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QUASI-UNIFORM TOPOLOGIES ON LOCAL OBSERVABLES

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Квазиравномерные топологии на локальных наблюдаемых

В работе введены квазиравномерные топологии на алгебрах локальных наблюдаемых. Для БКШ-модели показано, что с помощью этих топологий можно получить пополнение алгебры локальных наблюдаемых, на котором динамика описывается однопараметрической группой преобразований.

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Quasi-Uniform Topologies on Local Observables

In this paper quasi-uniform topologies on algebras of local observables are introduced. For the BCS-model it is shown that with the help of these topologies one gets a completion of the algebra of local observables on which the dynamics is given by a one-parameter group of transformations.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

In this paper we want to show that unnormable locally convex topologies on the $*$-algebra of local observables of a statistical system may be more appropriate to describe the dynamics and equilibrium states in the thermodynamical limit than the C^*-*norm topology in the usual algebraic approach to statistical* physics. In section 2 we describe the quasi-uniform topologies on algebras of unbounded operators and in section 3 we show that the dynamics of the BCS-model in the thermodynamical limit is given by a one-parameter group of transformations on the completion of the algebra of local observables with respect to the locally convex topology $\mathfrak{S}$ generated by uniform topologies.

Let me first repeat the scheme to handle equilibrium states of infinite systems in the algebraic approach /3/. The basic object is the $*$-algebra $\mathfrak{A}_\ell = \bigcup_V \mathfrak{A}_V$ of local observables, where $\mathfrak{A}_V$ is the observable $*$-algebra related to the bounded region (box) V. Since $\mathfrak{A}_V \subset \mathfrak{A}_{V'}$ for $V \subset V'$, the $*$-algebra $\mathfrak{A}$ is well-defined. We do not assume $\mathfrak{A}$ to be a normed algebra. Describing equilibrium states in statistical physics one usually starts with the "Gibbs Ansatz". For this one takes a Hilbert space $\mathcal{H}_V$ (Fock space) for every finite volume V and realizes the $*$-algebra $\mathfrak{A}_V$ as a $*$-algebra of (unbounded) operators on $\mathcal{H}_V$. The interactions, which are characteristic for the physical situation we want to describe are concentrated in the Hamiltonian H_V , which is an unbounded self-adjoint operator on $\mathcal{H}_V$ known explicitly from ordinary quantum mechanics. The Hamiltonian is in general of the form $H_V = H_o^V + H_{int}^V - \mu N$, where H_o^V is the free Hamiltonian, H_{int}^V, the interaction part, μ, the chemical potential and N, the number operator.

The "Gibbs Ansatz" consists in writing

$$\omega_V(A) = \mathrm{Tr}\, e^{-\beta H_V} A \,/\, \mathrm{Tr}\, e^{-\beta H_V} , \quad A \in \mathfrak{A}_V , \qquad (1.1)$$

where $\beta^{-1} = kT$, k, Boltzmann constant, T, temperature. We assume that the traces on the right-hand side of (1.1) are finite. ω_V is then a linear, positive and normed functional on the $*$-algebra $\mathfrak{A}_V$, i.e.

$$\begin{aligned} &\text{i)} \quad \omega_V \text{ linear functional on } \mathfrak{A}_V , \\ &\text{ii)} \quad \omega_V(A^+A) \geq 0 \quad \text{for} \quad A \in \mathfrak{A}_V , \qquad (1.2) \\ &\text{iii)} \quad \omega_V(I) = 1 . \end{aligned}$$

We assume $\mathfrak{A}_\ell$ to contain a unite element I , which is also an element of each $\mathfrak{A}_V$.

The dynamical evolution of the system in volume V is given by

$$\alpha_t^V(A) = e^{iH_V t} A \, e^{-iH_V t} , \quad t \in R^1 . \qquad (1.3)$$

We assume $\mathfrak{A}_V$ to be chosen so that $\alpha_t^V(A) \in \mathfrak{A}_V$ for $A \in \mathfrak{A}_V$.

Now one has to take the thermodynamical limit , i.e., the limit of ω_V when $V \to \infty$. Let us assume that

$$\omega(A) = \lim_{V \to \infty} \omega_V(A) \qquad (1.4)$$

exists. If all $\mathfrak{A}_V$ are C^*-algebras, then there exists a natural C^*-norm on $\mathfrak{A}_\ell$ and we can form the completion $\mathfrak{A} = \widetilde{\mathfrak{A}_\ell[\|\cdot\|]}$. Now we can ask whether for every A

$$\alpha_t(A) = \lim_{V \to \infty} \alpha_t^V(A) \tag{1.5}$$

exists in $\mathfrak{A}$. If so and if the limit exists with respect to the norm topology on $\mathfrak{A}$, then α_t is a one-parameter group of $*$-automorphisms of $\mathfrak{A}$, describing the dynamics of the physical system in the thermodynamical limit. The equilibrium state ω (1.4) on $\mathfrak{A}$ satisfies the KMS-condition (Kubo-Martin-Schwinger boundary condition) with respect to α_t , i.e., if we form both the functions $F_{AB}(t) = \omega(B\,\alpha_t(A))$, $G_{AB}(t) = \omega(\alpha_t(A)B)$ then for a dense set of elements A, B in $\mathfrak{A}$ there exists an analytic function $U_{AB}(z)$ in the strip $0 \leq \text{Im } z \leq i\beta$, continuous on the boundary so that

$$\begin{aligned} F_{AB}(t) &= U_{AB}(t) \\ G_{AB}(t) &= U_{AB}(t+\beta i). \end{aligned} \tag{1.6}$$

The importance of the dynamical one-parameter group α_t consists in the fact that one can characterize the equilibrium states of the statistical system by the KMS-condition (1.6). Also if ω is uniquely determined by (1.4) there can exist yet other KMS-states for a fix β , which are interpreted to describe other phases of the statistical system.

