] e^{i\vec{k}\vec{r}(t)} e^{-i\vec{k}\vec{r}_0(t,\tau)} \mathfrak{L}_{t_0}, \tag{3.3}$$

where

$$\vec{r}_0(t,\tau) = \vec{r}(t) - \frac{\vec{p}(t)}{m}(t-\tau).$$

It is to be pointed out that these expressions are multiplied by $\mathfrak{L}^2(k)$ -proportional to the small parameter.

We thus expect that the first order terms in α in the right-hand side of (2.20) are correctly evaluated. This is the only approximation we need and all we have to do further is to carry out the limiting processes $V \to \infty$, $t_0 \to -\infty$ and in conclusion to put $\epsilon \to 0$. But first let us disentangle the expressions $\mathcal{E}_{app}$, $\mathcal{E}^*_{app}$ given by (3.3).

Using the commutation properties between components of the the vectors $\vec{r}(t)$, $\vec{p}(t)$ one immediately obtains:

$$[\vec{k}\vec{r}_0(t,\tau), \vec{k}\vec{r}(t)] = [\vec{k}\{\vec{r}(t) - \frac{\vec{p}(t)}{m}(t-\tau), \vec{k}\vec{r}(t)] = \frac{i\hbar k^2}{m}(t-\tau).$$

As is well known, if the commutator

$[A, B]$

is a c -number then:

$$e^A e^B = e^{\frac{[A,B]}{2}} e^{A+B}.$$

This enables us to write:

$$e^{i\vec{k}\vec{r}_o(t,\tau)} \; e^{-i\vec{k}\vec{r}(t)} = e^{\frac{i\hbar k^2}{2m}(t-\tau)} \; e^{-\frac{i\vec{k}\vec{p}(t)}{m}(t-\tau)},$$

which leads to:

$$\mathcal{E}_{app} = e^{\frac{i\hbar k^2}{2m}(t-\tau)} \underset{(s,\Sigma)}{Sp}\; e^{-i(t-\tau)\frac{\vec{k}\vec{p}(t)}{m}} [\, f(\vec{p}(t) - \hbar\vec{k}) - f(\vec{p}(t))]\mathfrak{D}_{t_o}.$$

Let us go back to the relation:

$$\underset{(s,\Sigma)}{Sp}\; F(\vec{p}(t)\mathfrak{D}_{t_o}) = \int F(\vec{p})\, w_t(\vec{p})\, d\vec{p} \tag{1.21}$$

valid for arbitrary function $F(\vec{p})$ of momentum.
Taking here

$$F(\vec{p}) = e^{-i(t-\tau)\frac{\vec{k}\vec{p}}{m}} [\, f(\vec{p} - \hbar\vec{k}) - f(\vec{p})]$$

we obtain:

$$\begin{aligned} \mathcal{E}_{app} &= \int d\vec{p}\, e^{i(t-\tau)\left(\frac{\hbar k^2}{2m} - \frac{\vec{k}\vec{p}}{m}\right)} [\, f(\vec{p} - \hbar\vec{k}) - f(\vec{p})]\, w_t(\vec{p}) = \\ &= \int d\vec{p}\, e^{-i(t-\tau)\left(\frac{\hbar k^2}{2m} + \frac{\vec{k}\vec{p}}{m}\right)} f(\vec{p})\, w_t(\vec{p} + \hbar\vec{k}) \; - \\ &- \int d\vec{p}\, e^{i(t-\tau)\left(\frac{\hbar k^2}{2m} - \frac{\vec{k}\vec{p}}{m}\right)} f(\vec{p})\, w_t(\vec{p})\,. \end{aligned} \tag{3.4}$$

It is easy to see that $\mathcal{E}^*_{app}$ is the complex conjugate of $\mathcal{E}_{app}$. Therefore:

$$\begin{aligned} \mathcal{E}^*_{app} &= \int d\vec{p}\, e^{i(t-\tau)\left(\frac{\hbar k^2}{2m} + \frac{\vec{k}\vec{p}}{m}\right)} f(\vec{p})\, w_t(\vec{p} + \hbar\vec{k}) \; - \\ &= \int d\vec{p}\, e^{-i(t-\tau)\left(\frac{\hbar k^2}{2m} - \frac{\vec{k}\vec{p}}{m}\right)} f(\vec{p})\, w_t(\vec{p})\,. \end{aligned} \tag{3.5}$$

Substitute now these expressions into equation (2.20). Let us then pass to the limit $V \to \infty$ which amounts to replace the sums:

$$\frac{1}{V}\sum_{(k)} \ldots$$

by the corresponding integrals:

$$\frac{1}{(2\pi)^3}\int d\vec{k} \ldots$$

It is convenient to make the transformation:

$$\vec{k} \to -\vec{k}$$

in the integrals containing $w_t(\vec{p})$. We will also introduce the new time variable

$$t - \tau = \xi$$

instead of τ so that:

$$\int_{t_o}^{t} d\tau \ldots = \int_{0}^{t-t_o} d\xi\,.$$

In the limit

$$t_o \to -\infty$$

these integrals become

$$\int_0^\infty d\xi \ldots$$

In such a way we can write the equation of the first approximation in the form:

$$\int d\vec{p}\, f(\vec{p}) \{ \frac{\partial w_t(\vec{p})}{\partial t} - e^{-\epsilon t} \vec{E}(t) \frac{\partial w_t(\vec{p})}{\partial \vec{p}} \} =$$

$$= \frac{e^{2\epsilon t}}{(2\pi)^3} \int d\vec{p}\, f(\vec{p}) \int d\vec{k} \frac{\mathfrak{L}^2(k)}{2\hbar\omega_k} A_\epsilon(\vec{p},\vec{k}) ,$$

where

$$A_\epsilon(\vec{p},\vec{k}) = \int_0^\infty d\xi\, e^{-\epsilon\xi} \left((1+N_k) e^{i\omega_k\xi} + N_k e^{-i\omega_k\xi}\right) \times$$

$$\times \{ e^{-i\xi(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m})} w_t(\vec{p}+\hbar\vec{k}) - e^{i\xi(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m})} w_t(\vec{p}) \} +$$

$$+ \int_0^\infty d\xi\, e^{-\epsilon\xi} \left((1+N_k) e^{-i\omega_k\xi} + N_k e^{i\omega_k\xi}\right) \times$$

$$\times \{ e^{i\xi(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m})} w_t(\vec{p}+\hbar\vec{k}) - e^{-i\xi(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m})} w_t(\vec{p}) \} .$$

Because $f(\vec{p})$ is an arbitrary function of $\vec{p}$ this equation leads to:

$$\frac{\partial w_t(\vec{p})}{\partial t} - e^{\epsilon t} \vec{E}(t) \frac{\partial w_t(\vec{p})}{\partial \vec{p}} = \frac{e^{2\epsilon t}}{(2\pi)^3} \int d\vec{k} \frac{\mathfrak{L}^2(k)}{2\hbar\omega_k} A_\epsilon(\vec{p},\vec{k}) \quad (3.6)$$

Arranging the terms in the expression of $A_\epsilon(\vec{p},\vec{k})$ we find

$$A_\epsilon(\vec{p},\vec{k}) = \{(1+N_k) w_t(\vec{p}+\hbar\vec{k}) - N_k w_t(\vec{p})\} \times$$

$$\times \{ \int_0^\infty e^{-\epsilon\xi} e^{-i\xi(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m}-\omega_k)} d\xi + \int_0^\infty e^{-\epsilon\xi} e^{i\xi(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m}-\omega_k)} d\xi \} +$$

$$+ \{N_k w_t(\vec{p}+\hbar\vec{k}) - (1+N_k) w_t(\vec{p})\} \times$$

$$\times \{ \int_0^\infty e^{-\epsilon\xi} e^{-i\xi(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m}+\omega_k)} d\xi + \int_0^\infty e^{-\epsilon\xi + i\xi(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m}+\omega_k)} d\xi \},$$

or else

$$A_\epsilon(\vec{p},\vec{k}) = \frac{w_t(\vec{p}+\hbar\vec{k}) - e^{-\beta\hbar\omega_k} w_t(\vec{p})}{1-e^{-\beta\hbar\omega_k}} \Delta_\epsilon\left(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m}-\omega_k\right) +$$

$$+ \frac{w_t(\vec{p}+\hbar\vec{k})\, e^{-\beta\hbar\omega_k} - w_t(\vec{p})}{1-e^{-\beta\hbar\omega_k}} \Delta_\epsilon\left(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m}+\omega_k\right) ,$$

where

$$\Delta_\epsilon(z) = \int_{-\infty}^{\infty} e^{-\epsilon|\xi|} e^{i\xi z}\, d\xi .$$

Note also that

$$\lim_{\epsilon\to 0} \Delta_\epsilon\left(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m} \mp \omega_k\right) = 2\pi\delta\left(\frac{\hbar k^2}{2m}+\frac{\vec{k}\vec{p}}{m} \mp \omega_k\right) =$$

$$= 2\pi\delta\left\{ \frac{ \frac{(\vec{p}+\hbar\vec{k})^2}{2m} - \frac{\vec{p}^{\,2}}{2m} \mp \hbar\omega_k }{\hbar} \right\} = 2\pi h\,\delta\left(\frac{(\vec{p}+\hbar\vec{k})^2}{2m} - \frac{\vec{p}^{\,2}}{2m} \mp \hbar\omega_k \right).$$

For this reason:

$$\lim_{\epsilon\to 0} A_\epsilon(\vec{p},\vec{k}) = \frac{2\pi\hbar}{1-e^{-\beta\hbar\omega_k}}\{ w_t(\vec{p}+\hbar\vec{k}) -$$

$$- e^{-\beta\hbar\omega_k}\, w_t(\vec{p})\}\,\delta\left(\frac{(\vec{p}+\hbar\vec{k})^2}{2m} - \frac{\vec{p}^{\,2}}{2m} - \hbar\omega_k\right) +$$

$$+ \frac{2\pi\hbar}{1-e^{-\beta\hbar\omega_k}}\{ w_t(\vec{p}+\hbar\vec{k})\, e^{-\beta\hbar\omega_k} -$$

$$- w_t(\vec{p})\}\,\delta\left(\frac{(\vec{p}+\hbar\vec{k})^2}{2m} + \hbar\omega_k - \frac{\vec{p}^{\,2}}{2m}\right).$$

Let us now make the last step by putting $\epsilon \to 0$ in equation (3.6).

