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HYDRODYNAMIC DESCRIPTION OF SUPERFLUID SYSTEMS

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HYDRODYNAMIC DESCRIPTION OF SUPERFLUID SYSTEMS

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Гидродинамические описания сверхтекучей системы

Проблема взаимодействующей сверхтекучей Бозе-жидкости в модели Боголюбова рассмотрена в двух представлениях: частиц и квазичастиц (фононов). Найдена связь между этими представлениями. Показано, что в пределе малых импульсов гамильтониан Боголюбова, выраженный через операторы плотности конденсата и скорости, имеет квантовый гидродинамический вид. Каноническое u - v преобразование является в этом случае примером преобразования, связывающего канонические переменные частиц с фононами.

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Hydrodynamic Description of Superfluid Systems

The problem of consideration of interacting superfluid Bose systems either in terms on initial particles or quasiparticles (phonons) is examined with the help of simple Bogoliubov model. It appeared possible to join these two pictures. Namely it was demonstrated that for small momenta Bogoliubov Hamiltonian expressed by condensate density and velocity operators has a form of quantum hydrodynamical Hamiltonian. The canonical /u,v/ transformation is just a special example of transformation connecting particle canonical variables with phonon ones.

The investigations has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

In 1941 Landau /1/ obtained hydrodynamic equations describing in a phenomenological way superfluid helium 4. In the same paper density of particles and density of current have been regarded as operators and mass density $\hat{\varrho}(t,r)$ and velocity field $\underline{\hat{v}}(t,r)$ as quantized canonical variables. These variables enter the Hamiltonian describing the fluid. According to examination of the energy spectrum the lowest excitations in superfluid helium 4 are phonons with spectrum $\varepsilon(p) = cp$, where c is sound velocity and p is momentum.

The phonon-phonon interaction was considered in /2/ with the help of Hamiltonian describing four phonon processes. Just the Hamiltonian describing four phonon processes, named quantum hydrodynamical Hamiltonian (QHH), is a basic one for considerations of ref. /3/. It has a form

$$\hat{H} = \frac{1}{2}\sum_{p}\left(\varrho\,\underline{\hat{v}}_p\underline{\hat{v}}_{-p} + \frac{c^2}{\varrho}\hat{\varrho}_p\hat{\varrho}_{-p}\right) + \frac{1}{2}\sum_{p,q}\left(\underline{\hat{v}}_p\hat{\varrho}_q\underline{\hat{v}}_{-p+q} + \frac{c^2(2u-1)}{3\varrho^2}\hat{\varrho}_p\hat{\varrho}_q\hat{\varrho}_{-q-p}\right) \tag{1.1}$$

$$+\frac{c^2[(u-1)^2+w]}{12\rho^3}\sum_{p,q,r}\hat{\rho}_p\hat{\rho}_q\hat{\rho}_r\hat{\rho}_{-p-q-r},$$

where

$$u=\frac{\rho}{c}\frac{\partial c}{\partial\rho}\quad,\quad w=\frac{\rho^2}{c^2}\frac{\partial^2 c}{\partial\rho^2}. \tag{1.2}$$

The operators $\hat{\rho}_p$, $\hat{\underline{v}}_p$ are expressed in terms of operators of creation and annihilation b_k^+ , b_k of phonons

$$\hat{\rho}_p=\sqrt{\frac{\rho p}{2c}}\left(b_p^+ + b_{-p}\right),$$

$$\hat{\underline{v}}_p=\sqrt{\frac{c}{2\rho p}}\,\underline{p}\left(b_p-b_{-p}^+\right),\quad \hat{\underline{v}}_{-p}=-\sqrt{\frac{c}{2\rho p}}\,\underline{p}\left(b_{-p}-b_p^+\right). \tag{1.3}$$

Density and velocity operators satisfy commutation relations ($\hbar$=1)

$$\left[\hat{\rho}_p\,,\,\hat{\underline{v}}_k\right]=-\underline{k}\,\delta_{pk}. \tag{1.4}$$

By the way we will outline how to get Hamiltonian (1.1) using for simplicity classical considerations.