The existence of the dynamical automorphism group α_t can be proved for lattice models for a wide class of interactions / 9 /. But already for the simple nontrivial BCS-model a dynamical automorphism group in the above described sense does not exist / 11 , 12 / and this seems to be the general situation (see also / 2 /). A method to overcome these difficulties was introduced in / 1 /, where it was shown that under some conditions for every equilibrium state ω there is a one-parameter automorphism group α_t^ω on the W^*-algebra $\pi_\omega(\mathfrak{A})'' = \mathfrak{A}^\omega$, where π_ω is the GNS-representation of ω and ω satisfies the KMS-condition on $\mathfrak{A}^\omega$ with respect to α_t^ω. This result makes it possible to get structure results on the state ω, which are connected with the KMS-condition. Since α_t^ω depends on the state ω, this result is still too weak yielding the famous characterization of all equilibrium states of the infinite system by the KMS-condition with a fixed and uniquely determined automorphism group.

In what follows we shall discuss another method to get the dynamics as a one-parameter group of transformations of a more extended algebraic object $\mathfrak{A}$. We get this in the following way. First we choose an appropriate locally convex topology ξ on $\mathfrak{A}_\ell$ so that $\mathfrak{A}_\ell[\xi]$ becomes a topological $*$-algebra and form the completion $\mathfrak{A} = \widetilde{\mathfrak{A}_\ell[\xi]}$. Then we have

i) $\mathfrak{A}_\ell$ is a dense subspace of $\mathfrak{A}$,

ii) for $A \in \mathfrak{A}$, $B \in \mathfrak{A}_\ell$ the products $AB = (\lim A_i)\, B$, $BA = B\,(\lim A_i)$ are uniquely determined, where $A_i \in \mathfrak{A}_\ell$ is a sequence tending to A,

iii) the involution $A \to A^+$ is uniquely determined on $\mathfrak{A}$.

Let us remark that $\mathfrak{A}$ is in general not a topological algebra since the multiplication cannot ever be extended to $\mathfrak{A}$. This would be possible, e.g., if the multiplication $A,B \to AB$ in $\mathfrak{A}_\ell[\xi]$ is jointly continuous in both factors.

Now it may happen that for $A \in \mathfrak{A}_\ell$ $\lim\limits_{V \to \infty} \alpha_t^V(A) = \alpha_t(A)$

exists in $\mathfrak{A}$. If moreover $\alpha_t(A)$ is continuous on $\mathfrak{A}_\ell$, then $\alpha_t(A)$ can be extended to $\mathfrak{A}$. Furthermore, if the topology ξ is chosen appropriate it may happen that $\alpha_t(A)$ is a continuous one-parameter group of linear transformations and we can look for states ω satisfying the KMS-condition (1.6) for a dense set of $A \in \mathfrak{A}$ and $B \in \mathfrak{A}_\ell$.

In the next section we introduce different unnormable topologies on algebras of (unbounded) operators. In section 3 we shall show that with the help of these topologies the dynamics *of the BCS-model is given by a one-parameter transformation* group on the completion of the algebra of local observables.

2. Topologies on Unbounded Operators

In this section we devote to some properties of unnormable topological $*$-algebras. First we introduce topologies on algebras of unbounded operators which generalize the uniform topology (norm topology) on C^*-algebras to the case of unbounded operators.

Let $\mathcal{D}$ be a unitary space (incomplete Hilbert space) with the scalar product $\langle .,. \rangle$, $\mathcal{H}$ its completion. By $\mathcal{L}^+(\mathcal{D})$ we denote the set of all endomorphisms $A \in \mathrm{End}\,\mathcal{D}$ for which an $A^+ \in \mathrm{End}\,\mathcal{D}$ exists with $\langle \Psi, A\phi \rangle = \langle A^+\Psi, \phi \rangle$ for all $\phi, \Psi \in \mathcal{D}$. $\mathcal{L}^+(\mathcal{D})$ is a $*$-algebra with the usual algebraic operations with operators and the involution $A \to A^+$. If $\mathcal{D} = \mathcal{H}$, then $\mathcal{L}^+(\mathcal{D}) = \mathcal{B}(\mathcal{H})$ the C^*-algebra of all bounded operators on $\mathcal{H}$. We call a $*$-subalgebra $\mathcal{A}$ of $\mathcal{L}^+(\mathcal{D})$ containing the identity <u>Op^*-algebra</u>.

On $\mathcal{D}$ we define a locally convex topology t by the following system of seminorms

$$t: \qquad \|\phi\|_A = \|A\phi\| , \quad A \in \mathcal{L}^+(\mathcal{D}) . \qquad (2.1)$$

A domain in $\mathcal{H}$ is called a <u>closed domain</u>, if $\mathcal{D}[t]$ is a complete space. Then $\mathcal{D} = \bigcap_{A \in \mathcal{L}^+(\mathcal{D})} \mathcal{D}(\bar{A})$, where $\mathcal{D}(\bar{A})$ is the domain of the closure $\bar{A}$ of the operator A.