We then obtain our equation of the first approximation in the final form:

$$\frac{\partial w_t(\vec{p})}{\partial t} - \vec{E}(t)\,\frac{\partial w_t(\vec{p})}{\partial \vec{p}} = \qquad (3.7)$$

$$= \frac{1}{(2\pi)^2}\int d\vec{k}\,\frac{\mathfrak{L}^2(k)}{2\omega_k(1-e^{-\beta\hbar\omega_k})}\{ w_t(\vec{p}+\hbar\vec{k}) -$$

$$- e^{-\beta\hbar\omega_k}\, w_t(\vec{p})\}\,\delta\left(\frac{(\vec{p}+\hbar\vec{k})^2}{2m} - \frac{\vec{p}^{\,2}}{2m} - \hbar\omega_k\right) +$$

$$+ \frac{1}{(2\pi)^2}\int d\vec{k}\,\frac{\mathfrak{L}^2(k)}{2\omega_k(1-e^{-\beta\hbar\omega_k})}\{ w_t(\vec{p}+\hbar\vec{k})\, e^{-\beta\hbar\omega_k} - \qquad (3.7)$$

$$- w_t(\vec{p})\}\,\delta\left(\frac{(\vec{p}+\hbar\vec{k})^2}{2m} - \frac{\vec{p}^{\,2}}{2m} + \hbar\omega_k\right).$$

It is evident that this equation is just the usual Boltzmann equation. The integral terms in the right-hand side correspond respectively to one-phonon emission and absorption. Such Boltzmann equation was extensively studied for the investigation of the transport properties.

For the steady-state situation, when the electric field is time independent, (3.7) yields:

$$-\vec{E}\,\frac{\partial w(\vec{p})}{\partial \vec{p}} = \frac{1}{(2\pi)^2}\int d\vec{k}\,\frac{\mathfrak{L}^2(k)}{2\omega_k(1-e^{-\beta\hbar\omega_k})}\{ w(\vec{p}+\hbar\vec{k}) -$$

$$- e^{-\beta\hbar\omega_k}\, w(\vec{p})\}\,\delta\left(\frac{(\vec{p}+\hbar\vec{k})^2}{2m} - \frac{\vec{p}^{\,2}}{2m} - \hbar\omega_k\right) + \qquad (3.8)$$

$$+ \frac{1}{(2\pi)^2}\int d\vec{k}\,\frac{\mathfrak{L}^2(k)}{2\omega_k(1-e^{-\beta\hbar\omega_k})}\{ w(\vec{p}+\hbar\vec{k})\, e^{-\beta\hbar\omega_k} -$$

$$- w(\vec{p})\}\,\delta\left(\frac{(\vec{p}+\hbar\vec{k})^2}{2m} + \hbar\omega_k - \frac{\vec{p}^{\,2}}{2m}\right).$$

In an interesting case of low temperature, when the factor:

$$e^{-\beta\hbar\omega}$$

can be neglected, the resulting equation was considered /1/ by J.T.Devreese and R.Evrard for the Fröhlich model of the polaron.

They have found a very complicated behaviour of the distribution function $w(\vec{p})$ suggesting the existence of an essential singularity occuring at $E=0$.

In conclusion let us say a few words about one oversimplified way of approach to determine the dependence between the applied electric field and steady-state average velocity $\vec{v}$ of the electron.

Let us multiply both parts of (3.8) by $\vec{p}$ and integrate over the whole momentum space.

After some trivial transformations one obtains:

$$-\vec{E} = \frac{1}{(2\pi)^2}\int d\vec{k}\,\frac{\mathcal{L}^2(k)\,\hbar\vec{k}}{2\omega_k\,(1-e^{-\beta\hbar\omega_k})}\times$$

$$\times\int d\vec{p}\,w(\vec{p})\,\delta\Big(-\frac{(\hbar k)^2}{2m}+\hbar\frac{\vec{k}\vec{p}}{m}-\hbar\omega_k\Big)-$$

$$-\frac{1}{(2\pi)^2}\int d\vec{k}\,\frac{\mathcal{L}^2(k)\,\hbar\vec{k}}{2\omega_k\,(e^{\beta\hbar\omega_k}-1)}\times \tag{3.9}$$

$$\times\int d\vec{p}\,w(\vec{p})\,\delta\Big(\frac{(\hbar k)^2}{2m}+\hbar\frac{\vec{k}\vec{p}}{m}-\hbar\omega_k\Big).$$

Here, according to the notation of § 1:

$$\vec{E} = -e_c\vec{\mathcal{E}} \tag{3.10}$$

where $\vec{\mathcal{E}}$ denotes the external electric field.

So far it is an exact consequence of the Boltzmann equation.

We now make a rough approximation by taking the drifted maxwellian as a trial function for $w(\vec{p})$:

$$w(\vec{p}) = \rho_M(\vec{p}-m\vec{v})\,;\quad \rho_M(\vec{p}) = \Big(\frac{\beta}{2m\pi}\Big)^{3/2} e^{-\beta\frac{p^2}{2m}}$$

and substituting it into (3.9).

This leads to:

$$e_c\vec{\mathcal{E}} = \frac{1}{(2\pi)^2}\int d\vec{k}\,\frac{\mathcal{L}^2(k)\,\hbar\vec{k}}{2\omega_k\,(1-e^{-\beta\hbar\omega_k})}\times$$

$$\times\int d\vec{p}\,\rho_M(\vec{p})\,\delta\Big(-\frac{(\hbar k)^2}{2m}+\frac{\hbar\vec{k}\vec{p}}{m}-\hbar(\omega_k-\vec{k}\vec{v})\Big)-$$

$$-\frac{1}{(2\pi)^2}\int d\vec{k}\,\frac{\mathcal{L}^2(k)\,\hbar\vec{k}}{2\omega_k\,(e^{\beta\hbar\omega_k}-1)}\times \tag{3.11}$$

$$\times\int d\vec{p}\,\rho_M(\vec{p})\,\delta\Big(\frac{(\hbar k)^2}{2m}+\frac{\hbar\vec{k}\vec{p}}{m}-\hbar(\omega_k-\vec{k}\vec{v})\Big).$$

Note that:

$$\delta\left(-\frac{(\hbar k)^2}{2m}+\frac{\hbar\vec{k}\vec{p}}{m}-\hbar(\omega_k-\vec{k}\vec{v})\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{(\hbar k)^2}{2m}-\frac{\hbar\vec{k}\vec{p}}{m}+\hbar(\omega_k-\vec{k}\vec{v})\right)\xi}\,d\xi$$

$$\delta\left(\frac{(\hbar k)^2}{2m}+\frac{\hbar\vec{k}\vec{p}}{m}-\hbar(\omega_k-\vec{k}\vec{v})\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{(\hbar k)^2}{2m}+\frac{\hbar\vec{k}\vec{p}}{m}-\hbar(\omega_k-\vec{k}\vec{v})\right)\xi}\,d\xi$$

and

$$\int\rho_M(\vec{p})\,e^{-i\xi\frac{\hbar\vec{k}\vec{p}}{m}}\,d\vec{p}=e^{-\frac{(\hbar k)^2}{2m}\frac{\xi^2}{\beta}}$$

$$\int\rho_M(\vec{p})\,e^{i\xi\frac{\hbar\vec{k}\vec{p}}{m}}\,d\vec{p}=e^{-\frac{(\hbar k)^2}{2m}\frac{\xi^2}{\beta}}$$

Therefore from (3.11) we get:

$$e_c\vec{\mathcal{E}}=\int_{-\infty}^{\infty}d\xi\,\frac{1}{(2\pi)^3}\int\frac{\mathcal{L}^2(k)\hbar\vec{k}}{2\omega_k}\left(\frac{e^{i\hbar(\omega_k-\vec{k}\vec{v})\xi}}{1-e^{-\beta\hbar\omega_k}}-\frac{e^{-i\hbar(\omega_k-\vec{k}\vec{v})\xi}}{e^{\beta\hbar\omega_k}-1}\right)e^{-\frac{(\hbar k)^2}{2m}\left(\frac{\xi^2}{\beta}-i\xi\right)}\,d\vec{k}. \tag{3.12}$$

This approximate equation was obtained in the paper [2] by K.K.Thornberger and R.F.Feynman for the case of small interaction*. They have found that the mobility derived from (3.12) in the weak coupling limit does not agree with the mobility obtained from the standard Boltzmann treatment.

We see here that this disagreement is caused by the use of an inadequate approximation, that of the drifter maxwellian, in relation (3.9) which itself is an exact consequence of the Boltzmann equation.

The connection between formula (3.12) and the use of the drifted maxwellian as the trial form for the steady-state distribution function was also noted by J.T.Devreese (private communication).

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* In their system of unities and notation :

$$\hbar=1,\quad C_k=\frac{1}{\sqrt{V}}\left(\frac{1}{2\omega_k}\right)^{1/2}\mathcal{L}(k),\quad \vec{E}=e_c\vec{\mathcal{E}}$$

equation (3.12) has the form:

$$\vec{E}=\int_{-\infty}^{\infty}d\xi\sum_{(k)}|C_k|^2\,\vec{k}\left(\frac{e^{i(\omega_k-\vec{k}\vec{v})\xi}}{1-e^{-\beta\omega_k}}-\frac{e^{-i(\omega_k-\vec{k}\vec{v})\xi}}{e^{\beta\omega_k}-1}\right)e^{-\frac{k^2}{2m}\left(\frac{\xi^2}{\beta}-i\xi\right)}$$

– that of formula (17) in the mentioned paper.

4. Formulation of the Linear Model

We wish now to show that the results of the paper /2/ as well as of the previous paper /3/, concerning the calculation of the impedance, could de directly obtained without path-integral methods.

Our starting point resides in the exact equation (2.20) in which we choose:

$$f(\vec{p}) = \vec{p} \tag{4.1}$$

Denote the average momentum of the electron by:

$$\langle \vec{p} \rangle_t = \int \vec{p}\, w_t(\vec{p})\, d\vec{p}$$

Use also the notation

$$\underset{(S,\Sigma)}{\mathrm{Sp}}\, e^{i\vec{k}\vec{r}(\tau)}\, e^{-i\vec{k}\vec{r}(t)}\, \mathfrak{D}_{t_0} = \Phi_k(t,\tau,t_0) \tag{4.2}$$

We have

$$\underset{(S,\Sigma)}{\mathrm{Sp}}\, e^{i\vec{k}\vec{r}(t)}\, e^{-i\vec{k}\vec{r}(\tau)}\, \mathfrak{D}_{t_0} = \underset{(S,\Sigma)}{\mathrm{Sp}} \{ e^{i\vec{k}\vec{r}(\tau)}\, e^{-i\vec{k}\vec{r}(t)} \}^{+} \mathfrak{D}_{t_0} =$$

$$= \Phi_k^*(t,\tau,t_0).$$

Therefore (2.20) with (4.1) lead to:

$$- \{ \frac{d\langle \vec{p}\rangle_t}{dt} + e^{\epsilon t}\, \vec{E}(t) \} =$$

$$= \frac{1}{V} e^{2\epsilon t} \sum_{(k)} \frac{\mathfrak{L}^2(k)\vec{k}}{2\omega_k(1-e^{-\beta\hbar\omega_k})} \int_{t_0}^{t} d\tau \{ e^{i\omega_k(t-\tau)} + \tag{4.3}$$

$$+ e^{-i\omega_k(t-\tau)}\, e^{-\beta\hbar\omega_k} \} e^{-\epsilon(t-\tau)}\, \Phi_k(t,\tau,t_0) +$$

$$+ \frac{1}{V} e^{2\epsilon t} \sum_{(k)} \frac{\mathfrak{L}^2(k)\, \vec{k}}{2\omega_k(1-e^{-\beta\hbar\omega_k})} \int_{t_0}^{t} d\tau\, e^{-\epsilon(t-\tau)} \{ e^{-i\omega_k(t-\tau)} +$$

$$+ e^{i\omega_k(t-\tau)}\, e^{-\beta\hbar\omega_k} \} \Phi_k^*(t,\tau,t_0).$$

It is still an exact equation. It is clear, however, that in order to obtain explicit, though approximate, expressions for Φ_k we need to realy upon a suitable model hamiltonian, which would lead us to exactly soluble equations. To obtain a reasonable approach this model hamiltonian must be chosen so that the behaviour of its $\vec{r}(t)$ somehow simulates the behaviour of $\vec{r}(t)$ corresponding to the exact hamiltonian (1.1).

