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Преобразование Крамерса-Ванье для систем с симметрией $Z(n)$

Для систем с внутренней симметрией $Z(n)$ установлены соотношения дуальности, аналогичные симметрии Крамерса-Ванье двумерной модели Изинга.

Работа выполнена в Лаборатории ядерных проблем ОИЯИ.

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Kramers-Wannier Transform for $Z(n)$ Symmetric Systems

Duality relations analogous to the Kramers-Wannier symmetry of the plane Ising model are stated for the spin and gauge systems with isotopic symmetry $Z(n)$.

The investigation has been performed at the Laboratory of Nuclear Problems, JINR.

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Kramers and Wannier[1] have noted that the plane Ising model possesses an exact symmetry relating low and high temperature phases of this model. It turns out that the model can be equivalently described both in terms of the spin variable σ defined on the lattice (order parameter) and in terms of the dual variable μ (disorder variable[2]) which is associated with the dual lattice and is a spin variable too ($\mu = \pm 1$). The description of the low temperature phase in terms of σ is identical to that of the high temperature phase in terms of μ and vice versa.

There are two possible directions in which the Kramers-Wannier (K-W) symmetry can be generalized. The first is a wide class of models with the same isotopic symmetry as the Ising model ($Z(2)$ symmetry*)). These models may be called generalized gauge Ising models (see ref.[3]). Consider a simple cubic lattice in the d - dimensional space. The dual lattice is a simple cubic one too, its sites being situated in the centers of the initial lattice cells. Lattice elements of different dimensionality $0 \leq q \leq d$ may be considered (they are sites for $q = 0$, links for $q = 1$, plaquettes for $q = 2$ and so on), and the dual relation can be stated between the elements of the initial lattice of dimensionality q and $d-q$ dimensional ones of the dual lattice. For example, in the case of $d = 3$ the sites of the initial lattice are the centers of the cells of the dual one, and the initial lattice links pass through the centers of the dual lattice plaquettes. Generalized gauge Ising models are defined as follows: "Generalized gauge field"**) A (which is a spin variable $A = \pm 1$) is

*) The group $Z(n)$ may be defined as a set of integers $0, 1, \ldots n-1$, the group multiplication being the modulo n summation.

**) The fields under consideration are certain formal generalizations of the gauge field. However, the geometrical interpretation of these fields is not clear. For $q = 2$ the generalized gauge field is an ordinary one with the gauge symmetry $Z(2)$. Note, that all the gauge theories of commutative symmetry can be formally generalized in the same manner.

defined on the $q-1$ dimensional lattice elements and "generalized intensity" F which is associated with the q dimensional lattice element is defined as a product of A's corresponding to all $q-1$ dimensional elements that bound this q dimensional one. Defining a field σ on the $q-2$ dimensional elements of the lattice one introduces the "generalized gauge transformation" as a multiplication of A taken from a certain $q-1$ dimensional element by σ's taken from all its $q-2$ dimensional bounds. As a result of this gauge transform the intensity F is twice multiplied by each σ (because any boundary is of no boundary), therefore, the field intensities F are invariant under the generalized gauge transformations.

For these models the K-W symmetry is stated as follows: the model with intensities F defined on q dimensional elements of the lattice is dual to that with field intensities $\tilde{F}$ defined on the corresponding $d-q$ dimensional ones of the dual lattice. For example, the three-dimensional Ising model is dual to its gauge analog; in four dimensions the pure gauge Ising field is selfdual. Analogous relations can be stated for different mixed models; e.g., the model of Ising spins ($q=1$) interacting with the Ising gauge field ($q=2$) is selfdual in three dimensions, because the spin field is dual to the gauge field and vice versa. The survey of the problems considered can be found, e.g., in ref.[4].

The second way of generalizing the K-W symmetry is to consider systems of other commutative symmetry groups. It turns out now that the symmetry group of the dual system is not generally the same as that of the initial one. For example, the models of internal group $U(1)$ are dual to those of symmetry Z *)[5-7].