The dual space of $\mathcal{D}[t]$ we denote by $\mathcal{D}'[t']$, where t' is the strong topology on $\mathcal{D}'$. The Hilbert space $\mathcal{H}$ is canonical imbedded into $\mathcal{D}'[t']$. Hence, any dense domain $\mathcal{D} \subset \mathcal{H}$ defines in a canonical way a rigged Hilbert space

$$\mathcal{D}[t] \longrightarrow \mathcal{H} \longrightarrow \mathcal{D}'[t'] , \qquad (2.2)$$

where the scalar product $\langle F, \phi \rangle$ is defined for $F \in \mathcal{D}'$, $\phi \in \mathcal{D}$. In what follows we regard only such $\mathcal{D}$ for which $\mathcal{D}[t]$ is a reflexive space. Let $\mathcal{L}(\mathcal{D}, \mathcal{D}')$ be the linear space of all continuous maps of $\mathcal{D}[t]$ into $\mathcal{D}'[t']$. Further we write $\mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{D}, \mathcal{D})$ and $\mathcal{L}(\mathcal{D}') = \mathcal{L}(\mathcal{D}', \mathcal{D}')$. These last two spaces are algebras with respect to the usual operations with maps.

<u>Lemma 2.1</u> Let $\mathcal{D}[t]$ be a reflexive space. Then

i) if $A \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$, so the adjoint operator $A^+ \in \mathcal{L}(\mathcal{D}, \mathcal{D}')$ is uniquely defined by $\langle A\phi, \psi \rangle = \overline{\langle A^+\psi, \phi \rangle}$ and $A \to A^+$ is an involution on $\mathcal{L}(\mathcal{D}, \mathcal{D}')$,

ii) $\mathcal{L}(\mathcal{D})$, $\mathcal{L}(\mathcal{D}') \subset \mathcal{L}(\mathcal{D}, \mathcal{D}')$ and $\mathcal{L}(\mathcal{D})^+ = \mathcal{L}(\mathcal{D}')$,

iii) $\mathcal{L}^+(\mathcal{D})$ is a subspace of $\mathcal{L}(\mathcal{D})$ and it is $\mathcal{L}^+(\mathcal{D}) = \mathcal{L}(\mathcal{D}) \cap \mathcal{L}(\mathcal{D}')$.

<u>Proof:</u> i) is a consequence of the reflexivity of $\mathcal{D}[t]$.
ii) Since the topology t is stronger than t', this statement

is obvious. The second part is a consequence of the reflexivity of $\mathcal{D}[t]$. iii) By definition of t, any $A \in \mathcal{L}^+(\mathcal{D})$ is a continuous map of $\mathcal{D}[t]$ into itself and therefore $\mathcal{L}^+(\mathcal{D}) \subset \mathcal{L}(\mathcal{D})$. The last statement is now a consequense of i) and ii).

Let us still remark that for $\mathcal{D} = \mathcal{H}$ we have $\mathcal{L}^+(\mathcal{D}) = \mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{D}') = \mathcal{L}(\mathcal{D},\mathcal{D}') = \mathcal{B}(\mathcal{H})$, but if $\mathcal{D} \neq \mathcal{H}$ and $\mathcal{D}[t]$ reflexive, then all these spaces of operators are mutually different.

If E, F are two locally convex spaces, then the topology τ of uniformly bounded convergence on $\mathcal{L}(E,F)$ is defined by all seminorms

$$q_{\alpha,\mathcal{M}}(A) = \sup_{\phi\in\mathcal{M}} p_\alpha(A\phi), \qquad (2.3)$$

where p_α runs over the seminorms defining the topology of F and $\mathcal{M}$ runs over all bounded sets in E.

The topologies of uniformly bounded convergence on the spaces $\mathcal{L}(\mathcal{D},\mathcal{D}')$, $\mathcal{L}(\mathcal{D})$ and $\mathcal{L}(\mathcal{D}')$ we denote by $\tau_{\mathcal{D}}$, $\tau^{\mathcal{D}}$ and $\tau^{\mathcal{D}'}$. Let us describe the seminorms determining these topologies more explicitly /6, 7/.

$\tau_{\mathcal{D}}$: $\|A\|_{\mathcal{M}} = \sup_{\phi,\psi\in\mathcal{M}} |\langle A\psi, \phi\rangle|$, $\mathcal{M}$ bounded in $\mathcal{D}[t]$

$\tau^{\mathcal{D}}$: $\|A\|^{\mathcal{M},B} = \sup_{\phi\in\mathcal{M}} \|BA\phi\|$, $B \in \mathcal{L}^+(\mathcal{D})$, $\mathcal{M}$ bounded in $\mathcal{D}[t]$

$\tau^{\mathcal{D}'}$: $\|A\|^{\mathcal{N}',\mathcal{M}} = \sup_{\phi\in\mathcal{M},\,\psi\in\mathcal{N}'} |\langle A\psi, \phi\rangle|$, $\mathcal{M}$ bounded in $\mathcal{D}[t]$, $\mathcal{N}'$ bounded in $\mathcal{D}'[t']$

This definition of the topologies makes sense also for non-reflexive $\mathcal{D}[t]$.

<u>Lemma 2.2</u> /6, 7/ Let $\mathcal{D}[t]$ be reflexive. Then

i) the topology $\tau^{\mathcal{D}'}$ is given by the seminorms $\|A\|_+^{\mathcal{M},B} = \|A^+\|^{\mathcal{M},B}$, where B runs over all operators of $\mathcal{L}^+(\mathcal{D})$ and $\mathcal{M}$ over all bounded sets of $\mathcal{D}[t]$,

ii) $\mathcal{L}(\mathcal{D})[\tau^{\mathcal{D}}]$, $\mathcal{L}(\mathcal{D}')[\tau^{\mathcal{D}'}]$ are topological algebras of operators,

iii) $A \to A^+$ is a bijection between $\mathcal{L}(\mathcal{D})[\tau^{\mathcal{D}}]$ and $\mathcal{L}(\mathcal{D}')[\tau^{\mathcal{D}'}]$,

iv) $\mathcal{L}^+(\mathcal{D})[\tau_{\mathcal{D}}]$ is a locally convex $*$-algebra.

