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ON A CLASS OF EXACTLY SOLUBLE STATISTICAL MECHANICAL MODELS WITH NONPOLYNOMIAL INTERACTIONS

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Об одном классе точно решаемых моделей статистической механики с неполиномиальным взаимодействием

Метод аппроксимирующего гамильтониана Н.Н.Боголюбова (мл.) обобщается на случай моделей с неполиномиальным взаимодействием интенсивных операторов.

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On a Class of Exactly Soluble Statistical Mechanical Models with Nonpolynomial Interactions

The approximating Hamiltonian of N.N.Bogolubov, Jr. is generalized to models with nonpolynomial in intensive observables interactions. The original Hamiltonian is proved to be thermodynamically equivalent to a linear in the intensive observables trial Hamiltonian. We show that the exact expression for the free energy density in the thermodynamic limit can be obtained from a min-max principle for the system with trial Hamiltonian.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. *INTRODUCTION*

In our previous paper [1] we have proposed a further extension of the approximating Hamiltonian method of Bogolubov, Jr. [2, 3], which permits the asymptotically exact (i.e., exact in the thermodynamic limit) investigation of a general class of model systems with a nonpolynomial interaction term. The interaction is a function of the space average of some quasilocal operator (observable, see Haag [4] and the *Appendix*). Thus it is a function of an intensive observable of the system. In the case under consideration, the N-body Hamiltonian, defined in a region $\Lambda \subset \boldsymbol{R}^{\nu}$ $(\nu = \dim \boldsymbol{R}^{\nu})$ with a finite volume $|\Lambda|$, acts on the Hilbert space of states $\mathfrak{H}_\Lambda$ and has the form*

$$H_\Lambda = T_\Lambda - h|\Lambda| A_\Lambda - |\Lambda|\phi(A_\Lambda) . \tag{1.1}$$

Here $h \in \boldsymbol{R}^1$, T_Λ and A_Λ are *self-adjoint* operators satisfying the following conditions:

(i) A_Λ is an *intensive observable* generated by the space averaging of some uniformly bounded in $\mathfrak{H}_\Lambda$ self-adjoint quasilocal operator, i.e., there exists $M > 0$ such that for all $\Lambda \subset \boldsymbol{R}^{\nu}$, with $|\Lambda| < \infty$

$$\| A_\Lambda \|_{\mathfrak{H}_\Lambda} \le M;$$

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* For other types of nonpolynomial models (generalized Dicke-type models) see ref. [5].

(ii) the operator T_Λ which generally defines as *extensive observable* of the system, is such that there exists $K'>0$ satisfying

$$\| [T_\Lambda, A_\Lambda]_- \|_{\mathfrak{H}_\Lambda} \le K'$$

for all $\Lambda \subset R^\nu$ with $|\Lambda| < \infty$;

(iii) the operator-valued function $\phi(A_\Lambda)$ can be defined by the spectral representation

$$\phi(A_\Lambda) = \int_{-M}^{M+0} dE_\lambda (A_\Lambda) \phi(\lambda),$$

where $\phi(\lambda)$ is a twice differentiable function on $I = [-M, M]$ ($\phi(\lambda) \in C^2(I)$) such that there exists $K>0$ and the following inequality holds:

$$|\phi''(\lambda)| \le K,$$

without loss of generality we further assume $\phi(0) = \phi'(0) = 0$ (see (1.1));

(iv) the operator T_Λ generates the Gibbs semigroup $\{\exp(-\beta T_\Lambda)\}_{\beta>0}$ i.e., $\exp(-\beta T_\Lambda) \in$ Trace-class for all $\beta>0$;

(v) by virtue of conditions (i) and (iv) the operator

$$\Gamma_\Lambda(x) = T_\Lambda - x|\Lambda| A_\Lambda, \quad x \in R^1 \tag{1.2}$$

also generates the Gibbs semigroup; we require the existence of the thermodynamic limit $\text{t-lim}(\cdot) \equiv \lim_{|\Lambda|/N = v,\ N\to\infty,\ |\Lambda|\to\infty} (\cdot)$ (where $|\Lambda| \to \infty$ in the sense of Fisher [6]) for the free energy density

$$F_\Lambda(x) = -(\beta|\Lambda|)^{-1} \ln \operatorname{Tr} \exp(-\beta \Gamma_\Lambda(x)), \tag{1.3}$$

namely, for all $x \in R^1$, $\beta>0$, $v>0$ there exists function such that

$$\text{t-lim}\, F_\Lambda(x) = F(x), \quad F_\Lambda(x) \in C^\infty(R^1); \tag{1.4}$$

(vi) define the *approximating Hamiltonian*

$$H_{0,\Lambda}(a) = \Gamma_\Lambda(h + \phi'(a)) + |\Lambda| (a\phi'(a) - \phi(a)) \tag{1.5}$$

which depends on the real parameter $a \in I$; for the system with Hamiltonian (1.5) and all $\beta>0$ and $v>0$ the following *clustering property* must hold:

$$\text{t-lim}\{ \langle A_\Lambda^2 \rangle_{H_{0,\Lambda}(\bar a_\Lambda)} - \langle A_\Lambda \rangle^2_{H_{0,\Lambda}(\bar a_\Lambda)} \} = 0, \tag{1.6}$$

where $\bar a_\Lambda$ is determined from the equation

$$\min_{a \in S_\Lambda} f_\Lambda [H_{0,\Lambda}(a)] = f_\Lambda [H_{0,\Lambda}(\bar a_\Lambda)],$$
$$S_\Lambda = \{ a \in R^1 : \ a = \langle A_\Lambda \rangle_{H_{0,\Lambda}(a)} \}. \tag{1.7}$$