The present paper is devoted to demonstration that the models of internal symmetry $Z(n)$ are dual (in the sense mentioned above) to those of the same symmetry $Z(n)$. Under the special choice of the interaction between spins the duality turns out to be exact, i.e., K-W transform reduced to the change in temperature only as in the case of the Ising-like models.

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*) Z is a group of integers, the group multiplication being the numerical summation.

For the sake of simplicity at first consider a spin system of the global $Z(n)$ symmetry on the plane square lattice. Lattice sites will be numbered by the variable $x = \{x_1, x_2\}$ where x_1 and x_2 are integers. Dual lattice is a simple square one too, its sites are situated in the centers of the initial lattice cells and will be numbered by pair $\tilde{x} = \{\tilde{x}_1, \tilde{x}_2\}$ of half-integers. Furthermore we define two "unit vectors" Δx_μ, $\mu = 1,2$ as $\Delta x_1 = \{1, 0\}$, $\Delta x_2 = \{0, 1\}$ and a dual pair $\Delta\tilde{x}_\nu = \sum_{\mu=1} \epsilon_{\mu\nu}\Delta x_\mu$ $\epsilon_{\mu\nu}$ is the antisymmetrizator $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$; $\epsilon_{12} = 1$. The links of the initial lattice which ends in x and $x + \Delta x_\mu$ will be lettered by the pair x, μ and the dual lattice links with ends in $\tilde{x}$ and $\tilde{x} + \Delta\tilde{x}_\mu$ by the pair $\tilde{x}, \tilde{\mu}$. Note that link x, μ is dual to link $\tilde{x}, \mu$ provided $\tilde{x} = x + \{1/2, 1/2\}$. We shall represent the elements of group $Z(n)$ associated with the lattice site x by numbers $e^{i\phi_x}$, where $\phi_x = \frac{2\pi}{n} k_x$, $k_x = 0, 1, \ldots n-1$; in this case the group multiplication coincides with the usual production of these numbers. The configuration of the system will be denoted by $\{\phi_x\}$. Of course one should not distinguish between configurations which differ in $2\pi m_x$, where m_x are integers. Partition function of the model has the form

$$Z(T) = \sum_{\{\phi_x\}} \exp - U(\{\phi_x\}, T), \tag{1}$$

where T is a temperature-like parameter. For the function $U(\{\phi_x\}, T)$ we take

$$U(\{\phi_x\}, T) = -\sum_{x,\mu} \ln \sum_{m_{x,\mu}=-\infty}^{\infty} \exp - \frac{1}{2T}\left(\phi_x - \phi_{x+\Delta x_\mu} - 2\pi m_{x,\mu}\right)^2 \tag{2}$$

The reason of such unusual and complicated choice of U is that in this case its functional form remains unchanged under K-W transform except for the parameter T transformation*).

*) All the following considerations remain true under any other choice of the interaction energy, but the K-W transform leads to the change in the functional form of $U(\{\phi_x\})$.

Note, that as $T \to 0$ and $\phi_x - \phi_{x+\Delta x_\mu}$ are small

$$\mathcal{U}(\{\phi_x\}, T) \to \frac{1}{2T} \sum_{x,\mu} \left(\phi_x - \phi_{x+\Delta x_\mu}\right)^2 \tag{3}$$

and therefore it is natural to call T the temperature.

Assuming (2) we have

$$Z(T) = \sum_{\{\phi_x\}} \sum_{\{m_{x,\mu}\}} \exp -\frac{1}{2T}\left(\phi_x - \phi_{x+\Delta x_\mu} - 2\pi m_{x,\mu}\right)^2, \tag{4}$$

where the configuration of integers associated with all the links is denoted by $\{m_{x,\mu}\}$. It is convenient to define augmentations $\theta_{x,\mu} = \phi_{x+\Delta x_\mu} - \phi_x$ associated with the links of the lattice and to sum over ϕ_{x_0} (where x_0 is an arbitrary initial site) and over all configurations $\{\theta_{x,\mu}\}$. Of course, the following restriction should be satisfied:

$$(\mathrm{Rot}\,\theta)_{\tilde{x}} = \theta_{x,1} + \theta_{x+\Delta x_1,2} - \theta_{x+\Delta x_2,1} - \theta_{x,2} = 2\pi \ell_{\tilde{x}} \tag{5}$$

which takes into account the necessary requirement that the circulation of "the vector" $\theta_{x,\mu}$ around each lattice cell results in the initial group element. In eq. (5) the lattice "rotor" $(\mathrm{Rot}\,\theta)_{\tilde{x}}$ and the integers $\ell_{\tilde{x}}$ are attached to the sites of the dual lattice.

It is easy to check that*)

$$\sum_{p=0}^{n-1} e^{ip\phi} = \begin{cases} n & \text{if} \quad e^{i\phi} = 1 \\ 0 & \text{if} \quad e^{i\phi} \in Z(n) \text{ but } e^{i\phi} \neq 1 \end{cases} \tag{6}$$

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*) Eq. (6) is a particular example of the general formula $\sum_\nu \chi_\nu(G) = \begin{cases} n & \text{if } G = I \\ 0 & \text{if } G \neq I \end{cases}$ where ν numbers all the irreducible representations of the point group G, n is a number of its elements and $\chi_\nu(G)$ are characters of these representations. This formula together with the analogous one for the case of continuous group permits one to apply the method presented to systems with an arbitrary commutative symmetry.

Therefore the summation in eq.(4) can be carried out over $\{\theta_{x,\mu}\}$ provided the summand is multiplied by

$$\prod_{\tilde{x}} \frac{1}{n} \sum_{p_{\tilde{x}}=0}^{n-1} e^{i p_{\tilde{x}} (Rot\,\theta)_{\tilde{x}}} . \qquad (7)$$

Now eq.(4) may be written in the form

$$Z(T) \sim \sum_{\{\theta_{x,\mu}\}} \sum_{\{m_{x,\mu}\}} \sum_{\{p_{\tilde{x}}\}} \exp -\frac{1}{2T} \sum_{x,\mu} (\theta_{x,\mu} - 2\pi m_{x,\mu})^2 + \qquad (8)$$

$$+ i \sum_{x,\mu} \theta_{x,\mu} (p_{\tilde{x}} - p_{\tilde{x}+\Delta\tilde{x}_\mu}) .$$

In eq.(8) $\{p_{\tilde{x}}\}$ means the configuration of integers $p_{\tilde{x}} = 0, 1, \ldots n-1$. The summation over $m_{x,\mu}$ can be carried out and one obtains (the notation $\frac{2\pi}{n} K_{x,\mu} = \theta_{x,\mu} + 2\pi m_{x,\mu}$ is used):

$$Z(T) \sim \sum_{\{p_{\tilde{x}}\}} \sum_{\{K_{x,\mu}\}} \exp\Big\{ -\frac{2\pi^2}{Tn^2} \sum_{x,\mu} K_{x,\mu}^2 + \qquad (9)$$

$$+ \frac{2\pi i}{n} \sum_{x,\mu} K_{x,\mu} (p_{\tilde{x}} - p_{\tilde{x}+\Delta\tilde{x}_\mu}) \Big\} .$$

Using the well known identity

$$\sqrt{\alpha/\pi} \sum_{m=-\infty}^{\infty} e^{-\alpha m^2 + i m \phi} = \sum_{n=-\infty}^{\infty} e^{-\frac{1}{4\alpha}(\phi - 2\pi n)^2} \qquad (10)$$

one obtains

$$Z(T) \sim \sum_{\{\chi_{\tilde{x}}\}} \sum_{\{\ell_{\tilde{x},\mu}\}} \exp -\frac{1}{2T^*} \left(\chi_{\tilde{x}} - \chi_{\tilde{x}+\Delta\tilde{x}_\mu} - 2\pi \ell_{\tilde{x},\mu} \right)^2 , \qquad (11)$$

where $\{\ell_{\tilde{x},\mu}\}$ denotes the configuration of integers associated with all the links of the dual lattice and we use the notations