Let us still introduce $\tau_*^{\mathcal{D}} = \max(\tau^{\mathcal{D}}, \tau^{\mathcal{D}'})$ on $\mathcal{L}^+(\mathcal{D})$. Then $\mathcal{L}^+(\mathcal{D})$ becomes a locally convex $*$-algebra with respect to the topology $\tau_*^{\mathcal{D}}$. The relations between the different linear spaces of operators and their topologies are expressed by the following scheme.

$$\mathcal{L}^+(\mathcal{D})[\tau_*^{\mathcal{D}}] \begin{array}{c} \nearrow \;\mathcal{L}(\mathcal{D})[\tau^{\mathcal{D}}]\; \searrow \\ \searrow \;\mathcal{L}(\mathcal{D}')[\tau^{\mathcal{D}'}]\; \nearrow \end{array} \mathcal{L}(\mathcal{D},\mathcal{D}')[\tau_{\mathcal{D}}] \tag{2.5}$$

where $\longrightarrow$ denotes a continuous injection. If $\mathcal{D} = \mathcal{H}$ then all four spaces coincide with $\mathcal{B}(\mathcal{H})$ and all topologies with the operator norm topology. Between the four topologies $\tau_{\mathcal{D}}$ plays an exceptional role /4, 10/. Therefore we call it the <u>uniform topology</u> on $\mathcal{L}^+(\mathcal{D})$. The other topologies are called quasi-uniform topologies.

Now let $M \geq I$ be a selfadjoint operator in $\mathcal{H}$. We form $\mathcal{D}^\infty = \bigcap_k \mathcal{D}(M^k)$, where $\mathcal{D}(M^k)$ is the domain of the operator M^k. The canonical topology t on $\mathcal{D}^\infty$ is then defined by the norms

$$\|\phi\|_k = \|M^k \phi\| \quad , \quad k = 0,1,2,\ldots \quad . \tag{2.6}$$

$\mathcal{D}^\infty[t]$ is a F-space, i.e., a complete metric space. The fact that any $A \in \mathcal{L}^+(\mathcal{D})$ is already a continuous operator with respect to the topology defined by the norms (2.6) is a consequence of the closed graph theorem.

Let us describe the locally convex space $\mathcal{D}^\infty[t]$ and its dual space a little more explicit. If $M = \int_1^\infty \lambda \, dE_\lambda$ is the spectral decomposition of M, so we put $P_{(n)} = E_{n+1} - E_n$, $n = 1,2,\ldots$ Let $\mu_1 < \mu_2 < \cdots$ be the set of all natural numbers for which $P_\ell = P_{(\mu_\ell)} \neq 0$. Then $I = \sum_\ell P_\ell$. We put $\mathcal{H}_\ell = P_\ell \mathcal{H}$. Any $\phi \in \mathcal{H}$ has a unique decomposition $\phi = \sum_\ell \phi_\ell$ with $\phi_\ell \in \mathcal{H}_\ell$. $\mathcal{D}^\infty$ contains exactly all ϕ with

$$\|\phi\|_{(k)} = \Big(\sum_\ell \|\phi_\ell\|^2 \mu_\ell^{2k} \Big)^{1/2} < \infty, k = 0,1,2,\ldots \quad . (2.7)$$

Since for $\phi_\ell \in \mathcal{H}_\ell$ $\mu_\ell^k \|\phi_\ell\| \leq \|M^k \phi_\ell\| \leq (\mu_\ell+1)^k \|\phi_\ell\|$ we get for every $\phi \in \mathcal{D}^\infty$ $\|\phi\|_{(k)} \leq \|\phi\|_k \leq 2^k \|\phi\|_{(k)}$. Hence, both the systems of norms (2.6) and (2.7) are equivalent and define the topology t.

<u>Lemma 2.3</u> Let $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$ be a decreasing sequence of positive numbers so that $\sum_\ell a_\ell^2 \mu_\ell^{2k} < \infty$ for every $k = 0,1,2,\ldots$. Then $\mathcal{M}_{(a_n)} = \{\phi = \sum_n a_n \phi_n ; \|\phi_n\| \leq 1\}$ is a bounded set in $\mathcal{D}^\infty[t]$. The system of all such sequences (a_n) we denote by Γ_M. $\{\mathcal{M}_{(a_n)} ; (a_n) \in \Gamma_M\}$ is a total system of bounded sets in $\mathcal{D}^\infty[t]$.

<u>Proof:</u> Since $\|\mathcal{M}_{(a_n)}\|^2_{(k)} \leq \sum_\ell a_\ell^2 \mu_\ell^{2k} < \infty$ $\mathcal{M}_{(a_n)}$ is bounded. Now let $\mathcal{M}$ be an arbitrary bounded set. We put $a'_n = \sup_{\phi \in \mathcal{M}} \|P_n \phi\|$. Let $k \geq 0$ be arbitrary, then $a'_n \mu_n^k \leq \|\mathcal{M}\|_{(k)} < \infty$ for all n. We put $a_n = \sum_{i \geq n}^\infty a'_i$. Then a_n is decreasing and $\sum_\ell a_\ell^2 \mu_\ell^{2k} \leq \sum_\ell \mu_\ell^{-2} a_\ell \mu_\ell^{2k+2} \leq c \max\{a_n \mu_n^{2k+2}\}$. But $a_n \mu_n^{2k+2} \leq \sum_{i=n}^\infty a'_i \mu_i^{2k+2} \leq (\sum_i \mu_i^{-2}) \max_i \{a'_i \mu_i^{2k+4}\} < \infty$ Thus $(a_n) \in \Gamma_M$. Now let $\phi = \sum_\ell \phi_\ell \in \mathcal{M}$. Then $\|\phi_\ell\| \leq a'_\ell \leq a_\ell$ and therefore $\mathcal{M} \subset \mathcal{M}_{(a_n)}$.