Energy for unit volume of the fluid and for volume $V=1/\rho$ is

$$\tilde{E}=\rho\frac{v^2}{2}+\rho\varepsilon\quad,\quad \tilde{H}=\int\tilde{E}\,dV, \tag{1.5}$$

where the first term in expression for $\tilde{E}$ denotes kinetic energy and second internal energy (ε is energy for unit mass).

We are interested in a flow of fluid with small velocity $\underline{v}$. It produces deviation from equilibrium values $\rho_0,\ \varepsilon_0$ to $\rho=\rho_0+\rho'$ and $\varepsilon=\varepsilon_0+\varepsilon'$ (ρ' is of the same order of smallness like v or $\rho_0 v^2\sim\rho'^2$). Our aim is to develope $\rho\varepsilon$ in power series in keeping terms till ρ' in fourth power. It is convenient to use following thermodynamic relations

$$d(\rho\varepsilon)=\tilde{w}\,d\rho+\rho T ds, \tag{1.6}$$

where $\tilde{w}$ is enthalpy

$$\tilde{w}=\varepsilon+\frac{p}{\rho}\quad,\quad d\tilde{w}=Tds+\frac{1}{\rho}dp\,,\ \left(\frac{\partial\tilde{w}}{\partial p}\right)_s=\frac{1}{\rho}. \tag{1.7}$$

We remeber that

$$\left(\frac{\partial p}{\partial\rho}\right)_s=c^2. \tag{1.8}$$

From (1.5) - (1.7) we have

$$\left(\frac{\partial(\rho\varepsilon)}{\partial\rho}\right)_s=\tilde{w}_0\quad,\quad \left(\frac{\partial^2(\rho\varepsilon)}{\partial\rho^2}\right)_s=\left(\frac{\partial\tilde{w}}{\partial\rho}\right)_s=\left(\frac{\partial\tilde{w}}{\partial p}\right)_s\left(\frac{\partial p}{\partial\rho}\right)_s=\frac{c^2}{\rho},$$

$$\left(\frac{\partial^3(\rho\varepsilon)}{\partial\rho^3}\right)_s=\frac{\partial}{\partial\rho}\left(\frac{c^2}{\rho}\right)=\frac{c^2}{\rho^2}\left[2u-1\right], \tag{1.9}$$

$$\left(\frac{\partial^4(\rho\varepsilon)}{\partial\rho^4}\right)_s=\frac{\partial^2}{\partial\rho^2}\left(\frac{c^2}{\rho}\right)=2\,\frac{c^2}{\rho^2}\left[(u-1)^2+w\right],$$

where u, w are given by (1.2).

Term $\int\rho_0\varepsilon_0\,dV$ describes energy of a fluid in rest and do not describe phonons. Term $\int\tilde{w}_0\rho' dV=0$ because $\int\rho\,dV=\int\rho_0\,dV$. Therefore Hamiltonian of interacting phonons has a form

$$\hat{H}=\hat{H}_2+\hat{H}_3+\hat{H}_4,$$

$$\hat{H}_2=\frac{1}{2}\int\left[\rho^2 v^2+\frac{c^2}{\rho}\rho'^2\right]dV,$$

$$\hat{H}_3=\frac{1}{2}\int\left[\rho' v^2+\frac{1}{3}\frac{\partial}{\partial\rho}\left(\frac{c^2}{\rho}\right)\rho'^3\right]dV, \tag{1.10}$$

$$\hat{H}_4=\frac{1}{2}\int\frac{\partial^2}{\partial\rho^2}\left(\frac{c^2}{\rho}\right)_s\rho'^4\,dV.$$

In ref. /3/ the fundamental assumption is that superfluid helim is described with the help of QHH (1.1),i.e.,not in terms of particles (helium atoms) but in terms of quasiparticles (phonons). Under second very important assumption that average $\langle\hat{v}_k\rangle$ is equal to velocity of superfluid component the authors were able to derive Landau two-fluid equations for superfluid helium 4.