Here the use has been made of the notations

$$\langle \cdot \rangle_H = \operatorname{Tr}\{(\cdot) \exp(-\beta H)\} / \operatorname{Tr} \exp(-\beta H),$$

$$f_\Lambda[\cdot] = -(\beta|\Lambda|)^{-1} \ln \operatorname{Tr} \exp\{-\beta(\cdot)\},$$

for the thermal average and the free energy density, respectively.

R e m a r k 1.1. The clustering condition (1.6) corresponds to certain restrictions on the magnitude of the fluctuations of the intensive observable A_Λ in the system described by the approximating Hamiltonian (for further details see the *Appendix*).

P r o p o s i t i o n 1.1.[1] Let the Hamiltonian of the system be given by Eq. (1.1) and let conditions (i)-(vi) be satisfied, then

$$\text{t-lim} | f_\Lambda [H_\Lambda] - \min_{a \in S_\Lambda} f_\Lambda [H_{0,\Lambda}(a)] | = 0, \tag{1.8}$$

where $H_{0,\Lambda}(a)$ has been defined by Eq. (1.5).

Remark 1.2. As we have shown in paper [1]

$$\min_{a\in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)] = \min_{a\in R^1}\max_{b\in R^1} f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)], \tag{1.9}$$

where

$$\mathcal{H}_{0,\Lambda}(a,b) = \Gamma_{0,\Lambda}(h+\Phi_1'(a)+\Phi_2'(b)) + |\Lambda|\{a\Phi_1'(a) - \Phi_1(a) + b\Phi_2'(b) - \Phi_2(b)\}. \tag{1.10}$$

Here

$$\Phi_1(a) = \tilde{\phi}(a) + \frac{1}{2}La^2, \quad \Phi_2(b) = -\frac{1}{2}Lb^2, \ (L > 3K), \tag{1.11}$$

and the function $\tilde{\phi}(a) \in C^2(R^1)$ is a twice differentiable extension of $\phi(a)\in C^2(I)$ to R^1, which satisfies condition (iii).

In the general case (this means that approximating Hamiltonian (1.5) is not to be one-particle operator) the direct calculation of the thermodynamic limit $t\text{-}\lim\{\min_{a\in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)]\}$ is hardly practicable because of the absence of an explicit expression for $f_\Lambda[H_{0,\Lambda}(a)]$ at large but finite values N and $|\Lambda|$ as well as because of the lack of explicit information about the structure of the set S_Λ for $|\Lambda|\to\infty$. In the present paper it will be shown how to avoid these difficulties provided the limit function (see (v))

$$t\text{-}\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)] = F(h+\Phi_1'(a) - Lb) - \frac{1}{2}Lb^2 + a\Phi_1'(a) - \Phi_1(a) \tag{1.12}$$

is known.

Remark 1.3. Simultaneously with our work [1] the same problem has been studied by den Ouden, Capel and Perk[7]. They have considered the same Hamiltonian as (1.1) but containing an *analytic function* of a finite number of intensive (normalized [7]) self-adjoint operators $A_\Lambda^{(i)}$, $i = 1,2,\dots,n$, under stronger than (i)-(vi) restrictions on the operators T_Λ, $A_\Lambda^{(i)}$ and the function $\phi(\cdot)$ (see below remark 1.4). In the recent preprint [8] the same authors have given a convex-envelope formulation of the problem in the fixed-magnetization ensemble *.

This paper presents a further development of the approach proposed in [1] for systems with nonpolynomial interactions. In particular we shall give here a complete proof (see Sections 2 and 3) of the fact

$$t\text{-}\lim\{\min_{a\in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)]\} = \min_{a\in S}\{t\text{-}\lim f_\Lambda[H_{0,\Lambda}(a)]\} \tag{1.13}$$

important for practical applications of Proposition 1.1. In Eq. (1.13) the set S is defined by inequalities (1.14) (see below) which, as it was first shown in [7], replace the usual *self-consistence* equations (*molecular-field* equations). Below a new derivation of Eq. (1.14) is given which is based entirely on the analysis of the auxiliary two-parameter variational problem for the limit function (1.12). The important particular cases of *attractive:* $\phi''(a) > 0$ and *repulsive*: $\phi''(a) < 0$ interactions are also paid special attention (see Section 3). The main result of the present paper can be formulated as the following

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* The convex-envelope construction has been proposed by Lebowitz and Penrose [9] for mathematically rigorous derivation of the Van der Waals equation for classical gases with a long-range Kac-type potential (see also [10]). Generalization to quantum systems has been obtained by Lieb [11] (for further generalizations see [12] also the review article [13]).