<u>Lemma 2.4</u> The linear functionals $F \in \mathcal{D}^{\infty\prime}[t']$ are exactly given by the sequences $F = \{\psi_1, \psi_2, \psi_3, \ldots\}$, $\psi_\ell \in \mathcal{H}_\ell$, with $\|F\|_{(a_n)} = \sum \|\psi_\ell\| a_\ell < \infty$, $(a_n) \in \Gamma_M$ and it is for

$\phi \in \mathcal{D}^\infty[t] \quad \langle F, \phi\rangle = \sum_\ell \langle \Psi_\ell, \phi_\ell\rangle$. The seminorms $\| F\|_{(a_n)}$, $(a_n) \in \Gamma_M$, *define the topology* t'.

Proof: It is easy to see that any such sequence $F = \{\Psi_\ell\}$ defines a continuous linear functional on $\mathcal{D}^\infty[t]$. Let F be an arbitrary element of $\mathcal{D}^{\infty\prime}$. We construct the corresponding sequence $\{\Psi_\ell\}$. $\langle F, \phi_\ell\rangle$ is a linear continuous functional on $\mathcal{H}_\ell \subset \mathcal{D}^\infty$ and therefore it is of the form $\langle F, \phi_\ell\rangle = \langle \Psi_\ell, \phi_\ell\rangle$ with $\Psi_\ell \in \mathcal{H}_\ell$. Furthermore $\langle F, \phi\rangle \leq c \cdot \|\phi\|_{(k)}$ for a certain k and therefore $|\langle \Psi_\ell, \phi_\ell\rangle| \leq c\mu_\ell^k \|\phi_\ell\|$. Thus $\|\Psi_\ell\| \leq c\mu_\ell^k$ i.e. $\sum_\ell \|\Psi_\ell\| a_\ell < \infty$ for every $(a_n) \in \Gamma_M$. Now it is straightforward to show $\langle F, \phi\rangle = \sum_\ell \langle \Psi_\ell, \phi_\ell\rangle$.

Let $\mathcal{M}_{(a_n)}$ be an arbitrary bounded set of Lemma 2.3. Then $\sup_{\phi\in\mathcal{M}_{(a_n)}} |\langle F, \phi\rangle| = \sup_{\phi\in\mathcal{M}_{(a_n)}} |\sum_\ell \langle \Psi_\ell, \phi_\ell\rangle| = \sup_{\|\Omega_\ell\|\leq 1} |\sum_\ell a_\ell \langle \Psi_\ell, \Omega_\ell\rangle| = \sum_\ell a_\ell \|\Psi_\ell\| = \|F\|_{(a_n)}$. Since the system $\mathcal{M}_{(a_n)}$ is total, the seminorms $\|F\|_{(a_n)}$ define the topology t'.

If we put $P_\ell F = \Psi_\ell$, then the projektions P_ℓ are continuously extended to projections of $\mathcal{D}^{\infty\prime}$ onto $\mathcal{H}_\ell$.

Lemma 2.5 For $A \in \mathcal{L}(\mathcal{D}^\infty, \mathcal{D}^{\infty\prime})$ we define the matrix $A = (A_{lk})$ of operators $A_{lk} : \mathcal{H}_k \to \mathcal{H}_l$ by $A_{lk} = P_l A P_k$. Then

$$\|A\|_{(a_n)} = \sum_{k,l} \|A_{lk}\| a_l a_k < \infty \quad \text{for } (a_n) \in \Gamma_M, \tag{2.8}$$

where $\|A_{lk}\|$ is the usual norm of the bounded operator A_{lk}. Vice versa, any such matrix $(A_{lk}) = A$ with $\|A\|_{(a_n)} < \infty$ defines an operator of $\mathcal{L}(\mathcal{D}^\infty, \mathcal{D}^{\infty\prime})$. The system of seminorms $\|A\|_{(a_n)}$ defines the uniform topology $\mathcal{T}_{\mathcal{D}^\infty}$.

Using Lemma 2.3 and 2.4 this Lemma can be proved by standard estimations as in the proof of Lemma 2.3.

Quite analogous we get seminorms defining the other topologies of the spaces (2.5) Namely

$$
\begin{aligned}
\tau_{\mathcal{D}} &: \quad \|A\|_{(a_n)} &&= \sum_{k,l} \|A_{lk}\| \; a_l a_k \\
\tau^{\mathcal{D}} &: \quad \|A\|^{(a_n),m} &&= \sum_{k,l} \|A_{lk}\| \; l^m a_k \\
\tau^{\mathcal{D}'} &: \quad \|A\|_{+}^{(a_n),m} &&= \sum_{k,l} \|A_{lk}\| \; k^m a_l \\
\tau^{\mathcal{D}}_{*} &: \quad \|A\|_{*}^{(a_n),m} &&= \sum_{k,l} \|A_{lk}\| \; (l^m a_k + k^m a_l)
\end{aligned}
\tag{2.9}
$$

where (a_n) runs over all sequences of Γ_M and $m = 0,1,2,\ldots$.