Consider first the case when there is no external field

$$\vec{E} = 0 \tag{4.4}$$

and draw the attention to the hamiltonian:

$$H_L = \frac{p^2}{2m} + \frac{c^2 r^2}{2} + \sum_{(k)} \hbar\nu(k) b_k^+ b_k + \tag{4.5}$$

$$+ \frac{i}{\sqrt{V}} \sum_{(k)} \left(\frac{\hbar}{2\nu(k)}\right)^{1/2} \mathcal{J}(k)(\vec{k}\vec{r})(b_k + b_{-k}^+)$$

where $\mathcal{J}(k)$ are spherically symmetric functions of $\vec{k}$, $\nu(k)$ being also spherically symmetric are moreover essentially positive:

$\nu(k) > 0$.

Before the limiting process $V \to \infty$, when V is still finite the number $\mathcal{N}_V$ of terms in the sums over (k) is supposed also to be finite.

Then the corresponding Heisenberg equations:

$$\frac{d\vec{r}(t)}{dt} = \frac{\vec{p}(t)}{m};$$

$$\frac{d\vec{p}(t)}{dt} = -c^2\vec{r}(t) - \frac{i}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}\text{Л}(k)\vec{k}\left(b_k(t)+b^+_{-k}(t)\right)$$

$$\frac{db_k(t)}{dt} = -i\nu(k)b_k(t) - \frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2}\text{Л}(k)(\vec{k}\vec{r}(t)) \qquad (4.6)$$

$$\frac{db^+_{-k}(t)}{dt} = i\nu(k)b^+_{-k}(t) + \frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2}\text{Л}(k)(\vec{k}\vec{r}(t))$$

$$\vec{r}(t_0) = \vec{r}; \quad \vec{p}(t_0) = \vec{p}; \quad b_k(t_0) = b_k, \quad b^+_{-k}(t_0) = b^+_{-k}$$

constitute a finite linear system of ordinary differential equations with constant coefficients and thus are exactly soluble.

We now proceed to show that by a suitable choice of the constant c^2 the hamiltonian (4.5) becomes translation-invariant.

Let us start from the identity:

$$\sum_{(k)} h\nu(k)\left\{b^+_k + \frac{1}{\sqrt{V}} i(\vec{k}\vec{r})\frac{\text{Л}(k)}{\nu(k)\sqrt{2\hbar\nu(k)}}\right\}\left\{b_k - \frac{1}{\sqrt{V}} i(\vec{k}\vec{r})\frac{\text{Л}(k)}{\nu(k)\sqrt{2\hbar\nu(k)}}\right\} =$$

$$= \sum_{(k)}\hbar\nu(k)b^+_k b_k + \frac{i}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}\text{Л}(k)(\vec{k}\vec{r})b_k -$$

$$-\frac{i}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}\text{Л}(k)(\vec{k}\vec{r})b^+_k + \frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)}{2\nu^2(k)}(\vec{k}\vec{r})^2$$

and note that in virtue of the spherical symmetry of $\text{Л}(k), \nu(k)$

$$\frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)}{\nu^2(k)}(\vec{k}\vec{r})^2 = r^2\frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)}{3\nu^2(k)}k^2$$

For this reason

$$H_L = \frac{p^2}{2m} + \left(c^2 - \frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)}{3\nu^2(k)}k^2\right)\frac{r^2}{2} +$$

$$+\sum_{(k)}\hbar\nu(k)\left\{b^+_k + \frac{i}{\sqrt{V}}(\vec{k}\vec{r})\frac{\text{Л}(k)}{\nu(k)\sqrt{2\hbar\nu(k)}}\right\}\left\{b_k - \frac{i}{\sqrt{V}}(\vec{k}\vec{r})\frac{\text{Л}(k)}{\nu(k)\sqrt{2\hbar\nu(k)}}\right\}.$$

Therefore, if we choose:

$$c^2 = \frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)k^2}{3\nu^2(k)} \qquad (4.7)$$

then H_L becomes invariant with respect to the translation group

$$\vec{r} \to \vec{r} + \vec{R}, \; b_k \to b_k + \frac{i}{\sqrt{V}}(\vec{k}\vec{R})\frac{\text{Л}(k)}{\nu(k)(2\hbar\nu(k))^{1/2}} \qquad (4.8)$$

Such an invariance must lead to the existence of a conserved vector $\vec{\mathcal{P}}$

$$\frac{d\vec{\mathcal{P}}}{dt} = 0 \qquad (4.9)$$

which may be interpreted as a kind of the "total momentum". To find the expressions for $\vec{\mathcal{P}}$ let us note that from (4.6) it follows:

$$\frac{d(b_k(t)-b^+_{-k}(t))}{dt} = -i\nu(k)(b_k(t)+b^+_{-k}(t)) - \frac{2}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2} Л(k)(\vec{k}\vec{r}(t))$$

and hence

$$\frac{d}{dt}\frac{1}{\sqrt{V}}\sum_{(k)} Л(k)\left(\frac{\hbar}{2\nu(k)}\right)^{1/2} \frac{b_k(t)-b^+_{-k}(t)}{\nu(k)}\vec{k} =$$

$$= -i\frac{1}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2} Л(k)\vec{k}\,\{b_k(t)+b^+_{-k}(t)\} -$$

$$-\frac{1}{V}\sum_{(k)}\frac{Л^2(k)}{\nu^2(k)}(\vec{k}\vec{r}(t))\vec{k} = \frac{d\vec{p}(t)}{dt} + c^2\vec{r}(t) - \frac{1}{V}\sum_{(k)}\frac{Л^2(k)}{\nu^2(k)}(\vec{k}\vec{r}(t))\vec{k}$$

But here due to (4.7)

$$\frac{1}{V}\sum_{(k)}\frac{Л^2(k)}{\nu^2(k)}(\vec{k}\vec{r}(t))\vec{k} = \vec{r}(t)\,\frac{1}{V}\sum_{(k)}\frac{Л^2(k)k^2}{3\nu^2(k)} = c^2\vec{r}(t)$$

Therefore

$$\frac{d}{dt}\{\vec{p}(t) - \frac{1}{\sqrt{V}}\sum_{(k)}\frac{\vec{k}Л(k)}{\nu(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}(b_k(t)-b^+_{-k}(t))\} = 0$$

from which it follows that the conserved vector, we looked for, is

$$\vec{\mathcal{P}} = \vec{p} - \frac{1}{\sqrt{V}}\sum\frac{\vec{k}Л(k)}{\nu(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2}(b_k - b^+_{-k}) \qquad (4.10)$$

Let us introduce the external field thus replacing hamiltonian H_L by

$$\tilde{H}_L = H_L + \vec{E}(t)\vec{r} \qquad (4.11)$$

Because H_L commutes with $\vec{\mathcal{P}}$ and because

$$[\mathcal{P}_\beta, r_\gamma] = [p_\beta, r_\gamma] = -i\hbar\delta_{\beta\gamma}\ ;\ \beta,\gamma = 1,2,3$$

we see that now

$$\frac{d\vec{\mathcal{P}}(t)}{dt} = -\vec{E}(t) \qquad (4.12)$$

We may observe that for the hamiltonian (1.1) when (4.4) holds the translation group becomes:

$$\vec{r} \to \vec{r} + \vec{R}\ ,\quad b_k \to b_k e^{-i\vec{k}\vec{R}} \qquad (4.13)$$

In this situation the total momentum is given by:

$$\vec{\mathcal{P}} = \vec{p} + \sum_{(k)}\hbar\vec{k}\,b^+_k b_k \qquad (4.14)$$

When the external field is switched on then this $\vec{\mathcal{P}}$ also verifies equation (4.12).

Consider the hamiltonian (4.11) and the corresponding Heisenberg equations:

$$m^*\frac{d\vec{r}(t)}{dt} = \vec{p}(t)$$

$$(4.15)$$

$$\frac{d\vec{p}(t)}{dt} = -c^2\vec{r}(t) - \frac{i}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2} Л(k)\vec{k}\,(b_k(t) +$$

$$+ b^+_{-k}(t)) - \vec{E}(t)$$

$$\frac{db_k(t)}{dt} = -i\nu(k)b_k(t) - \frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2} \text{Л}(k)(\vec{k}\vec{r}(t))$$

$$\frac{db^+_{-k}(t)}{dt} = i\nu(k)b^+_{-k}(t) + \frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2} \text{Л}(k)(\vec{k}\vec{r}(t)) \qquad (4.16)$$

$$\vec{r}(t_0)=\vec{r}, \quad \vec{p}(t_0)=\vec{p}; \quad b_k(t_0)=b_k, \quad b^+_{-k}(t_0)=b^+_{-k}$$

from which it follows:

$$b_k(t) = b_k e^{-i\nu(k)(t-t_0)} - \frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2} \text{Л}(k)\int_{t_0}^{t} e^{-i\nu(k)(t-\tau)}(\vec{k}\vec{r}(\tau))d\tau$$

$$b^+_{-k}(t) = b^+_{-k} e^{i\nu(k)(t-t_0)} + \frac{1}{\sqrt{V}}\left(\frac{1}{2\hbar\nu(k)}\right)^{1/2} \text{Л}(k)\int_{t_0}^{t} e^{i\nu(k)(t-\tau)}(\vec{k}\vec{r}(\tau))d\tau$$