Theorem 1.1. Let the Hamiltonian of the system be given by Eq. (1.1) and let the operators T_Λ, A_Λ and the function $\phi(\cdot)$ satisfy conditions (i)-(vi), then
(a) $t\text{-}\lim f_\Lambda[H_\Lambda]$ exists for all $h \in R^1$, $\beta > 0$ and $v > 0$;
(b) $t\text{-}\lim f_\Lambda[H_\Lambda] = \min_{a \in S} \{t\text{-}\lim f_\Lambda[H_{0,\Lambda}(a)]\}$, where

$$S = \{a \in R^1 : -F'(h+\phi'(a)-0) \le a \le -F'(h+\phi'(a)+0)\}. \quad (1.14)$$

Remark 1.4. The above formulated theorem is a generalization of the result obtained in [7,8] under the condition that function is analytic on I and operators T_Λ, A_Λ satisfy certain "short-range" conditions. We extend this result to the case of the broader class of functions $\phi(a) \in C^2(I)$ (iii) and reduce the restrictions on the range of interactions included in T_Λ and A_Λ to the more general conditions (iv)-(vi). Particularly we do not need the boundedness of the intensive (normalized [7]) operator $|\Lambda|^{-1}T_\Lambda$. Thus T_Λ may correspond for example to the kinetic energy operator of particles enclosed in a region $\Lambda \subset R^\nu$.

The proof of Theorem 1.1 follows a line of reasoning different from [7] and is based essentially on Proposition 1.1 and the main Lemma 2.1 (see Section 2). The idea of our proof consists in the consecutive establishing of the following four relations:

$$t\text{-}\lim |f_\Lambda[H_\Lambda] - \min_{a \in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)]| = 0; \quad (1)$$

$$\min_{a \in S_\Lambda} f_\Lambda[H_{0,\Lambda}(a)] = \min_{a \in R^1} \max_{b \in R^1} f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]; \quad (2)$$

$$t - \lim\{\min_{a \in R^1} \max_{b \in R^1} f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)] = \\ = \min_{a \in R^1} \max_{b \in R^1} \{t - \lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\}; \quad (3)$$

$$\min_{a \in R^1} \max_{b \in R^1} \{t\text{-}\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\} = \min_{a \in S} \{t\text{-}\lim f_\Lambda[H_{0,\Lambda}(a)]\}. \quad (4)$$

Equations (1) (Proposition 1.1) and (2) have been obtained in [1] and the proof of Eqs. (3) and (4) is given respectively in Sections 2 and 3 of the present paper. The combination of equalities (2)-(4) gives (1.13) and of (1)-(4) gives the statement (b) of Theorem 1.1.

2. THE MAIN LEMMA

We start with the proof of Eq. (3) (see Section 1) which is the content of the following main

Lemma 2.1. Let $\{f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\}$ be a sequence of functions generated by the two-parameter family of Hamiltonians (1.10) with operators T_Λ and A_Λ satisfying conditions (i)-(vi) (Section 1). Then

(a) $\min_{a \in R^1} \max_{b \in R^1} \{t\text{-}\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\} = f(\beta, v, h)$

exists for all $h \in R^1$, $\beta > 0$ and $v > 0$;

(b) $f(\beta, v, h) = t - \lim\{\min_{a \in R^1} \max_{b \in R^1} f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]$.

Proof. (a) Let us denote $z = h + \Phi_1'(a) - Lb$. Then (see (1.3) and (1.10)) one has

$$f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)] = F_\Lambda(z) + a\Phi_1'(a) - \Phi_1(a) - \frac{1}{2}Lb^2. \quad (2.1)$$

Conditions (i) and (v) (Section 1) imply the *uniform equicontinuity* of the family $\{F_\Lambda(x)\}$, since

$$|F_\Lambda(x') - F_\Lambda(x'')| \le M|x' - x''| \quad (2.2)$$

for arbitrary $x', x'' \in R^1$. Hence, in the thermodynamic limit we obtain that the limit function $F(x)$ (see (1.4)) obeys the Lipschitz condition

$$|F(x') - F(x'')| \le M|x' - x''|. \quad (2.3)$$

Using (2.3) one easily verifies that for all fixed $a \in R^1$ the function $t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]$ reaches the absolute maximum with respect to $b \in R^1$ on the bounded interval $|b| \le 2M$. Denote by $\bar{b}(a)$ the point at which the maximum of $t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]$ is attained, and by $\bar{b}_\Lambda(a)$ the corresponding point for the function $f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]$.
On the other hand, the uniform equicontinuity of the family $\{F_\Lambda(x)\}$ and the pointwise convergence (1.4) imply the uniform convergence of $\{F_\Lambda(x)\}$ to $F(x)$ on every bounded set from R^1 (see, e.g., ref. /14/). Hence, for all $a, b \in R^1$ and arbitrary fixed $D>0$ such that $|h+\Phi_1'(a) - Lb| \le D$ we find

$$|t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)] - f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]| \le \delta_\Lambda(D),$$

where $t-\lim \delta_\Lambda(D) = 0$. Thus, for every fixed $a \in R^1$ one has:

$$f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b})] \ge f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}(a))] \ge$$
$$\ge t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}(a))] - \delta_\Lambda(D_a), \tag{2.4}$$

where $D_a = |h| + |\Phi_1'(a)| + 2LM$. Similarly:

$$t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}(a))] \ge \{t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\}\Big|_{b=\bar{b}_\Lambda(a)} \ge$$
$$\ge f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}_\Lambda(a))] - \delta_\Lambda(D_a)$$

and, taking into account (2.4), we obtain

$$|\max_{b\in R^1}\{t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\} - f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}_\Lambda(a))]| \le \delta_\Lambda(D_a). \tag{2.5}$$