Using the estimation $\max_{l,k}\{\|A_{lk}\| \; a_l a_k\} \leq \|A\|_{(a_n)} \leq$ $\leq \left(\sum_{\ell} \mu_\ell^{-2}\right)^2 \max_{l,k} \{\|A_{lk}\| \; (a_l \mu_l^2)(a_k \mu_k^2)\}$ we see that the topology $\tau_{\mathcal{D}^\infty}$ is also defined by the seminorms

$$\tau_{\mathcal{D}} : \quad \|A\|'_{(a_n)} = \max_{l,k} \|A_{lk}\| \; a_l a_k \quad , \quad (a_n) \in \Gamma_M \,. \tag{2.10}$$

In the same way we can also substitute the sum by the maximum in (2.9) .

We end up this section with the following Lemma, which can be proved with the help of the seminorms (2.9) (see also / 5, 6 /).

<u>Lemma 2.6</u>

i) All four locally convex spaces of operators in (2.5) are complete for $\mathcal{D}^\infty$.

ii) $\mathcal{L}^+(\mathcal{D}^\infty)$ is dense in $\mathcal{L}(\mathcal{D}^\infty, \mathcal{D}^{\infty\prime})$ $[\tau_{\mathcal{D}^\infty}]$.

3. The Dynamics of the BCS-Model

In this section we show that the dynamics of the BCS-Model is given by a one-parameter group of linear transformations α_t of a locally convex space $\mathfrak{A}$, which we obtain from the algebra $\mathfrak{A}_\ell$ of local observables by completion with respect to a certain locally convex topology . In deriving this result we shall make extensive use of the treatment of Thirring and Wehrl /11,12/ on the BCS-model.

We shall use the quasi-spin formulation in which the BCS-Hamiltonian is

$$H_{\Omega} = \varepsilon \sum_{p=1}^{\Omega} (1 - \sigma_p^z) - \frac{2g}{\Omega} \sum_{p,p'=1}^{\Omega} \sigma_p^- \sigma_{p'}^+ \qquad (3.1)$$

Ω is the number of pair states, g, an interaction constant, ε, the kinetic energy which we assume to be independent of p . $\sigma_p = (\sigma_p^x, \sigma_p^y, \sigma_p^z)$ are the Pauli matrices, $\sigma^{\pm} = \frac{1}{2}(\sigma^x \pm i\sigma^y)$ and $[\sigma_p, \sigma_{p'}] = 0$ for $p \neq p'$. If we introduce the total spin $S_{\Omega} = \frac{1}{2}\sum_{p=1}^{\Omega} \sigma_p = (S_{\Omega}^x, S_{\Omega}^y, S_{\Omega}^z)$ the Hamiltonian becomes

$$H_{\Omega} = \varepsilon\,(\Omega - 2\,S_{\Omega}^z) - \frac{2g}{\Omega}\,(\,S_{\Omega}^2 - S_{\Omega}^z\,(S_{\Omega}^z + 1)\,)\,. \qquad (3.2)$$

Let $\mathcal{H}_{\infty} = \prod_p \otimes C_p^2$ be the infinite tensor product of the 2-dimensional spaces /8/ . Following /11/ we choose special unit vectors in $\mathcal{H}_{\infty}$. For a real unit three-vector $n = (n_1, n_2, n_3)$ we denote by $|n\rangle$ a vector in C^2 which is characterized by $(\sigma\, n)\,|n\rangle = |n\rangle$. This determines $|n\rangle$ up to a phase factor. The scalar product of two such vectors is given by

$$\langle n\,|\,n'\rangle = e^{i\varphi} \sqrt{\frac{1 + (n\,n')}{2}}\,, \qquad (3.3)$$

where $(n\,n') = n_1 n_1' + \ldots + n_3 n_3'$. Let $\{n\} = \{n_1, n_2, \ldots\}$ be a set of such three-vectors then by

$$|\{n\}\rangle = \prod_p \otimes |n_p\rangle \qquad (3.4)$$

we denote unit vectors in $\mathcal{H}_{\infty}$. Further $\mathcal{H}_{\{n\}}$ denotes the separable Hilbert space generated by all vectors $|\{n'\}\rangle$ which are equivalent to $|\{n\}\rangle$ /8 /. One can choose a special base in $\mathcal{H}_{\{n\}}$ which one gets from $|\{n\}\rangle$ by flipping a finite number of spins. For this one chooses two three-vectors n^1 , n^2 which form together with n an orthonormal base and put $n^{\pm}=\frac{1}{2}(n^1 \pm i n^2)$. Then $(\sigma n^+)\,|n\rangle = 0$ and if we put $|m,n\rangle = (\sigma n^-)^m |n\rangle$ then

$$(\sigma n)\,|m,n\rangle \;=\; (-1)^m\,|m,n\rangle \quad , \quad m = 0,1\,. \tag{3.5}$$

Now $|\{m\}\{n\}\rangle = \prod_p \otimes |m_p,n_p\rangle$, $m_p = 0,1$, $\sum m_p < \infty$, form a denumerable orthonormal base in $\mathcal{H}_{\{n\}}$. On $\mathcal{H}_\infty$ we define the selfadjoint operator M by

$$M\,|\{m\}\{n\}\rangle = (\sum_p m_p + 1)\,|\{m\}\{n\}\rangle\,. \tag{3.6}$$

Let $\mathcal{D} \equiv \mathcal{D}^\infty = \bigcap_{k=0}^{\infty} \mathcal{D}(M^k)$ and $\mathcal{D}_{\{n\}} = \mathcal{D} \cap \mathcal{H}_{\{n\}}$ By $\mathcal{T}_{\{n\}} = \mathcal{T}_{\mathcal{D}_{\{n\}}}$ we denote the uniform topology on $\mathcal{L}(\mathcal{D}_{\{n\}},\mathcal{D}'_{\{n\}})$.