The substitution of these expressions into eq.(4.15) yields:

$$\frac{d\vec{p}(t)}{dt} + c^2\vec{r}(t) + i\frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)\vec{k}}{2\nu(k)}\int_{t_0}^{t} d\tau\,(\vec{k}\vec{r}(\tau))\{e^{i\nu(k)(t-\tau)} - e^{-i\nu(k)(t-\tau)}\} =$$

$$= -\frac{i}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2} \text{Л}(k)\vec{k}(b_k e^{-i\nu(k)(t-t_0)} + b^+_{-k} e^{i\nu(k)(t-t_0)}) -$$

$$-\vec{E}(t)$$

Let us perform the integration by parts:

$$i\int_{t_0}^{t} d\tau\,(\vec{k}\vec{r}(\tau))\{e^{i\nu(k)(t-\tau)} - e^{-i\nu(k)(t-\tau)}\} =$$

$$= -\frac{1}{\nu(k)}\int_{t_0}^{t}(\vec{k}\vec{r}(\tau))\frac{d}{d\tau}\{e^{i\nu(k)(t-\tau)} + e^{-i\nu(k)(t-\tau)}\}d\tau =$$

$$= -2\frac{(\vec{k}\vec{r}(t))}{\nu(k)} + 2\frac{(\vec{k}\vec{r})}{\nu(k)}\cos\nu(k)(t-t_0) +$$

$$+ \frac{2}{\nu(k)}\int_{t_0}^{t} d\tau\,(\vec{k}\frac{d\vec{r}(\tau)}{d\tau})\cos\nu(k)(t-\tau)$$

and recall that:

$$-\frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)\vec{k}}{\nu^2(k)}(\vec{k}\vec{r}(t)) = -\frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)k^2}{3\nu^2(k)}\vec{r}(t) = -c^2\vec{r}(t)$$

$$\frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)\vec{k}}{\nu^2(k)}(\vec{k}\frac{d\vec{r}(\tau)}{d\tau})\cos\nu(k)(t-\tau) =$$

$$= \frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)k^2}{3\nu^2(k)}\cos\nu(k)(t-\tau)\frac{d\vec{r}(\tau)}{d\tau}$$

We then obtain:

$$\frac{d\vec{p}(t)}{dt} + \frac{1}{m^*}\int_{t_0}^{t} d\tau\,K(t-\tau)\vec{p}(\tau) = -\vec{r}K(t-t_0) -$$

$$-\frac{i}{\sqrt{V}}\sum_{(k)}\left(\frac{\hbar}{2\nu(k)}\right)^{1/2} \text{Л}(k)\vec{k}(b_k e^{-i\nu(k)(t-t_0)} + b^+_{-k} e^{i\nu(k)(t-t_0)}) - \qquad (4.17)$$

$$-\vec{E}(t)$$

where

$$K(t-\tau)=\frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)k^2}{3\nu^2(k)}\cos\nu(k)(t-\tau). \tag{4.18}$$

Consider the averaging of this equation over the initial statistical operator

$$\mathfrak{D}_{t_0}=\rho(S)\,\mathfrak{D}(\Sigma) \tag{4.19}$$

and denote:

$$m^*\langle\vec{v}(t)\rangle=\langle\vec{p}(t)\rangle=\underset{(S,\Sigma)}{\mathrm{Sp}}\,\vec{p}(t)\,\mathfrak{D}_{t_0}$$

$$\langle\vec{r}\rangle=\underset{(S,\Sigma)}{\mathrm{Sp}}\,\vec{r}\,\mathfrak{D}_{t_0}=\underset{(S)}{\mathrm{Sp}}\,\vec{r}\rho(S)$$

Because:

$$\langle b_k\rangle=0,\quad \langle b^+_{-k}\rangle=0$$

eq. (4.17) leads to:

$$m^*\frac{d\langle\vec{v}(t)\rangle}{dt}+\int_{t_0}^{t}d\tau\langle\vec{v}(\tau)\rangle K(t-\tau)=-\langle\vec{r}\rangle K(t-\tau)-\vec{E}(t). \tag{4.20}$$

Here $\langle\vec{v}(t)\rangle$ is the averaged velocity of the particle.

Let us investigate now the situation when $\vec{E}(t)$ is a periodic function of t, multiplied by the factor $e^{\epsilon t}$ $(\epsilon>0)$, corresponding to the disappearance of the external electric field at $t\to-\infty$, and when we look for the steady-state solution of (4.20) i.e. the solution represented by the product of the factor $e^{\epsilon t}$ and of the periodic function. As (4.20) is a linear equation we can restrict ourselves to the simplest expression:

$$\vec{E}(t)=\vec{E}_\omega e^{(-i\omega+\epsilon)t}. \tag{4.21}$$

Really if $\vec{E}(t)$ would be a sum of such terms with different ω then the resulting steady-state solution of (4.20) should be the sum of solutions for the case (4.21).

We thus will examine the equation:

$$m^*\frac{d\langle\vec{v}(t)\rangle}{dt}+\int_{-\infty}^{t}d\tau\langle\vec{v}(\tau)\rangle K(t-\tau)=-\vec{E}_\omega e^{(-i\omega+\epsilon)t}.$$

By substituting here:

$$\langle\vec{v}(t)\rangle=\vec{v}_\omega e^{(-i\omega+\epsilon)t}$$

one obtains:

$$\{m^*(-i\omega+\epsilon)+\int_0^\infty K(t)e^{(i\omega-\epsilon)t}dt\}\vec{v}_\omega=-\vec{E}_\omega.$$

Definition (4.18) leads to:

$$\int_0^\infty K(t)e^{(i\omega-\epsilon)t}dt=\frac{1}{V}\sum_{(k)}\frac{\text{Л}^2(k)k^2}{6\nu^2(k)}\left\{\frac{1}{\epsilon-i(\omega+\nu(k))}+\frac{1}{\epsilon-i(\omega-\nu(k))}\right\}.$$

Denote

$$\sum_{(k)}\frac{\text{Л}^2(k)k^2}{6\nu^2(k)}\{\delta(\nu(k)-\Omega)+\delta(\nu(k)+\Omega)\}=I(\Omega) \tag{4.22}$$

Then:

$$I(-\Omega)=I(\Omega);\quad I(\Omega)\geq 0$$

$$\int_0^\infty K(t)e^{(i\omega-\epsilon)t}dt=i\int_{-\infty}^{\infty}I(\Omega)\frac{d\Omega}{\omega+i\epsilon-\Omega}. \tag{4.23}$$

Therefore:

$$\{m^*(-i\omega+\epsilon)+i\int_{-\infty}^{\infty} I(\Omega)\frac{d\Omega}{\omega+i\epsilon-\Omega}\}<\vec{v}(t)> = -\vec{E}_\omega e^{(-i\omega+\epsilon)t}$$

But in virtue of (3.10)

$$\vec{E}_\omega = -e_c\vec{\mathcal{E}}_\omega$$

and by the definition of the current:

$$j_\omega(t) = -e_c <\vec{v}(t)>$$

Hence

$$\{m^*(-i\omega+\epsilon)+i\int_{-\infty}^{\infty} I(\Omega)\frac{d\Omega}{\omega+i\epsilon-\Omega}\}j_\omega(t) = e_c^2\mathcal{E}_\omega e^{(-i\omega+\epsilon)t} \tag{4.24}$$

Let us now perform the limit $V\to\infty$ assuming that for any real ω and positive ϵ

$$\int_{-\infty}^{\infty} I(\Omega)\frac{d\Omega}{\omega+i\epsilon-\Omega} \underset{V\to\infty}{\to} \int_{-\infty}^{\infty} J(\Omega)\frac{d\Omega}{\omega+i\epsilon-\Omega}. \tag{4.25}$$

After taking such a limit, put in eq. (4.24) $\epsilon\to 0$. We then obtain:

$$j_\omega(t) = \frac{1}{Z_+(\omega)}e_c^2\vec{\mathcal{E}}_\omega e^{-i\omega t},$$

where

$$Z_+(\omega) = -m^* i\omega + i\int_{-\infty}^{\infty} J(\Omega)\frac{d\Omega}{\omega-\Omega+i0}. \tag{4.26}$$

Taking here electronic charge e_c as unity we see that expression (4.26) represents just the impedance, corresponding to frequency $-\omega$.

As we will see later, because of limiting processes, all expressions we need, including (4.2), depend only upon the function $J(\Omega)$ and not upon the particular choice of $\nu(k)$, $Л(k)$.

Therefore first we will take a suitable expression for $J(\Omega)$. Let us choose $J(\Omega)$ in the following way:

I) $J(\Omega)$ is an analytic function of the complex variable regular in the stripe:

$$|\mathrm{Im}\,\Omega| \leqq \eta_0$$

II) $J(\Omega) = J(-\Omega)$

III) $|J(\omega)| \leqq \frac{C}{|\Omega|^2}$ for $|\Omega| \geqq \omega_0$; $\omega_0, C = \text{const}$

IV) For real Ω, $J(\Omega) > 0$. (4.27a)

We then take the expressions for $Л(k)$, $\nu(k)$ such that*: $\nu(k) > 0$

$$\frac{1}{V}\sum_{\nu(k)\geqq\omega}\frac{Л^2(k)k^2}{6\nu^2(k)} \leqq \frac{C_1}{\omega},\ C_1 = \text{const., independent of } V \tag{4.27b}$$

$$\frac{1}{V}\sum_{\nu(k)<\omega}\frac{Л^2(k)k^2}{6\nu^2(k)} \to \int_0^\omega J(\Omega)d\Omega,\quad 0<\omega<\infty \tag{4.27c}$$

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* One of the possibilities of finding such expressions for $Л(k), \nu(k)$ is the following one:

We take

$$\vec{k} = \left(\frac{2\pi n_1}{L}, \frac{2\pi n_2}{L}, \frac{2\pi n_3}{L}\right),\quad L^3 = V$$

(n_1, n_2, n_3) being integers both positive and negative, assuming that

$$n_1^2 + n_2^2 + n_3^2 \neq 0$$

thus excluding the zero value $\vec{k}$ from sums over (k)

Then put:

$$\nu(k) = s|k|,\quad Л^2(k) = 2\pi^2\frac{s^3}{|k|^2}J(s|k|),$$

where s is a positive constant independent of V.

In the considered situation it is clear that relation (4.25) holds for any fixed $\epsilon>0$ and that the convergence there is uniform with respect to ω ($-\infty<\omega<\infty$).