On the other hand Bogoliubov /4/ obtained Landau equations starting from Hamiltonian describing particles. In spirit of Bogoliubov paper, in ref. /5/, operators of mass density of condensate $\hat{\rho}^c$ and velocity of condensate (superfluid velocity) of condensate (superfluid velocity) $\hat{\underline{v}}^c$ have been introduced. They are built from the operators of creation and annihilation of particles

$$\hat{\rho}^c_{-p} = m\sqrt{\rho_0}\,(a_{-p} + a^+_p),$$
$$\hat{\underline{v}}^c_p = \frac{\underline{p}}{m}\,\frac{1}{2\sqrt{\rho_0}}(a_p - a^+_{-p}), \tag{1.11}$$

and satisfy commutation relations

$$[\hat{\rho}^c_{-p}\ ,\ \hat{\underline{v}}^c_k] = -\underline{p}\,\delta_{pk}\,. \tag{1.12}$$

In (1.11) ρ_0 denotes density of Bose-condensate at equilibrium. Under assumption about existence of Bose condensate

$$a^+_0 a_0 = N_0 \sim N\ ,\quad \frac{a^+_0 a_0}{V} = \rho_0\ ,\quad \frac{a_0}{\sqrt{V}} = \frac{a_0}{\sqrt{V}} = \sqrt{\rho_0}\ . \tag{1.13}$$

It is justified to separate /5/, in the expression for density and current, terms of the form (1.11). Namely

$$\hat{\rho}_k = m\frac{1}{\sqrt{V}}\sum_p a^+_{p-\frac{k}{2}}\,a_{p+\frac{k}{2}} = m\sqrt{\rho_0}\,(a_k + a^+_{-k})$$
$$+\, m\frac{1}{\sqrt{V}}\sum_{p\neq \pm k/2} a^+_{p-\frac{k}{2}}\,a_{p+\frac{k}{2}} = \hat{\rho}^c_k + \hat{\rho}^d_k\ ,$$

$$\hat{\underline{j}}_k = \frac{1}{\sqrt{V}}\sum_p \underline{p}\,a^+_{p-\frac{k}{2}}\,a_{p+\frac{k}{2}} = \underline{k}\,\frac{\sqrt{\rho_0}}{2}(a_k - a^+_{-k}) \tag{1.14}$$
$$+\,\frac{1}{\sqrt{V}}\sum_{p\neq\pm\frac{k}{2}} \underline{p}\,a^+_{p-\frac{k}{2}}\,a_{p+\frac{k}{2}} = \hat{\underline{j}}^c_k + \hat{\underline{j}}^d_k = m\rho_0\,\hat{\underline{v}}^c_k + \hat{\underline{j}}^d_k\,.$$

Comparison of (1.3) with (1.11) suggests very simple transformation from canonical variables ($\hat{\rho}_p$, $\hat{\underline{v}}_p$) to canonical variables ($\hat{\rho}^c_{-p}$, $\hat{\underline{v}}^c_p$). Namely

$$\hat{\rho}_p = \frac{1}{\gamma_p}\,\hat{\rho}^c_{-p} = U\hat{\rho}^c_{-p}U^+\ ,$$
$$\hat{\underline{v}}_p = \gamma_p\,\hat{\underline{v}}^c_p = U\,\hat{\underline{v}}^c_p U^+\ , \tag{1.15}$$

where

$$\gamma_p = m\sqrt{\frac{2\rho_0 c}{m\rho p}}\ . \tag{1.16}$$

The aim of our present considerations is to try to examine connection between (1.3) and (1.11) on the basis of a simple Bogoliubov model /6/ describing nonperfect Bose gas with a weak interaction between particles.