Thus:

$$t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}(a))] = \max_{b\in R^1}\{t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\} =$$
$$= t-\lim\{\max_{b\in R^1} f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\} = t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}_\Lambda(a))]. \tag{2.6}$$

Consider now the sequence $\{f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}_\Lambda(a))]\}_\Lambda$, $a \in R^1$. In ref. /1/ we have shown that the functions $\{\bar{b}_\Lambda(a)\}$ are continuously differentiable with respect to $a \in R^1$ and:

$$\bar{b}_\Lambda(a) \equiv \langle A_\Lambda \rangle_{\mathcal{H}_{0,\Lambda}(a,\bar{b}_\Lambda(a))}. \tag{2.7}$$

By differentiating the above identity with respect to the variable $a \in R^1$ and making use of (1.11) and condition (iii) (Section 1) we conclude that

$$0 \le \frac{d\bar{b}_\Lambda(a)}{da} \le \frac{\Phi_1''(a)}{L} \le (1+\frac{K}{L}). \tag{2.8}$$

Hence, and from inequality (2.2), condition (iii) and the existence of the limit (1.4), it follows that the limit function $t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}(a))]$ is *continuous* in $a \in R^1$.
Further, from the estimate

$$f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}_\Lambda(a))] - f_\Lambda[\mathcal{H}_{0,\Lambda}(0,\bar{b}_\Lambda(0))] \ge$$
$$\ge -M(4K+3L)|a| - \frac{1}{2}(L-3K)a^2 \tag{2.9}$$

in which we have taken into account the fact that $|\bar{b}_\Lambda(a)| \le M$ (see (2.7) and condition (i))as well as (2.8) and the inequality

$$a\Phi_1'(a) - \Phi_1(a) \ge \frac{1}{2}(L-3K)a^2 \tag{2.10}$$

it follows that the function $t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}_\Lambda(a))]$ attains its absolute minimum in the bounded interval $|a| \le R = 2M(4K+3L)/(L-3K)$. Let $\bar{a}$, $|\bar{a}| \le R$, denote the point that provides the absolute minimum value of the function $t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar{b}_\Lambda(a))]$ on R^1. Then from (2.6) we obtain the existence of:

$$\min_{a\in R^1}\max_{b\in R^1}\{t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\} = t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(\bar a,\bar b_\Lambda(\bar a))]. \tag{2.11}$$

(b) Let us return now to estimate (2.5). For all $a\in[-R,R]$ we have

$$|t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar b(a))] - f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar b_\Lambda(a))]| \le \delta_\Lambda(\tilde D), \tag{2.12}$$

where $\tilde D = \max_{|a|\le R} D_a$ is finite. The estimate (2.9) implies also the existence of the point $a=\bar a_\Lambda$ $(|\bar a_\Lambda|\le R)$ that provides the absolute minimum value of the function $f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar b_\Lambda(a))]$ on R^1. Therefore from (2.12) and the definition of the points $a=\bar a$, $a=\bar a_\Lambda$ we obtain

$$\begin{aligned} t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(\bar a,\bar b(\bar a))] - f_\Lambda[\mathcal{H}_{0,\Lambda}(\bar a_\Lambda,\bar b_\Lambda(\bar a_\Lambda))] &\le \\ \le \{t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar b(a))]\}\big|_{a=\bar a_\Lambda} - f_\Lambda[\mathcal{H}_{0,\Lambda}(\bar a_\Lambda,\bar b_\Lambda(\bar a_\Lambda))] &\le \\ \le \delta_\Lambda(\tilde D). \end{aligned} \tag{2.13}$$

Similarly,

$$\begin{aligned} f_\Lambda[\mathcal{H}_{0,\Lambda}(\bar a_\Lambda,\bar b_\Lambda(\bar a_\Lambda))] - t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(\bar a,\bar b(\bar a))] &\le \\ \le f_\Lambda[\mathcal{H}_{0,\Lambda}(\bar a,\bar b_\Lambda(\bar a))] - t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(\bar a,\bar b(\bar a))] &\le \\ \le \delta_\Lambda(\tilde D). \end{aligned} \tag{2.14}$$

From (2.13) and (2.14) we find

$$|f_\Lambda[\mathcal{H}_{0,\Lambda}(\bar a_\Lambda,\bar b_\Lambda(\bar a_\Lambda))] - t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(\bar a,\bar b(\bar a))]| \le \delta_\Lambda(\tilde D). \tag{2.15}$$

Hence, in the thermodynamic limit we get

$$\begin{aligned} \min_{a\in R^1}\max_{b\in R^1}\{t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\} &= \\ = t-\lim\{\min_{a\in R^1}\max_{b\in R^1} f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\} \end{aligned} \tag{2.16}$$

which completes the proof of the lemma.