Let us introduce the function of the complex variable w :

$$\Delta(w)=i\int_{-\infty}^{\infty} J(\Omega)\frac{d\Omega}{w-\Omega}. \qquad (4.28)$$

We see that it is regular for

$$|\operatorname{Im} w|>0.$$

Because of the properties (4.27) it is easy to see that

$$\Delta(w)=\lim_{V\to\infty} i\int_{-\infty}^{\infty} I(\Omega)\frac{d\Omega}{w-\Omega}, \qquad \operatorname{Im} w\neq 0 \qquad (4.29)$$

Here, in virtue of (4.22):

$$i\int_{-\infty}^{\infty} I(\Omega)\frac{d\Omega}{w-\Omega}=i\sum_{(k)}\frac{\Pi^2(k)k^2}{6\nu^2(k)}\left(\frac{1}{w-\nu(k)}+\frac{1}{w+\nu(k)}\right).$$

Hence this function is analytic in the whole complex plane and its only singularities are poles on the real axis:

$$w=\pm\nu(k)$$

The limiting function however has the cut over all the real axis:

$$\Delta(\omega+i0)-\Delta(\omega-i0)=2\pi J(\omega)>0.$$

So, in fact, we have two analytic functions:

$$\Delta_+(w)=i\int_{-\infty}^{\infty} J(\Omega)\frac{d\Omega}{w-\Omega}, \qquad \text{for } \operatorname{Im} w\geq 0$$

$$\Delta_-(w)=i\int_{-\infty}^{\infty} J(\Omega)\frac{d\Omega}{w-\Omega}, \qquad \text{for } \operatorname{Im} w\leq 0 \qquad (4.30)$$

In view of (IV) these functions are simply related to each other:

$$\Delta_-(w)=-\Delta_+(-w), \qquad \text{for } \operatorname{Im} w<0. \qquad (4.31)$$

Therefore we need to examine only one of them, e.g. $\Delta_+(w)$. Denote

$$\operatorname{Re} w=\omega, \qquad \operatorname{Im} w=y>0. \qquad (4.32)$$

Then, for any fixed $\omega_1>0$:

$$\Delta_+(\omega+iy)=i\int_{|\Omega-\omega|>\omega_1} J(\Omega)\frac{d\Omega}{\omega+iy-\Omega}+i\int_{\omega-\omega_1}^{\omega+\omega_1} J(\Omega)\frac{\omega-\Omega-iy}{(\omega-\Omega)^2+y^2}d\Omega$$

But

$$\int_{\omega-\omega_1}^{\omega+\omega_1}\frac{\omega-\Omega}{(\omega-\Omega)^2+y^2}d\Omega=-\int_{-\omega_1}^{\omega_1}\frac{\Omega}{\Omega^2+y^2}d\Omega=0$$

and thus:

$$\Delta_+(\omega+iy)=i\int_{|\Omega-\omega|>\omega_1} J(\Omega)\frac{d\Omega}{\omega+iy-\Omega}+i\int_{\omega-\omega_1}^{\omega+\omega_1}\frac{J(\Omega)-J(\omega)}{(\omega-\Omega)^2+y^2}(\omega-\Omega)d\Omega+$$

$$+\int_{\omega-\omega_1}^{\omega+\omega_1} J(\Omega)\frac{y}{(\omega-\Omega)^2+y^2}d\Omega \qquad (4.33)$$

from which it follows:

$$\Delta_+(\omega)=\lim_{y\to 0}\Delta_+(\omega+iy)=i\int_{|\Omega-\omega|>\omega_1} J(\Omega)\frac{d\Omega}{\omega-\Omega}+$$

$$+i\int_{\omega-\omega_1}^{\omega+\omega_1}\frac{J(\Omega)-J(\omega)}{\omega-\Omega}d\Omega+\pi J(\omega) \qquad (4.34)$$

So, $\Delta_+(\omega)$ is the analytic function also on the real axis. By using (4.33) it is easy to show that:

$$|\Delta_+(w)| \leqq \frac{\text{const}}{|w|}, \quad |w| \to \infty. \tag{4.35}$$

We further have:

$$\Delta_+(\omega) = \Delta_-(\omega) + 2\pi J(\omega) = -\Delta_+(-\omega) + 2\pi J(\omega).$$

In view of condition (I) the function

$$-\Delta_+(-w) + 2\pi J(w)$$

is analytic for

$$0 > \operatorname{Im} w \geqq -\eta_0. \tag{4.36}$$

As it coincides with $\Delta_+(w)$ on the real axis, we see that $\Delta_+(w)$ previously defined for $\operatorname{Im} w \geqq 0$ can be analytically continued in the region (4.36).

So, we may write:

$$\Delta_+(w) = -\Delta_+(-w) + 2\pi J(w), \quad \text{for} \quad 0 \geqq \operatorname{Im} w \geqq -\eta_0. \tag{4.37}$$

It can also be established that inequality (4.35) holds everywhere for

$$\operatorname{Im} w \geqq -\eta_0. \tag{4.38}$$

Consider now the impedance function

$$Z_+(w) = -im^* w + \Delta_+(w)$$

in the domain (4.38) and note that on the upper half-plane and on the real axis it can have no zeroes because in view of (4.33)

$$\operatorname{Re} Z_+(w) > 0 \quad \text{for} \quad \operatorname{Im} w \geqq 0$$

Therefore the zeroes of this function in the considered domain (4.38) if they exist at all must be confined to region (4.36). But

$$\Delta_+(w) \to 0, \quad |w| \to \infty$$

and for this reason the zeroes of $Z_+(w)$ can be found only in the bounded region:

$$|\operatorname{Re} w| \leqq \text{const}, \quad 0 > \operatorname{Im} w \geqq -\eta_0. \tag{4.39}$$

As is well known, an analytic function can have only a finite number of zeroes in such a bounded region. If zero-points are really contained in (4.30) take a quantity $\eta > 0$ so that $-\eta$ is greater than the imaginary parts of all these points. If on the other hand the region (4.39) contains no zeroes of $Z_+(w)$ at all, take $\eta = \eta_0$. In any case we see that by choosing an appropriate value of $\eta > 0$ we can attain the situation when the region

$$\operatorname{Im} w \geqq -\eta \tag{4.40}$$

does not contain any zeroes of the impedance function $Z_+(w)$.

Therefore the admittance function

$$\frac{1}{Z_+(w)}$$

is a regular analytic function in region (4.40).

Its behaviour at infinity is given by

$$\frac{1}{Z_+(w)} = \frac{1}{-m^*iw + \Delta_+(w)} = -\frac{1}{m^*iw} + \frac{\Delta_+(w)}{m^*iw(-m^*iw + \Delta_+(w))} = \tag{4.41}$$

$$= -\frac{1}{m^*iw} + O\left(\frac{1}{w^3}\right), \quad |w| \to \infty.$$

In conclusion let us give an example. Take:

$$\Delta_+(w) = i\frac{k_0^2}{2}\left\{\frac{1}{w - \nu_0 + i\gamma} + \frac{1}{w + \nu_0 + i\gamma}\right\}, \ \gamma > 0, \ \operatorname{Im} w > -\gamma, \tag{4.42}$$

$$\Delta_-(w) = -\Delta_+(-w) = i\frac{k_0^2}{2}\left\{\frac{1}{w + \nu_0 - i\gamma} + \frac{1}{w - \nu_0 - i\gamma}\right\}, \operatorname{Im} w < \gamma.$$

Then

$$J(\omega) = \frac{1}{2\pi}\{\Delta_+(\omega) - \Delta_-(\omega)\} = \frac{k_0^2}{2\pi}\{\frac{\gamma}{(\omega-\nu_0)^2+\gamma^2} + \frac{\gamma}{(\omega+\nu_0)^2+\gamma^2}\}. \quad (4.42)$$

In this example all our conditions are satisfied. The same situation would be attained also in the case when instead of one term (4.42) a finite sum of such expressions was considered.

After these rather lengthy discussions of the analyticity of impedance and admittance function, let us return to our fundamental equation (4.17) in which we take:

$$\vec{E}(t) = \sum_{(\omega)} \vec{E}_\omega e^{-i\omega t} . \quad (4.43)$$

We will try to solve it by the Laplace transform method. Thus both parts of (4.17) will be multiplied by

$$e^{iwt} , \quad w = \Omega + i\delta \quad (4.44)$$

and integrated over t:

$$\int_{t_0}^{\infty} dt\, e^{iwt} \frac{d\vec{p}(t)}{dt} + \frac{1}{m^*}\int_{t_0}^{\infty} dt\, e^{iwt} \int_{t_0}^{t} d\tau K(t-\tau)\vec{p}(\tau) =$$

$$= -\vec{r}\int_{t_0}^{\infty} dt\, e^{iwt} K(t-t_0) - \sum_{(\omega)} \vec{E}_\omega \int_{t_0}^{\infty} e^{i(w-\omega)t} dt - \quad (4.45)$$

$$- \frac{i}{\sqrt{V}}\sum_{(k)} \left(\frac{\hbar}{2\nu(k)}\right)^{1/2} \Pi(k)\vec{k}\{b_k \int_{t_0}^{\infty} dt\, e^{i(w-\nu(k))t} e^{i\nu(k)t_0} +$$

$$+ b^+_{-k} \int_{t_0}^{\infty} dt\, e^{i(w+\nu(k))t} e^{-i\nu(k)t_0} \} .$$

But

$$\int_{t_0}^{\infty} dt\, e^{iwt} \frac{d\vec{p}(t)}{dt} = -\vec{p}\, e^{iwt_0} - iw\int_{t_0}^{\infty} dt\, e^{iwt}\, \vec{p}(t) ,$$

$$\int_{t_0}^{\infty} dt\, e^{iwt} \int_{t_0}^{t} d\tau K(t-\tau)\vec{p}(\tau) = \int_0^{\infty} K(t)\, e^{iwt}\, dt \int_{t_0}^{\infty} e^{iwt}\, \vec{p}(t)\, dt.$$

Therefore:

$$\frac{1}{m^*}\{-im^*w + \int_0^{\infty} K(t)\, e^{iwt}\, dt\} \int_{t_0}^{\infty} \vec{p}(t)\, e^{iwt}\, dt =$$

$$= \vec{p}\, e^{iwt_0} - \vec{r}\, e^{iwt_0} \int_0^{\infty} K(t)\, e^{iwt} dt + \sum_{(\omega)} \vec{E}_\omega \frac{e^{i(w-\omega)t_0}}{i(w-\omega)} +$$

$$+ \frac{1}{\sqrt{V}}\sum_{(k)} \left(\frac{\hbar}{2\nu(k)}\right)^{1/2} \Pi(k)\vec{k}\{\frac{b_k e^{iwt_0}}{w-\nu(k)} + \frac{b^+_{-k} e^{iwt_0}}{w+\nu(k)}\} .$$

Here, because of (4.23)

$$\int_0^{\infty} K(t)\, e^{iwt}\, dt = i\int_{-\infty}^{\infty} I(\nu)\frac{d\nu}{w-\nu} .$$