Let $\mathfrak{A}_\ell$ be the $*$-algebra of lokal observables generated by all σ_p^i , $i = x,y,z$. $\mathfrak{A}_\ell$ is in a natural way realized as an operator algebra on $\mathcal{D}_{\{n\}}$ and $\mathfrak{A}_\ell \subset \mathcal{L}^+(\mathcal{D}_{\{n\}})$. Therefore we have on $\mathfrak{A}_\ell$ the uniform topology $\mathcal{T}_{\{n\}}$.

Let Q be the system of all vectors $|\{n\}\rangle$ for which

$$\lim_{\Omega\to\infty} \frac{1}{\Omega}\sum_{p=1}^{\Omega} n_p \;=\; \eta\, n \quad , \quad 0 \le \eta \le 1 \;,\; n = (n_1,n_2,n_3) \tag{3.7}$$

exists. On $\mathfrak{A}_\ell$ we define the locally convex topology

$$\xi \;=\; \sup_{\{n\}\in Q} \mathcal{T}_{\{n\}} \quad , \tag{3.8}$$

the weakest locally convex topology which is stronger than all $\mathcal{T}_{\{n\}}$, $\{n\} \in Q$. Now we can state our main theorem.

<u>Theorem 3.1</u> Let $\mathfrak{A} = \widetilde{\mathfrak{A}_\ell[\xi]}$ be the completion of $\mathfrak{A}_\ell$ with respect to the locally convex topology ξ . For every local observable $A \in \mathfrak{A}_\ell$ it exists

$$\lim_{\Omega\to\infty} e^{iH_\Omega t}\, A\, e^{-iH_\Omega t} \;=:\; \alpha_t(A) \;\in\; \mathfrak{A}\;. \tag{3.9}$$

α_t is an one-parameter group of linear transformations on $\mathfrak{A}$.

We prove this Theorem in some steps. First we prove a Lemma which strengthens Lemma 1 of /11/.

Lemma 3.2 Let $\{n\} \in Q$ and $s_\Omega = \frac{1}{\Omega} S_\Omega$, then
$\lim_{\Omega\to\infty} s_\Omega = \frac{1}{2}\eta n$ in $\mathcal{L}^+(\mathcal{D}_{\{n\}})[\mathcal{T}_{\{n\}}]$.

Proof: For simplicity we drop the index $i = x,y,z$ of the spin coordinate. Since $n(\sigma n) + 2n^-(\sigma n^+) + 2n^+(\sigma n^-) = \sigma$ we get for $|\{m\}\rangle \equiv |\{m\}\{n\}\rangle$ $s_\Omega|\{m\}\rangle = (\Delta_1^\Omega + \Delta_2^\Omega + \Delta_3^\Omega)|\{m\}\rangle$,
where

$$\begin{aligned}
\Delta_1^\Omega|\{m\}\rangle &= \Big(\frac{1}{2\Omega}\sum_{\rho=1}^{\Omega}(-1)^{m_\rho} n_\rho\Big)|\{m\}\rangle \\
\Delta_2^\Omega|\{m\}\rangle &= \Big(\frac{1}{\Omega}\sum_{\rho=1}^{\Omega}\frac{1-(-1)^{m_\rho}}{2}\, n_\rho^-(\sigma n_\rho^+)\Big)|\{m\}\rangle \qquad (3.10)\\
\Delta_3^\Omega|\{m\}\rangle &= \Big(\frac{1}{\Omega}\sum_{\rho=1}^{\Omega}\frac{1+(-1)^{m_\rho}}{2}\, n_\rho^+(\sigma n_\rho^-)\Big)|\{m\}\rangle .
\end{aligned}$$

Let $\|\cdot\|_{(a_k)}$ be a seminorm (2.8) for the topology $\mathcal{T}_{\{n\}}$. Since M has the eigenvalues 1,2,3,... the $(a_n) \in \Gamma_M$ satisfies the condition $\sum_k a_k k^r < \infty$ for every $r = 0,1,2,\ldots$. By $\|A\|_{1,k}$ we denote the norm of the operator $P_1 A P_k = A_{1k}$. From (3.10) it is easy to see that

$$\|\Delta_1^\Omega - \tfrac{1}{2}\eta n\|_{1,k} \le \delta_{1k}\Big(\Big|\frac{1}{2\Omega}\sum_{\rho=1}^{\Omega} n_p - \tfrac{1}{2}\eta n\Big| + \frac{k}{\Omega}\Big)$$

Now let $\phi = \sum_{\{m\}} x_{\{m\}}|\{m\}\rangle$ with $|\{m\}\rangle \in \mathcal{H}_k$. Then

$$\|P_1\Delta_2^\Omega\phi\|^2 = \frac{1}{\Omega^2}\sum_{\{m\},\{m'\}}\sum_{\rho,\rho'}^{\Omega} \bar{x}_{\{m'\}}x_{\{m\}}\,\overline{n_{\rho'}^-}\,n_\rho^- \langle\{m'\}|(\sigma_{\rho'} n^+)^* P_1(\sigma_\rho n^+)|\{m\}\rangle .$$

The coefficients of the quadratic form are only 0 or 1 , where any ray or column of the coefficient matrix contains at most k times the 1 . Therefore the largest eigenvalue is less than k and we get

$$\|P_1\Delta_2^\Omega\phi\|^2 \le \frac{k}{\Omega^2}\sum_{\{m\}\rho}|x_{\{m\}}|^2|n_p^-|^2 \le \frac{k}{\Omega}\|\phi\|^2 .$$

Hence $\|\Delta_2^\Omega\|_{1k} \le \left(\frac{k}{\Omega}\right)^{1/2}$. Quite analogous one proves $\|\Delta_3^\Omega\|_{1k} \le \left(\frac{k+1}{\Omega}\right)^{1/2}$.