$$\chi_{\tilde{x}} = \frac{2\pi}{n}\rho_{\tilde{x}} \quad \text{and}$$

$$TT^* = \frac{4\pi^2}{n^2} . \tag{12}$$

Finally, the following remark should be mentioned. Throughout the calculations presented the summation over ϕ_{x_0} (which can be reduced to the multiplication by n) was systematically dropped, one superfluous summation over χ was performed. To understand the latter point, imagine the lattice to be finite and to form a closed figure, say a torus. Now single out any lattice cell. The condition $Rot\,\theta = 0$ in all other cells automatically ensures it in the cell taken and one of the summations over ρ's in eq.(7) turns out to be unnecessary. This summation results in the multiplication by n only, because the dual system is $Z(n)$ symmetric.

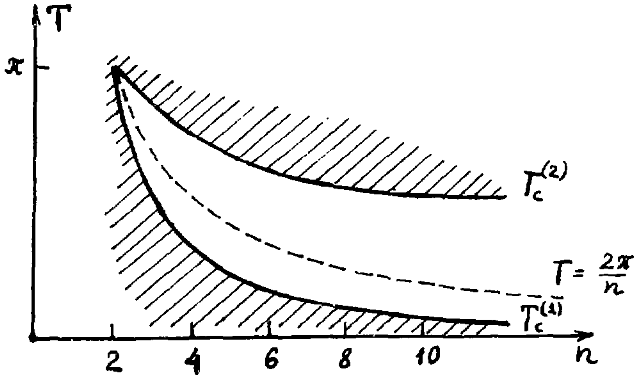
The derivation of the duality relations in the case of the generalized gauge systems with symmetry $Z(n)$ can be made in a similar manner. Note at first that under the generalized gauge invariance one should not distinguish between configurations which differ in gauge transformation only. Therefore in the partition function the summation over the "potentials" A can be replaced by that over the "intensities" F provided its lattice "rotor" is constrained to be zero. The latter is associated with the $q+1$ dimensional lattice elements and can be attached to the dual $d-q-1$ dimensional elements as well. Ensuring the constraint by the expressions of the type (7), where ρ's are functions of $d-q-1$ dimensional elements of the dual lattice, one immediately gets the required relations.

Note that for $n=2$ model (1) is reduced to the plane Ising model, the parameters T and K (K is the inverse temperature of the Ising model) being related by

$$e^{-2K} = \frac{\sum\limits_{m=-\infty}^{\infty} \exp - \frac{\pi^2}{2T}(2m+1)^2}{\sum\limits_{m=-\infty}^{\infty} \exp - \frac{2\pi^2}{T} m^2} . \tag{13}$$

It is easy to check that $T \leftrightarrow T^*$ corresponds to $K \leftrightarrow K^*$, where $\operatorname{Sh} 2K \operatorname{Sh} 2K^* = 1$.

The plane Ising model reveals a single phase transition point, its position can be therefore determined from the equation $K = K^*$. For the case $n \geq 3$ the situation seems to be different: there are two phase transition points, their positions $T_c^{(1)}$ and $T_c^{(2)}$ being related by $T_c^{(1)} T_c^{(2)} = \frac{4\pi^2}{n^2}$. The presumable phase diagram in the T-n plane is drawn in the figure. Under the K-W



transform the upper and lower phases (shaded regions in the figure) turn into each other. In the lower phase the symmetry is broken and small fluctuations occur around a certain group element and in the upper phase the symmetry of the dual system is broken. Between these two phases there is an intermediate region $T_c^{(1)} < T < T_c^{(2)}$. It looks like true that the properties of this phase are analogous to those of Berezinski phase [7] of the XY - model, i.e., the symmetry is not broken but the system possesses a transverse rigidity and large distance asymptotics of all the correlation functions are power-like, the exponents being the continuous functions of the temperature. In this case the phase transition at $T = T_c^{(2)}$ corresponds to vanishing the transverse rigidity and analogous to that of the XY - model.

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