Denote:

$$-im^*w + i\int_{-\infty}^{\infty} I(\nu)\frac{d\nu}{w-\nu} = Z^{(V)}(w) , \quad (4.46)$$

$$i\int_{-\infty}^{\infty} I(\nu)\frac{d\nu}{w-\nu} = \Delta^{(V)}(w) .$$

We then obtain:

$$\int_{t_0}^{\infty} \vec{p}(t)\, e^{iwt}\, dt = \frac{m^*\vec{p}\, e^{iwt_0}}{Z^{(V)}(w)} - m^* r \frac{\Delta^{(V)}(w)}{Z^{(V)}(w)} e^{iwt_0} -$$

$$-i \sum_{(\omega)} m^* \vec{E}_\omega \frac{e^{iwt_0}\, e^{-i\omega t_0}}{(w-\omega) Z^{(V)}(w)} + \tag{4.47}$$

$$+ \frac{1}{\sqrt{V}} \sum_{(k)} m^* \left(\frac{\hbar}{2\nu(k)}\right)^{1/2} \frac{\Pi(k)\vec{k}\, e^{iwt_0}}{Z^{(V)}(w)} \left\{ \frac{b_k}{w-\nu(k)} + \frac{b^+_{-k}}{w+\nu(k)} \right\}.$$

But, as is well known:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\delta - i\Omega)t} \left\{ \int_{t_0}^{\infty} f(t)\, e^{(i\Omega-\delta)t}\, dt \right\} d\Omega .$$

$$t > t_0$$

For this reason, by using the notation :

$$\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(\delta-i\Omega)(t-t_0)}}{(\Omega+i\delta-\nu) Z^{(V)}(\Omega+i\delta)} d\Omega = f(\nu, \delta, t-t_0) ,$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(\delta-i\Omega)(t-t_0)}}{Z^{(V)}(\Omega+i\delta)} d\Omega = \tag{4.48}$$

$$= \frac{1}{2\pi m^*} \int_{-\infty}^{\infty} \left\{ -\frac{1}{i\Omega-\delta} + \frac{\Delta^{(V)}(\Omega+i\delta)}{(i\Omega-\delta) Z^{(V)}(\Omega+i\delta)} \right\} e^{(\delta-i\Omega)(t-t_0)} d\Omega = g_0(\delta, t-t_0)$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Delta^{(V)}(\Omega+i\delta)}{Z^{(V)}(\Omega+i\delta)} e^{(\delta-i\Omega)(t-t_0)} d\Omega = g_1(\delta, t-t_0) ,$$

from (4.47) we get:

$$\vec{p}(t) = \vec{p}^{(S)}(t) + \vec{p}^{(E)}(t) + \vec{p}^{(\Sigma)}(t) ,$$

$$\vec{p}^{(S)}(t) = m^*\vec{p}\, g_0(\delta, t-t_0) - m^*\vec{r}\, g_1(\delta, t-t_0) ,$$

$$\vec{p}^{(E)}(t) = -m^* \sum_{(\omega)} \vec{E}_\omega f(\omega, \delta, t-t_0)\, e^{-i\omega t_0} , \tag{4.49}$$

$$\vec{p}^{(\Sigma)}(t) = \frac{-im^*}{\sqrt{V}} \sum_{(k)} \left(\frac{\hbar}{2\nu(k)}\right)^{1/2} \Pi(k)\vec{k} \{ b_k f(\nu(k), \delta, t-t_0) + b^+_{-k} f(-\nu(k), \delta, t-t_0) \}.$$

It is to be stressed that function (4.48) depends essentially on V.

In virtue of our choice which has led us to conditions (I)-(IV) and (4.27) we can perform the limiting process $V \to \infty$. Till yet δ could be an arbitrary positive quantity.

We now choose:

$$\delta = \frac{\eta}{2} . \tag{4.50}$$

On the other hand it is easy to see that:

$$g_0(\delta, t-t_0) \to \psi_0(t-t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(\delta-i\Omega)(t-t_0)}}{Z_+(\Omega+i\delta)} d\Omega ,$$

$$g_1(\delta, t-t_0) \to \psi_1(t-t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Delta_+(\Omega+i\delta)}{Z_+(\Omega+i\delta)} e^{(\delta-i\Omega)(t-t_0)} d\Omega , \tag{4.51}$$

$$f(\nu, \delta, t-t_0) \to \phi(\nu, t-t_0) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(\delta-i\Omega)(t-t_0)}}{(\Omega+i\delta-\nu) Z_+(\Omega+i\delta)} d\Omega .$$

$$V \to \infty$$

Because of the identity:

$$\frac{1}{Z(\Omega+i\delta)} = \frac{1}{\delta-i\Omega} + \frac{\Delta(\Omega+i\delta)}{(i\Omega-\delta)\,Z(\Omega+i\delta)}$$

and because δ is now fixed (4.50) it can also be shown that the convergence:

$$|f(\nu,\delta,t-t_0)-\phi(\nu,t-t_0)| \to 0, \quad \nu|f(\nu,\delta,t-t_0)-\phi(\nu,t-t_0)| \to 0 \qquad (4.52)$$
$$V \to \infty$$

is uniform with respect to real ν, when

$$|t-t_0| \leqq T$$

T being any constant independent of V.

Let us proceed to study the behaviour of the limit functions ψ_0, ψ_1, ϕ for $t-t_0 \to \infty$.

It will be convenient to recall that the functions

$$\Delta_+(\Omega+i\delta), \quad \frac{1}{Z_+(\Omega+i\delta)}$$

are regular analytic functions of Ω in the region where

$$\mathrm{Im}\,\Omega \geqq -\delta-\eta = -3\delta.$$

Therefore the integration occuring in the expressions of ψ_0, ψ_1 can be shifted from the real axis to the axis

$$(-3i\delta-\infty, \; -3i\delta+\infty),$$

which amounts to the change of the variable:

$$\Omega \to \Omega - 3i\delta$$

For this reason:

$$\psi_1(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Delta_+(\Omega-i\eta)}{Z_+(\Omega-i\eta)}\, e^{-i\Omega(t-t_0)}\, d\Omega\, e^{-\eta(t-t_0)},$$

$$\psi_0(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\Omega(t-t_0)}}{Z_+(\Omega-i\eta)}\, d\Omega\, e^{-\eta(t-t_0)} =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\Delta_+(\Omega-i\eta)}{(\eta+i\Omega)Z_+(\Omega-i\eta)}\, e^{-i\Omega(t-t_0)}\, d\Omega e^{-\eta(t-t_0)}$$

because:

$$\int_{-\infty}^{\infty} \frac{e^{-i\Omega(t-t_0)}}{\eta+i\Omega}\, d\Omega = 0, \qquad \text{for } t > t_0 .$$

Therefore by taking into account previously formulated inequalities we get:

$$\begin{aligned} |\psi_1(t-t_0)| &\leqq K_1 e^{-\eta(t-t_0)} \\ |\psi_0(t-t_0)| &\leqq K_0 e^{-\eta(t-t_0)} \quad ; t > t_0 \end{aligned} \qquad (4.53)$$

where K_0, K_1 are constants.

Let us apply the same procedure to the expression of $\phi(\nu,t-t_0)$. We must only note that in the region

$$\mathrm{Im}\,\Omega + \delta \geqq -\eta$$

the function under the sign of the integral (4.51) has a pole

$$\Omega = \nu - i\delta.$$

Hence:

$$\phi(\nu,t-t_0) = \frac{e^{-i\nu(t-t_0)}}{Z_+(\nu)} + \frac{i}{2\pi} e^{-\eta(t-t_0)} \int_{-\infty}^{\infty} \frac{e^{-i\Omega(t-t_0)}}{(\Omega+i\delta-\nu)\,Z_+(\Omega-i\eta)}\, d\Omega = \qquad (4.54)$$

$$= \frac{e^{-i\nu(t-t_0)}}{Z_+(\nu)} + \frac{i}{2\pi} e^{-\eta(t-t_0)} \int_{-\infty}^{\infty} \frac{\Delta_+(\Omega-i\eta)e^{-i\Omega(t-t_0)}}{(\Omega-i\eta-\nu)(i\Omega+\eta)Z_+(\Omega-i\eta)}\, d\Omega,$$

because

$$\int_{-\infty}^{\infty} \frac{e^{-i\Omega(t-t_0)}}{(\Omega - i\eta - \nu)(\Omega - i\eta)} d\Omega = 0, \quad t > t_0 .$$

Expression (4.54) leads to:

$$|\phi(\nu, t-t_0) - \frac{e^{-i\nu(t-t_0)}}{Z_+(\nu)}| \leqq K_2 e^{-\eta(t-t_0)} , \tag{4.55}$$

$$|\nu\phi(\nu, t-t_0) - \nu \frac{e^{-i\nu(t-t_0)}}{Z_+(\nu)}| \leqq K_3 e^{-\eta(t-t_0)} .$$

$$t > t_0$$

Here K_2, K_3 are the constants.

We now return to the initial statistical operator

$$\mathfrak{L}_{t_0} = \rho(S)\, \mathfrak{L}(\Sigma).$$

Take a positive number c independent of V and note that the eigenvalues of the operator

$$\vec{p}^{\,2} + c^2 \vec{r}^{\,2} \tag{4.56}$$

are

$$(2n_1 + 1) c\hbar + (2n_2 + 1) c\hbar + (2n_3 + 1) c\hbar ,$$

where n_1, n_2, n_3 are non-negative integers.

Let us also take a positive constant K independent of V so that

$$K^2 \gg c\hbar .$$

We now choose $\rho(S)$ as a positive operator with the usual normalization

$$\mathrm{Sp}_{(S)}\, \rho(S) = 1$$

in such a way that its eigenfunctions be orthogonal to all eigenfunctions of (4.56), for which:

$$(2n_1 + 1) c\hbar + (2n_2 + 1) c\hbar + (2n_3 + 1) c\hbar > K^2 .$$

Denote

$$\theta(z) = \begin{cases} 1, & z \geqq 0 \\ 0, & z < 0 \end{cases} .$$

We then see:

$$\mathrm{Sp}_{(S)}\, \theta(K^2 - \vec{p}^{\,2} - c^2 \vec{r}^{\,2})\, \rho(S) = 1 .$$

For this reason operator (4.56) can be treated as a bounded operator

$$\vec{p}^{\,2} + c^2 \vec{r}^{\,2} \leqq K^2$$

and thus

$$|\vec{p}| \leqq K, \quad |\vec{r}| \leqq \frac{K}{c} .$$

Here $|\dots|$ denotes as usual the norm of the opeartor.