These three estimations together give us

$$\| s_{\Omega} - \tfrac{1}{2}\eta n \|_{(a_k)} \leq \sum_{l} a_l^2 \left| \frac{1}{2\Omega}\sum_{p=1}^{\Omega} n_p - \tfrac{1}{2}\eta n \right| + \frac{1}{\Omega}\sum_{l} a_l \sum_{k} a_k\, k + \frac{1}{\sqrt{\Omega}}\sum_{l} a_l \sum_{k} a_k \left(\sqrt{k} + \sqrt{k+1} \right) .$$

Since the right-hand side of the last formula tends to 0 for $\Omega \to \infty$, the Lemma is completly proved.

Lemma 3.3

i) s_{Ω}^{i} , $i = x,y,z$, $\Omega = 1,2,\ldots$, is a set of equicontinuous maps on $\mathcal{D}_{\{n\}}[t]$.

ii) If $\mathcal{M}$ is a bounded set in $\mathcal{D}_{\{n\}}[t]$, then there exists a bounded set $\mathcal{M}'$ so that $s_{\Omega}^{i}\, \mathcal{M} \subset \mathcal{M}'$ for all i, Ω .

Proof: It is straightforward to see that for any k there exists a constant c independent of p so that $\| M^k \sigma_p \phi \| \leq c \| M^k \phi \|$. Therefore $\| s_{\Omega}^{i} \phi \|_k \leq c/2 \| \phi \|_k$ for all i, Ω . This proves i). ii) is a consequence of i) .

Lemma 3.4 The "mean quasi-spin" s_{Ω} converges in $\mathfrak{A}$ to s with respect to the topology ξ .

Furthermore, $\lim_{\Omega \to \infty} (s_{\Omega}^{i})^n = (s^i)^n$ for every degree $n = 1,2,\ldots$.

Proof: Since s_{Ω}^{i} is a Cauchy sequence with respect to every topology $\Upsilon_{\{n\}}$ (Lemma 3.2), it is also a Chauchy sequence with respect to ξ . Therefore $\lim_{\Omega \to \infty} s_{\Omega}^{i} = s^i$ exists in $\mathfrak{A}$.
Now let $\mathcal{M}$ be a bounded set in $\mathcal{D}_{\{n\}}[t]$ and $\| \cdot \|_{\mathcal{M}}$ the corresponding seminorm for the topology $\Upsilon_{\{n\}}$. Then

$$\| s_{\Omega'}^2 - s_{\Omega}^2 \|_{\mathcal{M}} \leq \| s_{\Omega'} (s_{\Omega'} - s_{\Omega}) \|_{\mathcal{M}} + \| (s_{\Omega'} - s_{\Omega}) s_{\Omega} \|_{\mathcal{M}}$$
$$\leq 2 \| s_{\Omega'} - s_{\Omega} \|_{\mathcal{M}_1} ,$$

where $\mathcal{M}_1 = \mathcal{M} \cup \mathcal{M}'$ and $\mathcal{M}'$ is the bounded set of Lemma 3.3 ii).

Therefore s^2_Ω is also a Cauchy sequence with respect to the topology ξ . Thus $\lim_{\Omega\to\infty} s^2_\Omega = s^2$ exists in $\mathfrak{A}$. For the higher powers one proves the convergence by induction.

Let us still remark that the powers $(s^1)^n$ in $\mathfrak{A}$ are defined by the limit. As we outlined in the introduction we have not a product of two arbitrary elements of $\mathfrak{A}$.

Now we are in the position to give the proof of Theorem 3.1. Take

$$\alpha^\Omega_t(\sigma_p) = e^{itH_\Omega}\,\sigma_p\,e^{-itH_\Omega} = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}[H_\Omega,\sigma_p]_n ,$$

where $[H_\Omega,\sigma_p]_{n+1} = [H_\Omega,[H_\Omega,\sigma_p]_n]$ and $[\cdot,\cdot]_1 \equiv [\cdot,\cdot]$ is the usual commutator. It is straightforward to check that $[H_\Omega,\sigma_p]_n = R_n(\sigma_p, s_\Omega)$, where $R_n(\cdot,\cdot)$ is a polynom of degree not higher than n in every variable (see also /12/). From Lemma 3.4 we get $R_n(\sigma_p, s_\Omega) \to R_n(\sigma_p, s)$ for $\Omega\to\infty$. If $\|\cdot\|_\mu$ is a seminorm of the topology $\mathcal{T}_{\{n\}}$, then $R(\sigma_p,s_\Omega) \leq c^n$ where c is independent of Ω . Therefore $\alpha^\Omega_t(\sigma_p)$ converges in $\mathfrak{A}$ to a limit which we denote by $\alpha_t(\sigma_p)$. One can yet check by quite analogous considerations that α_t is an one-parameter group of transfomations on $\mathfrak{A}$.

Finally let us remark that the commutator of s with any local observable $A \in \mathfrak{A}_\ell$ is well-defined and $[s,A] = 0$. So we have the situation that the completion $\mathfrak{A}$ of $\mathfrak{A}_\ell[\xi]$ contains elements commuting with every element of the algebra $\mathfrak{A}_\ell$, although the centre of $\mathfrak{A}_\ell$ is trivial .

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