C o r o l l a r y 2.1. If the function $F(x)$ is known, Lemma 2.1 gives the thermodynamic limit of the free energy density for the model (1.1) in terms of the two-parameter variational problem (see (1.8), (1.9) and (2.16)):

$$t-\lim f_\Lambda[H_\Lambda] = \min_{a\in R^1}\max_{b\in R^1}\{t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\}. \tag{2.17}$$

This result generalizes the *mini-max principle* due to Bogolubov, Jr. /2, 3/ for the models with the nonpolynomial interaction (1.1).

3. PROOF OF THEOREM 1.1

(a) Proposition 1.1 (see (1.8)), Remark 1.2 and Lemma 2.1 imply the existence of $t-\lim f_\Lambda[H_\Lambda]$. The fact that $\min_{a\in R^1}\max_{b\in R^1} f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]$ is independent of the choice of the auxiliary parameter $L>3K$ in Eq. (1.11) follows from (1.9).

(b) Note that the functions $\{F_\Lambda(x)\}$ and consequently the function $F(x)$ (see (1.2)-(1.4)) are *convex* on R^1. Therefore the left derivative, $F'(x-0)$, and the right derivative, $F'(x+0)$, exist for all $x\in R^1$. Hence, the condition for maximum with respect to $b\in R^1$ in (2.17) is equivalent (taking into account Eq. (2.1)) to the inequalities

$$-F'(h+\Phi_1'(a)-Lb-0) \le b \le -F'(h+\Phi_1'(a)-Lb+0). \tag{3.1}$$

From the monotone non-increasing of the left and right hand sides of (3.1) with the increase of $b\in R^1$ it follows that for *each* $a\in R^1$ the solution $b=\bar b(a)$ of inequalities (3.1) is *unique*. For $b=\bar b(a)$ we have

$$-F'(\bar z(a)-0)\le \bar b(a)\le -F'(\bar z(a)+0)\ , \tag{3.2}$$

where

$$\bar z(a)\equiv h+\Phi_1'(a)-L\bar b(a)\ . \tag{3.3}$$

It should be emphasized that the uniqueness of $\bar b(a)$ (or $\bar b_\Lambda(a)$, , which is the solution of inequalities (3.1) with $F_\Lambda(h+\Phi_1'(a)-Lb\pm 0)$)) is an immediate consequence of the *strict convexity* of the function $t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]$ (or $f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]$) with respect to $b\in R^1$. Furthermore, from the uniform in $b\in K$ (for any compact set $K\subset R^1$) convergence of the sequence $\{f_\Lambda[\mathcal{H}_{0,\Lambda}(a,b)]\}$ (see Proof (a) of Lemma 2.1) and from the uniqueness of the points $\bar b_\Lambda(a)$ and $\bar b(a)$ it follows that for every $a\in R^1$ one has:

$$t-\lim \bar b_\Lambda(a)=\bar b(a)\ . \tag{3.4}$$

We need now some properties of the function $\bar b(a)$. Integrating inequalities (2.8) over the interval $[a_1,a_2]$ and proceedings to the thermodynamic limit we find that

$$0\le \bar b(a_2)-\bar b(a_1)\le \frac{1}{L}[\Phi_1'(a_2)-\Phi_1'(a_1)]\ , \tag{3.5}$$

i.e., $\bar b(a)$ is Lipschitz-continuous (see (1.11) and condition (iii), Section 1) monotone non-decreasing function of $a\in R^1$.

Consider now the conditions for the determination of the points $\{a_n\}$ which correspond to the local minima of the function $t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar b(a))]$. By definition of the point $a_m\in\{a_n\}$, there exists a neighbourhood $\Sigma(a_m)$ of a_m, such that for all $a\in\Sigma(a_m)$

$$t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar b(a))]-t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a_m,\bar b(a_m))]\ge 0\ .$$

Hence, by using (1.12) and the concavity of $\Phi_1(a)$, it is easy to obtain the inequality

$$F(\bar z(a))-F(\bar z(a_m))-L\bar b(a_m)[\bar b(a)-\bar b(a_m)]+ \\ +a[\Phi_1'(a)-\Phi_1'(a_m)]\ge 0\ . \tag{3.6}$$

Next, from the convexity of the function $F(x)$ on R^1 it follows that for any $x_1\le x_2$

$$(x_2-x_1)F'(x_2-0)\le F(x_2)-F(x_1)\le (x_2-x_1)F'(x_1+0)\ . \tag{3.7}$$

If $a_m\le a$, $a\in\Sigma(a_m)$, then from (3.3) and (3.5) we have $\bar z(a_m)\le \bar z(a)$ and from (3.6), in view of (3.2) and (3.7), we obtain

$$0\le[\Phi_1'(a)-\Phi_1'(a_m)][a+F'(\bar z(a_m)+0)]- \\ -L[\bar b(a)-\bar b(a_m)][\bar b(a_m)+F'(\bar z(a_m)+0)]\le \\ \le \Phi_1''(\xi_a)(a-a_m)(a-\bar b(a_m))\ , \tag{3.8}$$

where $\xi_a\in(a_m,a)$, $\Phi_1''(\xi_a)>0$ (see (1.11)). Hence, for all $a\in\{a\in\Sigma(a_m):\ a_m\le a\}$ one has:

$$a\ge \bar b(a_m)\ . \tag{3.9}$$

By similar arguments, for all $a\in\{a\in\Sigma(a_m):\ a\le a_m\}$ (now $\bar z(a)\le\bar z(a_m)$), we find

$$a\le \bar b(a_m)\ . \tag{3.10}$$

Combining inequalities (3.9) and (3.10) we conclude that $a_m=\bar b(a_m)$. We have thus proven the following important fact: Every point $a=a_m$, which corresponds to a local minimum of the function $t-\lim f_\Lambda[\mathcal{H}_{0,\Lambda}(a,\bar b(a))]$, satisfies the equation

$$a = \bar{b}(a)\,. \tag{3.11}$$

We observe now that on the set S of all the solutions of equation (3.11):

$$S = \{ a \in \boldsymbol{R}^1 : a = \bar{b}(a) \} \tag{3.12}$$

the Hamiltonians $\mathcal{H}_{0,\Lambda}(a, \bar{b}(a))$ (1.10) and $H_{0,\Lambda}(a)$ (1.5) coincide, therefore

$$\min_{a \in \boldsymbol{R}^1} \{ t-\lim f_\Lambda [\mathcal{H}_{0,\Lambda}(a, \bar{b}(a))] \} =$$

$$= \min_{a \in S} \{ t-\lim f_\Lambda [\mathcal{H}_{0,\Lambda}(a, \bar{b}(a))] \} =$$

$$= \min_{a \in S} \{ t-\lim f_\Lambda [H_{0,\Lambda}(a)] \}\,. \tag{3.13}$$

The definition of the set S (3.12) can be re-formulated in terms of the linearized system $\Gamma_\Lambda(x)$ (see (1.2)-(1.4)). To this end we notice that if $\hat{a} \in S$, then from (3.2) it follows that

$$-F'(h + \phi'(\hat{a}) - 0) \le \hat{a} \le -F'(h + \phi'(\hat{a}) + 0) \tag{3.14}$$

and, conversely, if (3.14) holds, then $b = \hat{a}$ satisfies inequalities (3.1) for $a = \hat{a}$. Hence, by the uniqueness of the point $\bar{b}(a)$, we get $\bar{b}(\hat{a}) = \hat{a}$. Therefore

$$S = \{ a \in \boldsymbol{R}^1 :\ -F'(h + \phi'(a) - 0) \le a \le -F'(h + \phi'(a) + 0) \}$$

which (see (2.17) and (3.13)) completes the proof of Theorem 1.1.

C o r o l l a r y 3.1. Let the function $\phi(\cdot)$ in the initial Hamiltonian (1.1) correspond to an *attractive type* of interaction, i.e., let for all $a \in \boldsymbol{I}$

$$\phi''(a) > 0. \tag{3.15}$$

Then

$$\min_{a \in S} \{ t-\lim f_\Lambda [H_{0,\Lambda}(a)] \} = \min_{a \in \boldsymbol{R}^1} \{ t-\lim f_\Lambda [H_{0,\Lambda}(a)] \} =$$

$$= t-\lim f_\Lambda [H_{0,\Lambda}(\tilde{a})]\,, \tag{3.16}$$

where $a = \tilde{a}$ satisfies the self-consistence equation (1.7), taken in the thermodynamic limit:

$$a = t-\lim \langle A_\Lambda \rangle_{H_{0,\Lambda}(a)}\,. \tag{3.17}$$

Really, from the Bogolubov inequality, the spectral representation (iii) (Section 1) and condition (3.15) it follows that

$$f_\Lambda [H_{0,\Lambda}(a)] - f_\Lambda [H_\Lambda] \ge$$

$$\ge \frac{1}{2} \left\langle \int_{-M}^{M+0} dE_\lambda(A_\Lambda)\, \phi''(\xi_\Lambda)(\lambda - a)^2 \right\rangle_{H_{0,\Lambda}(a)} \ge 0\,, \tag{3.18}$$

where $\xi_\Lambda \in (-M, M)$. Hence, taking into account that $S \subset \boldsymbol{R}^1$ we have

$$t-\lim f_\Lambda [H_\Lambda] \le \min_{a \in \boldsymbol{R}^1} \{ t-\lim f_\Lambda [H_{0,\Lambda}(a)] \} \le$$

$$\le \min_{a \in S} \{ t-\lim f_\Lambda [H_{0,\Lambda}(a)] \}\,. \tag{3.19}$$

Since the function $t-\lim f_\Lambda [H_{0,\Lambda}(a)]$ is continuous (see (1.3)-(1.5)) and the set S is bounded ($S \subset \boldsymbol{I}$, because $|F'(x \pm 0)| \le M$, , see (i) and (v), Section 1) and closed (see (3.5) and (3.12)), it reaches the minimum on some subset of the set S. From (3.19) and Theorem 1.1 it follows that equality (3.16) must holf for any point $\tilde{a}$ belonging to this subset. Next, taking into account (3.15) and the existence of the left and right derivatives of the function $t-\lim f_\Lambda[H_{0,\Lambda}(a)]$ (see (1.3)-(1.5)) the minimum condition for $\tilde{a}\in R^1$ takes the form

$$F'(h+\phi'(\tilde{a})-0)+\tilde{a}\leq 0,$$
$$F'(h+\phi'(\tilde{a})+0)+\tilde{a}\geq 0. \tag{3.20}$$