In view of (4.51), (4,53)

$$\vec{p}^{(S)}(t) \underset{V\to\infty}{\to} m^*\{\psi_0(t-t_0)\vec{p} - \psi_1(t-t_0)\vec{r}\} , \tag{4.57}$$

$$|\lim_{V\to\infty} \vec{p}^{(S)}(t)| \leqq m^* K \{K_0 + \frac{K_1}{c}\} e^{-\eta(t-t_0)} .$$

Note also that the vector $\vec{p}^{(E)}(t)$ is an ordinary c-vector and that in virtue of (4.51), (4.55):

$$|\lim_{V\to\infty} \vec{p}^{(E)}(t) - m^* \vec{v}(t)| < \sum_{(\omega)} |\vec{E}_\omega| K_2 e^{-\eta(t-t_0)} , \tag{4.58}$$

$$\vec{v}(t) = - \sum_{(\omega)} \frac{\vec{E}_\omega}{Z_+(\omega)} e^{-i\omega t} .$$

We see that here $\vec{v}(t)$ is just the average velocity corresponding to the steady-state solution.

Consider at last the vector $\vec{p}^{(\Sigma)}(t)$ and denote its components by $p_j^{(\Sigma)}(t)$, $j = 1,2,3$.

We have:

$$\underset{(S,\Sigma)}{Sp}\ p_j^{(\Sigma)}(t)\, p_{j'}^{(\Sigma)}(\tau)\mathfrak{L}_{t_0} = \underset{(\Sigma)}{Sp}\ p_j^{(\Sigma)}(t)\, p_{j'}^{(\Sigma)}(\tau)\,\mathfrak{L}(\Sigma).$$

Here us usual

$$\mathfrak{L}(\Sigma) = \text{const}\ e^{-\beta\sum_{(k)}\hbar\nu(k)\, b_k^+ b_k} \quad . \qquad (4.59)$$

By using the spherical symmetry of functions $Л(k)$, $\nu(k)$ we obtain:

$$\underset{(\Sigma)}{Sp}\ p_j^{(\Sigma)}(t)\, p_{j'}^{(\Sigma)}(\tau)\,\mathfrak{L}(\Sigma) = (m^*)^2\delta_{j,j'}\, F(t,\tau,t_0) \ , \qquad (4.60)$$

where

$$F(t,\tau,t_0) = \frac{1}{V}\sum_{(k)}\frac{\hbar}{6\nu(k)}Л^2(k)k^2\Big\{\frac{1}{1-e^{-\beta\hbar\nu(k)}}f(\nu(k),\delta,t-t_0)\,f(-\nu(k),\delta,\tau-t_0)+$$

$$+\frac{e^{-\beta\hbar\nu(k)}}{1-e^{-\beta\hbar\nu(k)}}f(-\nu(k),\delta,t-t_0)\,f(\nu(k),\delta,\tau-t_0)\Big\},$$

$$\delta_{j,j'} = \begin{cases} 1, & j=j' \\ 0, & j\neq j' \end{cases} \quad .$$

From (4.22) it now follows:

$$F(t,\tau,t_0) = \int_0^\infty d\nu I(\nu)\frac{\hbar\nu}{1-e^{-\beta h\nu}}\{f(\nu,\delta,t-t_0)f(-\nu,\delta,\tau-t_0) +$$

$$+ e^{-\beta\hbar\nu}f(-\nu,\delta,t-t_0)f(\nu,\delta,\tau-t_0)\} .$$

Let us take into account (4.27),(4.51), (4.52) and pass here to the limit:

$$\underset{V\to\infty}{F(t,\tau,t_0)} \to \int_0^\infty d\nu J(\nu)\frac{\hbar\nu}{1-e^{-\beta\hbar\nu}}\{\phi(\nu,t-t_0)\phi(-\nu,\tau-t_0) +$$

$$+ e^{-\beta\hbar\nu}\phi(-\nu,t-t_0)\phi(\nu,\tau-t_0)\}$$

from which because of (4.55) we obtain:

$$|\lim_{V\to\infty}F(t,\tau,t_0) - F(t-\tau)| \leq \tilde{K}(e^{-\eta(t-t_0)} + e^{-\eta(\tau-t_0)}) \ , \qquad (4.61)$$

$$\tilde{K} = \text{const}$$

where:

$$F(t-\tau) = \int_0^\infty d\nu J(\nu)\frac{\hbar\nu}{1-e^{-\beta\hbar\nu}}\{\frac{e^{-i\nu(t-\tau)} + e^{-\beta\hbar\nu + i\nu(t-\tau)}}{Z_+(\nu)Z_+(-\nu)}\}. \qquad (4.62a)$$

Because $J(\nu) = J(-\nu)$ this relation can also be written as:

$$F(t-\tau) = \int_{-\infty}^\infty d\nu J(\nu)\frac{\hbar\nu}{1-e^{-\beta\hbar\nu}}\ \frac{e^{-i\nu(t-\tau)}}{Z_+(\nu)Z_+(-\nu)}. \qquad (4.62b)$$

But:

$$2\pi J(\nu) = \Delta_+(\nu) - \Delta_-(\nu) = Z_+(\nu) - Z_-(\nu) = Z_+(\nu) + Z_+(-\nu)$$

from which it follows:

$$\frac{1}{Z_+(\nu)Z_+(-\nu)}J(\nu) = \frac{1}{2\pi}\ \frac{Z_+(\nu)+Z_+(-\nu)}{Z_+(\nu)Z_+(-\nu)} = \frac{1}{2\pi}\{\frac{1}{Z_+(\nu)} - \frac{1}{Z_-(\nu)}\}.$$

Because $J(\nu)$ is real and positive we have:

$$2\pi J(\nu) = \text{Re}\, Z_+(\nu) + \text{Re}\, Z_+(-\nu) \ ,$$

$$0 = \text{Im}\, Z_+(\nu) + \text{In}\, Z_+(-\nu) \ .$$

On the other hand:

$$Z_+(\nu) = i \int_{-\infty}^{\infty} \frac{J(\Omega)}{\Omega - \nu + i0} d\Omega$$

and thus

$$\mathrm{Re}\, Z_+(\nu) = \pi J(\nu) = \mathrm{Re}\, Z_+(-\nu).$$

Therefore

$$Z_+(-\nu) = Z_+^*(\nu),$$

$$Z_+(\nu) Z_+(-\nu) = |Z_+(\nu)|^2$$

and so we can write:

$$F(t-\tau) = \int_{-\infty}^{\infty} G(\nu) \frac{\hbar\nu}{1-e^{-\beta h\nu}} e^{-i\nu(t-\tau)} d\nu,$$

$$G(\nu) = \frac{1}{2\pi}\left\{\frac{1}{Z_+(\nu)} - \frac{1}{Z_-(\nu)}\right\} = \frac{J(\nu)}{|Z_+(\nu)|^2}. \tag{4.62c}$$

It is interesting to point out that the equilibrium average

$$\langle p_j(t)\, p_{j'}(\tau)\rangle_{eq} = \lim_{V\to\infty} \underset{(S,\Sigma)}{Sp}\, p_j(t)\, p_{j'}(\tau) \mathfrak{L}_{eq}(S,\Sigma)$$

corresponding to hamiltonian (4.5):

$$\mathfrak{L}_{eq}(S,\Sigma) = Z^{-1} e^{-\beta H_L}; \quad Z = \underset{(S,\Sigma)}{Sp}\, e^{-\beta H_L}$$

is equal to

$$\delta_{j,j'} (m^*)^2 F(t-\tau)$$

or, otherwise:

$$\langle p_j(t)\, p_{j'}(\tau)\rangle_{eq} = \lim_{t_0\to-\infty} \lim_{V\to\infty} \underset{(\Sigma)}{Sp}\, p_j^{(\Sigma)}(t)\, p_{j'}^{(\Sigma)}(\tau) \mathfrak{L}_{t_0} =$$

$$= \lim_{t_0\to-\infty} \lim_{V\to\infty} \underset{(S,\Sigma)}{Sp}\, p_j'(t)\, p_{j'}'(\tau) \mathfrak{L}_{t_0}.$$

Consider now the probability density of momentum distribution of the particle S for the hamiltonian $\bar{H}_L$ of (4.11) and use the general equation (1.21).
Then:

$$\int e^{-i\vec{\lambda}\vec{p}} w_t(\vec{p})\, d\vec{p} = \underset{(S,\Sigma)}{Sp}\, e^{-i\vec{\lambda}\vec{p}(t)} \mathfrak{L}_{t_0}.$$

Because $\vec{p}^{(\Sigma)}(t)$, $\vec{p}^{(E)}(t)$ are defined by (4.49) one obtains:

$$\int e^{-i\vec{\lambda}\vec{p}} w_t(\vec{p})\, d\vec{p} = \underset{(S)}{Sp}\, \{ \underset{(\Sigma)}{Sp}\, e^{-i\vec{\lambda}\vec{p}^{(\Sigma)}(t)} \mathfrak{L}(\Sigma)\} \times$$

$$\times\, e^{-i\vec{\lambda}(\vec{p}^{(S)}(t)+\vec{p}^{(E)}(t))} \rho(S).$$

From the fact that $p_j^{(\Sigma)}(t)$ are linear forms in bose amplitudes it follows:

$$\underset{(\Sigma)}{Sp}\, e^{-i\vec{\lambda}\vec{p}^{(\Sigma)}(t)} \mathfrak{L}(\Sigma) = e^{-\frac{\langle \vec{\lambda}\vec{p}^{(\Sigma)}(t)\rangle^2}{2}},$$

where

$$\langle \vec{\lambda}\vec{p}^{(\Sigma)}(t)\rangle^2 = \underset{(\Sigma)}{Sp}\, (\vec{\lambda}\vec{p}^{(\Sigma)}(t))^2 \mathfrak{L}(\Sigma) - (m^*)^2 \lambda^2 F(t,t,t_0).$$

We thus have:

$$\tilde{w}_t(\vec{\lambda}) = \int e^{-i\vec{\lambda}\vec{p}} \, w_t(\vec{p})\, d\vec{p} =$$

$$= \underset{(S)}{\mathrm{Sp}}\, \rho(S) \exp\{-\frac{\lambda^2 (m^*)^2}{2} F(t,t,t_0) - i\vec{\lambda}(\vec{p}^{(S)}(t) + \vec{p}^{(E)}(t)\},$$

which yields:

$$w_t(\vec{p}) = \frac{1}{(2\pi)^3} \int e^{i\vec{\lambda}\vec{p}} \, \tilde{w}_t(\vec{\lambda}) d\vec{\lambda} =$$

$$= (\frac{1}{2\pi m^*})^3 (\frac{2\pi}{F(t,t,t_0)})^{3/2} \underset{(S)}{\mathrm{Sp}}\, \rho(S) \exp\{- \frac{(\vec{p} - \vec{p}^{(S)}(t) - \vec{p}^{(E)}(t))^2}{2(m^*)^2 F(t,t,t_0)}\}.$$

The limiting process $V \to \infty$ leads to:

$$\lim_{V\to\infty} w_t(\vec{p}) =$$

$$= (\frac{1}{2\pi m^*})^3 (\frac{2\pi}{a(t,t_0)})^{3/2} \underset{(S)}{\mathrm{Sp}}\, \rho(S) \exp\{- \frac{(\vec{p} - \vec{p}^{(S)}_\infty(t) - \vec{p}^{(E)}_\infty(t))^2}{2a(t,t_0)(m^*)^2}\}, \quad (4.63)$$

where:

$$a(t,t_0) = \lim_{V\to\infty} F(t,t,t_0),$$

$$\vec{p}^{(S)}_\infty(t) = \lim_{V\to\infty} \vec{p}^{(S)}(t),$$

$$\vec{p}^{(E)}_\infty(t) = \lim_{V\to\infty} \vec{p}^{(E)}(t).$$

Therefore from (4.57), (4.58), (4.61), (4.62c) it now follows:

$$\lim_{V\to\infty} w_t(p) \to (\frac{1}{m^*})^3 (\frac{1}{2\pi F(0)})^{3/2} \exp\{- \frac{(\vec{p} - m^*\vec{v}(t))^2}{2(m^*)^2 F(0)}\},$$

$$t - t_0 \to \infty \quad (4.64)$$

$$F(0) = \int_{-\infty}^{\infty} G(\nu) \frac{\hbar\nu}{1 - e^{-\hbar\beta\nu}} d\nu.$$

We thus see that after the limit $V \to \infty$ has been carried out the momentum distribution function exhibits irreversible behaviour and tends to a steady-state form when $t - t_0 \to \infty$.