2. Hamiltonian of a nonperfect Bose gas as a quantum hydrodynamical Hamiltonian

Let us consider system of nonperfect Bose gas with Hamiltonian

$$\hat{H}_B = \sum_p \left(\frac{p^2}{2m} + V_0\rho_0\right) a^+_p a_p$$
$$+\ \frac{1}{2}V_0\rho_0 \sum_{p\neq 0}\left(a^+_{-p}a^+_p + a_p a_{-p}\right)\,. \tag{2.1}$$

Here V_0 describes interaction between particles, a^+_p, a_p are creation and annihilation operators of particles. (We underline now that our "superfluid" is described in terms of initial particles). Operators of particles are expressed by

$$a_k = \frac{1}{2m\sqrt{\rho_0}}\,\hat{\rho}^c_k + m\sqrt{\rho_0}\,\underline{k}\,\hat{\underline{v}}^c_k/k^2\ , \tag{2.2}$$

$$a_{-k} = \frac{1}{2m\sqrt{\rho_0}}\hat{\rho}^c_k - m\sqrt{\rho_0}\,\underline{k}\,\hat{\underline{v}}^c_{-k}/k^2 ,$$

$$a^+_k = \frac{1}{2m\sqrt{\rho_0}}\hat{\rho}^c_{-k} + m\sqrt{\rho_0}\,\underline{k}\,\hat{\underline{v}}^c_k/k^2 ,$$

$$a^+_{-k} = \frac{1}{2m\sqrt{\rho_0}}\hat{\rho}^c_{-h} - m\sqrt{\rho_0}\,\underline{k}\,\hat{\underline{v}}^c_{-k}/k^2 .$$

One can put into (2.1) (see ,e.g., ref. /6/)

$$V_0\rho_0 = mc^2 \tag{2.3}$$

and have for (2.1) expression

$$\hat{H}_B = \sum_p\Big[\Big(\frac{p^2}{2m} + 2mc^2\Big)\frac{1}{4m^2\rho_0}\hat{\rho}^c_{-p}\hat{\rho}^c_p + \frac{m\rho_0}{2}\hat{\underline{v}}^c_{-p}\hat{\underline{v}}^c_p\Big] \tag{2.4}$$

At sufficiently low temperatures we can drop terms with higher p . In this case $p^2/2m \ll 2mc^2$. Now Hamiltonian (2.1) or (2.4) has form of the QHH (1.1)

$$\hat{H}_B = \frac{1}{2}\sum_p\Big[\rho^m_0\,\hat{\underline{v}}^c_{-p}\hat{\underline{v}}^c_p + \frac{c^2}{\rho^m_0}\hat{\rho}^c_{-p}\hat{\rho}^c_p\Big] , \tag{2.5}$$

$$\rho^m_0 = m\rho_0 .$$

It means that starting from the Hamiltonian of particles we can get quantum hydrodynamical Hamiltonian (QHH).

With the help of Hamiltonian (2.5) one can find equations of motion for operators $\hat{\rho}^c$ and $\hat{\underline{v}}^c$

$$i\frac{\partial\hat{\rho}^c_k}{\partial t} = [\hat{\rho}^c_k, \hat{H}_B] = \rho^m_0\,\underline{k}\,\hat{\underline{v}}^c_k$$

$$i\frac{\partial\hat{\underline{v}}^c_k}{\partial t} = [\hat{\underline{v}}^c_k, \hat{H}_B] = c^2\frac{\hat{\rho}_k}{\rho^m_0}\underline{k} = \mu\underline{k}\frac{\hat{\rho}_k}{\rho^m_0} , \tag{2.6}$$

where μ denotes chemical potential per unit mass /7/

$$\mu = \frac{V_0\rho_0}{m} . \tag{2.7}$$

The first of the eqs (2.6) is the continuity equation for condensate, and second (of the form $\partial v_s/\partial t \sim \nabla\mu$) is the Landau equation for superfluid velocity.

Now we should like to change from particles to quasiparticles using the known Bogoliubov /u,v/ transformation.

3. Transformation of canonical variables by Bogoliubov transformation

We change now from operators a^+_p , a_p to new Bose amplitudes β^+_p , β_p describing quasiparticles (Bogoliubov phonons)

$$\begin{aligned} a_p &= u(p)\beta_p + v(p)\beta^+_{-p} , \\ a^+_{-p} &= u(p)\beta^+_{-p} + v(p)\beta_p , \\ u^2(p) &- v^2(p) = 1 . \end{aligned} \tag{3.1}$$