On the other hand, by definition $\tilde{a}\in S$. Therefore, (3.14) and (3.20) imply the differentiability of the function $t-\lim f_\Lambda[H_{0,\Lambda}(a)]$ at the point $a=\tilde{a}$:

$$\tilde{a}=-F'(h+\phi'(\tilde{a})). \tag{3.21}$$

Equality (3.17) is then a consequence of the Griffiths lemma/15/ about the convergence of the derivatives of the convergent sequence $\{F_\Lambda(x)\}$ of convex functions at the points of differentiability of $\{F_\Lambda(x)\}$ and the limit function $F(x)$:

$$-F'(h+\phi'(\tilde{a}))=t\text{-}\lim\{-F'_\Lambda(h+\phi'(\tilde{a}))\}=$$

$$=t-\lim\langle A_\Lambda\rangle_{H_{0,\Lambda}(\tilde{a})}. \tag{3.22}$$

R e m a r k 3.1. As was shown in ref./1/, in the case of attractive interaction theorem 1.1 holds without the clustering condition (1.6). This specific property of attraction has been exploited in paper /16/ for the particular case of $\phi(a)=\frac{J}{2}a^2$, $J>0$. The result of this paper can be generalized now to the case of an arbitrary twice differentiable function $\phi(a)$, such that $\phi''(a)>0$, for $a\in I$.

R e m a r k 3.2. If the interaction in Hamiltonian 1.1 is not purely attractive, then the clustering property is *essential*. In the case of $\phi(a)=\frac{J}{2}a^2$, $J<0$, this question has been discussed in/17/ (see also /3/). Den Ouden et al./7/ have made an attempt to replace the clustering

condition by a "short-range interaction" condition for the operators T_Λ and A_Λ simultaneously. In/18/ it has been assumed that the bounded self-adjoint operators T_Λ and A_Λ are one-particle operators, then the clustering property follows trivially.

C o r o l l a r y 3.2. Let the function $\phi(\cdot)$ in (1.1) correspond to a *repulsive type* of interaction, i.e., for all $a\in I$ one has

$$\phi''(a)<0. \tag{3.23}$$

Then

$$\min_{a\in S}\{t-\lim f_\Lambda[H_{0,\Lambda}(a)]\}=\max_{a\in R^1}\{t-\lim f_\Lambda[H_{0,\Lambda}(a)]\}=$$
$$=t-\lim f_\Lambda[H_{0,\Lambda}(\bar{a})], \tag{3.24}$$

where

$$\bar{a}=t-\lim\bar{a}_\Lambda \tag{3.25}$$

and $\bar{a}_\Lambda$ is the *unique* solution of the self-consistence equation for the finite system (compare (1.7) for S_Λ):

$$a=\langle A_\Lambda\rangle_{H_{0,\Lambda}(a)}. \tag{3.26}$$

Really, by virtue of the convexity of function $F(x)$ and condition (3.23), the set S contains *only one* point $a=\bar{a}$. Hence, using Theorem 1.1, we obtain

$$\min_{a\in S}\{t-\lim f_\Lambda[H_{0,\Lambda}(a)]\}=t-\lim f_\Lambda[H_{0,\Lambda}(\bar{a})]=t-\lim f_\Lambda[H'_\Lambda]. \tag{3.27}$$

Next, taking into account the spectral representation (iii), (see Section 1), the Bogolubov inequality and (3.23) we get

$$f_\Lambda[H_\Lambda]-f_\Lambda[H_{0,\Lambda}(a)]\geq -\frac{1}{2}\langle\int_{-M}^{M+0}dE_\lambda(A_\Lambda)\phi''(\xi_\Lambda)(\lambda-a)^2\rangle_{H_\Lambda}\geq 0,$$

where $\xi_\Lambda \in (-M, M)$. Therefore

$$t-\lim f_\Lambda[H_{0,\Lambda}(a)] \le \max_{a \in R^1} \{t-\lim f_\Lambda[H_{0,\Lambda}(a)]\} \le$$

$$\le t-\lim f_\Lambda[H_\Lambda]. \qquad (3.28)$$

Thus Eq. (3.24) is a direct consequence of (3.27) and (3.28). Equality (3.25) follows from the uniform on any bounded interval of R^1 convergence of the sequence $\{f_\Lambda[H_{0,\Lambda}(a)]\}$ to the limit function $t-\lim f_\Lambda[H_{0,\Lambda}(a)]$ (see Proof (a) of Lemma 2.1) and from the uniqueness (due to (3.23)) of the points $\bar{a}_\Lambda$ and $\bar{a}$.