In the particular case when $\vec{E} = 0$ this form corresponds to the equilibrium distribution

$$(\frac{1}{m^*})^3 (\frac{1}{2\pi F(0)})^{3/2} \exp\{- \frac{\vec{p}^2}{2(m^*)^2 F(0)}\} \quad (4.65)$$

which has a "quasi-maxwellian" behaviour. We say "quasi" because instead of the temperature module $\theta = \frac{1}{\beta}$ in (4.65) there stands

$$m^* \int_{-\infty}^{\infty} G(\nu) \frac{\hbar\nu}{1 - e^{-\beta\hbar\nu}} d\nu.$$

We may also note that the general steady-state distribution function is obtained from the equilibrium function (4.65) by introducing the drift velocity $\vec{v}(t)$. When

$$\vec{E}(t) = \vec{E} = \text{const}$$

this velocity is also constant:

$$\vec{v} = - \frac{\vec{E}}{Z_+(0)} = - \frac{\vec{E}}{\pi J(0)}. \quad (4.66)$$

Let us now pass to our main aim - to evaluate expression (4.2) in our model, based on the hamiltonian $\bar{H}_L$.

We first note that in our model Heisenberg's equations are linear and it is easy to see that the commutators

$$[r_j(t),\ r_{j'}(\tau)]\ ;\quad j, j' = 1,2,3$$

are c-numbers.

For this reason:

$$e^{i\vec{k}\vec{r}(\tau)}\ e^{-i\vec{k}\vec{r}(t)} = e^{-i\vec{k}(\vec{r}(t)-\vec{r}(\tau))}\ \exp\frac{[\vec{k}\vec{r}(\tau),\vec{k}\vec{r}(t)]}{2} =$$

$$= e^{-i\vec{k}\int_\tau^t \frac{\vec{p}(S)}{m^*}ds}\ \exp\frac{[\vec{k}\vec{r}(\tau),\ \vec{k}\vec{r}(t)]}{2}$$

which leads to:

$$\Phi_k(t,\tau,t_0) =$$

$$= \left\{ e^{-i\vec{k}\int_\tau^t \frac{\vec{p}^{(E)}(S)}{m^*}ds}\ \exp\frac{[\vec{k}\vec{r}(\tau),\ \vec{k}\vec{r}(t)]}{2} \right\} \times \tag{4.67}$$

$$\times\ \underset{(\Sigma)}{\mathrm{Sp}}\ e^{-i\vec{k}\int_\tau^t \frac{\vec{p}^{(\Sigma)}(S)}{m^*}ds}\ \mathfrak{D}(\Sigma)\ \underset{(S)}{\mathrm{Sp}}\ e^{-i\vec{k}\int_\tau^t \frac{\vec{p}^{(S)}(S)}{m^*}ds}\ \rho(S).$$

To disentangle the commutators let us note that:

$$[r_j(t), r_{j'}(\tau)] = [r_j(t), r_{j'}(\tau) - r_{j'}(t)] = -\frac{1}{m^*}\int_\tau^t [r_j(t), p_{j'}(s)]ds.$$

We further have:

$$[r_j(t), p_{j'}(s)] = [r_j(t) - r_j(s), p_{j'}(s)] + i\hbar\delta_{j,j'} =$$

$$= i\hbar\delta_{j,j'} + \frac{1}{m^*}\int_s^t [p_j(\sigma), p_{j'}(s)]d\sigma$$

and therefore

$$[r_j(t), r_{j'}(\tau)] = -i\hbar\delta_{j,j'}\frac{(t-\tau)}{m^*} - \left(\frac{1}{m^*}\right)^2 \int_\tau^t ds \int_s^t [p_j(\sigma), p_{j'}(s)]d\sigma =$$

$$= -i\hbar\delta_{j,j'}\frac{(t-\tau)}{m^*} - \left(\frac{1}{m^*}\right)^2 \int_\tau^t ds \int_s^t d\sigma [p_j^{(\Sigma)}(\sigma), p_{j'}^{(\Sigma)}(s)] - \tag{4.68}$$

$$- \left(\frac{1}{m^*}\right)^2 \int_\tau^t ds \int_s^t d\sigma [p_j^{(S)}(\sigma), p_{j'}^{(S)}(s)].$$

Because the considered commutators are c-numbers we may write, in virtue of (4.60):

$$[p_j^{(\Sigma)}(\sigma), p_{j'}^{(\Sigma)}(s)] = \underset{(\Sigma)}{\mathrm{Sp}}\ [p_j^{(\Sigma)}(\sigma),\ p_{j'}^{(\Sigma)}(s)]\mathfrak{D}(\Sigma) =$$

$$= \delta_{j,j'}(m^*)^2\{F(\sigma,s,t_0) - F(s,\sigma,t_0)\}.$$

On the other hand (4.49) yields:

$$\frac{1}{(m^*)^2}[p_j^{(S)}(\sigma), p_{j'}^{(S)}(s)] = i\hbar\delta_{j,j'}\{g_0(\delta,\sigma-t_0)g_1(\delta,s-t_0) -$$

$$- g_0(\delta,s-t_0)g_1(\delta,\sigma-t_0)\}$$

We thus obtain:

$$[(\vec{k}\vec{r}(t)),\ (\vec{k}\vec{r}(\tau))] = -i\hbar k^2\frac{(t-\tau)}{m^*} +$$

$$+ k^2 \int_\tau^t ds \int_s^t d\sigma \{F(s,\sigma,t_0) - F(\sigma,s,t_0) +$$

$$+ i\hbar [g_0(\delta,s-t_0) g_1(\delta,\sigma-t_0) - g_0(\delta,\sigma-t_0) g_1(\delta,s-t_0)]\}. \quad (4.69)$$

It is also easy to see that:

$$\mathop{Sp}_{(\Sigma)} e^{-i\vec{k}\int_\tau^t \frac{\vec{p}^{(\Sigma)}(s)\,ds}{m^*}} = \exp\{-\frac{k^2}{2}\int_\tau^t ds \int_\tau^t d\sigma F(s,\sigma,t_0)\}. \quad (4.70)$$

Relations (4.69), (4.70) are to be substituted into expression (4.67).

By taking the limit $V\to\infty$ and computing the asymptotic part of (4.67) for $t_0\to-\infty$ we obtain:

$$\Phi_k(t,\tau,-\infty) =$$

$$= e^{-i\int_\tau^t (\vec{k}\vec{v}(s))ds} \exp k^2\{i\hbar\frac{t-\tau}{2m^*} + \frac{1}{2}\int_\tau^t ds \int_s^t d\sigma (F(\sigma-s) - F(s-\sigma)) -$$

$$- \frac{1}{2}\int_\tau^t ds \int_\tau^t d\sigma F(s-\sigma)\}.$$

But, because of (4.62c)

$$\frac{1}{2}\int_\tau^t ds \int_\tau^t d\sigma F(s-\sigma) = \int_{-\infty}^{\infty} G(\nu)\frac{\hbar\nu}{1-e^{-\beta\hbar\nu}} \frac{1-\cos\nu(t-\tau)}{\nu^2} d\nu$$

$$F(\sigma-s) - F(s-\sigma) = \int_{-\infty}^{\infty} G(\nu)\hbar\nu e^{-i\nu(\sigma-s)} d\nu = -i\int_{-\infty}^{\infty} G(\nu)\hbar\nu \sin\nu(\sigma-s) d\nu$$

$$\int_\tau^t ds \int_s^t d\sigma \{F(\sigma-s) - F(s-\sigma)\} = i\hbar\int_{-\infty}^{\infty} G(\nu)\{\frac{\sin\nu(t-\tau)}{\nu} - (t-\tau)\}d\nu$$

Thus we are led to:

$$\Phi_k(t,\tau,-\infty) =$$

$$= e^{-i\int_\tau^t (\vec{k}\vec{v}(s))ds} \exp\{i\hbar k^2(\frac{t-\tau}{2m^*} + \frac{1}{2}\int_{-\infty}^{\infty} G(\nu)[\frac{\sin\nu(t-\tau)}{\nu} - (t-\tau)]d\nu) - \quad (4.70)$$

$$- \hbar k^2 \int_{-\infty}^{\infty} G(\nu)\frac{h\nu}{1-e^{-\beta\hbar\nu}} \frac{1-\cos\nu(t-\tau)}{\nu^2} d\nu\}$$

This expression is to be substituted into equation (4.3) in which also the limits:

1) $V\to\infty$, 2) $t_0\to-\infty$, 3) $\epsilon\to 0$

must be carried out.

In such a way we obtain the general equation from which the results of mentioned papers /2,3/ could be obtained. Note that there $m^*=m$ and the function (4.42) is used with $\gamma\to 0$. We may also note that by substituting $f(p)=p^2$ into (2.20) the equation for the rate of change of the electron kinetic energy will be obtained. Such an equation can be disentangled just as previously by using the linear model, and be applied to verify the results following from (4.3) with (4.71).

Acknowledgements

I would like to thank Prof. J.T.Devreese for stimulating discussions.

I also wish to thank Prof. V.A.Meshcheryakov's group at the Laboratory of Theoretical Physics and the Publishing Department of JINR for the their help in preparing the manuscript for printing.

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Received by Publishing Department
on August 2, 1978.