The canonical variables transform

$$\hat{\underline{v}}^c_p = (u(p) - v(p))\frac{\underline{p}}{m}\frac{1}{2\sqrt{\rho_0}}(\beta_p - \beta^+_{-p}) = (u(p) - v(p))\hat{\underline{v}}_p ,$$

$$\hat{\rho}^c_{-p} = (u(p) - v(p))m\sqrt{\rho_0}(\beta_{-p} + \beta^+_p) = (u(p) + v(p))\hat{\rho}_p . \tag{3.2}$$

We see that we must now calculate coefficients ($u(p) - v(p)$) and $(u(p) + v(p))$ of transformation (3.2) in the limit of small p. One can find them from standard condition that in transformed Hamiltonian (2.1) coefficient of $(\beta_p\beta_{-p}+\beta^+_{-p}\beta^+_p)$ vanishes. This leads to equations

$$4\varepsilon^2(p)\,u^2(p)\,v^2(p) - m^2c^4 = 0$$
$$u^2(p) - v^2(p) = 1 \quad , \quad \varepsilon(p) = cp \,. \tag{3.3}$$

For small p

$$v^2(p) = -\tfrac{1}{2}\left(1-\sqrt{\frac{m^2c^2}{p^2}+1}\right) = \tfrac{1}{2}\,\frac{mc}{p}\left(1-\frac{p}{mc}\right),$$
$$u^2(p) = \tfrac{1}{2}\left(1+\sqrt{\frac{m^2c^2}{p^2}+1}\right) = \tfrac{1}{2}\,\frac{mc}{p}\left(1+\frac{p}{mc}\right). \tag{3.4}$$

This gives

$$v(p) = \sqrt{\frac{mc}{2p}}\left(1-\tfrac{1}{2}\frac{p}{mc}\right),$$
$$u(p) = \sqrt{\frac{mc}{2p}}\left(1+\tfrac{1}{2}\frac{p}{mc}\right). \tag{3.5}$$

Finally we have

$$u(p) - v(p) = \sqrt{\frac{p}{2mc}} = \frac{1}{\alpha_p},$$
$$u(p) + v(p) = \sqrt{\frac{2mc}{p}} = \alpha_p \,. \tag{3.6}$$

Transformation (3.2) has a form

$$\hat{\underline{v}}^c_p = \frac{1}{\alpha_p}\hat{\underline{v}}_p \quad , \quad \hat{\varsigma}^c_{-p} = \alpha_p\hat{\varsigma}_p \,. \tag{3.7}$$

Independently of approximation it is evidently a canonical transformation.

In our model $N_0 \approx N$, i.e., $\varsigma_0 \approx \varsigma$ and

$$\gamma_p = m\sqrt{\frac{2c\varsigma_0}{mp\varsigma}} \approx \sqrt{\frac{2mc}{p}} = \alpha_p \,. \tag{3.8}$$

We see that in our model the transformation which connects canonical variables of particles and of phonons is just known as /u,v/ transformation from particles to Bogoliubov phonons.

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References:

/1/ L.D.Landau, J.Phys (USSR), 5, 71 (1941).
/2/ I.M.Khalatnikov, L.D.Landau, JETP, 19, 637 (1949).
/3/ K.Miyaka, K.Yamada, "Quantum Hydrodynamics and Two Fluid Dynamics" , preprint (1976); Prog.Theor.Phys., 57, 1133(1977).
/4/ N.N.Bogoliubov, "On Hydrodynamics of a Superfluid" Dubna preprint (1963) "Lectures on Quantum Statistics" (Gordon and Breach, New York (1970)) Vol.2 Ch.3.
/5/ Z.M.Galasiewicz, Bull. Acad.Polon. Sci., 24, 353 (1976).
/6/ I.M.Khalatnikov, "An Introduction to the Theory of Superfluidity" (W.A.Benjamin, Inc. New York, (1965)).
/7/ S.T.Beliaev, JETP, (USSR), 34 417, 433 (1958); see also Z.M. Glasiewicz, "Superfluid Hydrodynamics for Weakly Interacting Bose-Systems", XIV-th Winter School of Theoretical Physics; "Collective Phenomena in Solids and Helium", Acta Univ. Wratislaviensis (1977).

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