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APPENDIX

1. Let the region $\Lambda \subset R^\nu$ (or Z^ν) be of finite volume with respect to the usual Lebesgue measure on R^ν: $\mu(\Lambda) = |\Lambda| < \infty$ (or with respect to the corresponding discrete measure on Z^ν). Consider the local C^*-algebra of the observables $\mathfrak{A}_\Lambda$, contained in the domain Λ, that is the algebra of all bounded operators acting on the Hilbert space of states $\mathfrak{H}_\Lambda$ /6/. If $x \in \Lambda$, then the operator-valued function $A: x \to A(x) \in \mathfrak{A}_\Lambda$ is called a local observable (local operator). Alongside with that it is convenient to define "qiasi-local quantities" (Haag /4/). Let the continuous function $f_Q(x,y)$ be such that there exists $Q>0$ and $f_Q(x,y) = 0$ for $|x-y| > Q$, then

$$A_Q(y) = \int_\Lambda dx f_Q(x,y) A(x)$$

is called a *quasi-local operator*.

Further, denote the group of translations of the space R^ν (the lattice Z^ν) by $G = \{R^\nu\}$ ($\{Z^\nu\}$). Let $g_x \in G$, then there exists a representation $g_x \to \tau_x$ of the group G into the group of automorphism of the quasi-local algebra $\mathfrak{A} = \overline{\bigcup_{\Lambda \subset R^\nu} \mathfrak{A}_\Lambda}$ which acts on the operators $A(x_0)$ (or $A_Q(x_0)$)) as follows:

$$\tau_x A(x_0) = A(x_0 + x)$$

for arbitrary $x_0, x \in R^\nu$. The group G is locally compact and abelian, therefore there exists an invariant Haar measure dg on this group. For $G = \{R^\nu\}$ it coincides (up to a constant factor) with the usual Lebesgue measure, for $G = \{Z^\nu\}$ - with the corresponding discrete measure. Thus, the *space average* of the local (quasi-local) operator $A(x_0) \in \mathfrak{A}$ over the region $\Lambda \subset R^\nu$ (or Z^ν) is defined for arbitrary $x_0 \in \Lambda$ as

$$\dot{A}_\Lambda = \frac{1}{|\Lambda|} \int_{G_\Lambda(x_0)} dg_x \tau_x A(x_0). \qquad (A.1)$$

The operator $\dot{A}_\Lambda$ is called *intensive* /1/ (or *normalized* /7/) operator. Here $G_\Lambda(x_0) \subset G$ is such that for all $\tau_x \in G_\Lambda(x_0)$ we have $x_0 + x \in \Lambda$. For $G = \{Z^\nu\}$ the corresponding discrete measure dg induces summation over the sublattice $\Lambda \subset Z^\nu$. A similar construction for $\{\Lambda_\alpha\}: \Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \dots$ and $|\Lambda_\alpha| \to \infty$ is called "averaging operation" /19/, M- filter /20/ or M-net /6/ (see also /21/).

2. With the notion of the space-average (or M-filter, M-net) of quasi-local operators one can formulate such a property of the infinite system states $\rho(\cdot)$ as the *weak clustering* /19-21/:

$$\lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \int_{G_\Lambda(x_0)} dg_x \rho(\tau_x A(x_0) \cdot B) = \rho(A(x_0)) \cdot \rho(B), \qquad (A.2)$$

for arbitrary $A(x_0), B \in \mathfrak{U}$. This property is necessary for the G-invariant state $\rho(\cdot)$ to correspond to a *pure phase* (see [6] and [19–21]).

In the present work we have used a *clustering property* (see (vi), Section 1), which is obviously *weaker* than (A.2), since (vi) involves *only one* intensive operator in interaction Hamiltonian (1.1). This means that for such a model the infinite system states generated by approximating Hamiltonian (1.5) may not correspond to pure phases. Thus, the condition (vi) is just a restriction on the fluctuations of the intensive operator A_Λ.

A trivial example, when the clustering property (vi) takes place, corresponds to the case of one-particle operators T_Λ and A_Λ (see [2,3] and also [18,23]). It can easily be verified that the infinite system states generated by approximating Hamiltonian (1.5) for all $a \in R^1$ are G-invariant and weakly clustering.

Now, let $\Lambda \subset Z^2$, $|\Lambda| < \infty$, and the operator

$$T_\Lambda = -\frac{J}{2} \sum_{\substack{(i,j)\subset\Lambda \\ |i-j|=1}} \sigma_i \sigma_j , \quad J > 0$$

describes the square Ising model $(\sigma_i = \pm 1)$ with nearest neighbour interaction. Let the space average A_Λ be

$$A_\Lambda = \frac{1}{|\Lambda|} \sum_{i\in\Lambda} A(i)$$

where $A(0)$ denotes the quasi-local operator $\frac{1}{2}\sigma_0 \cdot \sum_{\substack{j\in\Lambda \\ |0-j|=1}} \sigma_j$ and

$$A(i) = \tau_i A(0) = \frac{1}{2}\sigma_i \sum_{\substack{j\in\Lambda \\ |i-j|=1}} \sigma_j .$$

Then the infinite system states generated by approximating Hamiltonian (1.5) are known to be not weakly clustering for some domain of the variables $\beta > 0$, $h \in R^1$ and $a \in R^1$.

Nevertheless, the clustering property (vi) takes place because the fluctuations in (1.6) are proportional to $|\Lambda|^{-1} c_\Lambda(\beta, a, h)$, where $c_\Lambda(\beta, a, h)$ is the specific heat capacity, which according to [23] is bounded above by $O(\ln|\Lambda|)$ for $|\Lambda| \to \infty$.

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