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N.N.Bogolubov

ON THE STOCHASTIC PROCESSES IN THE DYNAMICAL SYSTEMS

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Боголюбов Н.Н. E17 - 10514

О стохастических процессах в динамических системах

В работе рассмотрены стохастические процессы в динамических системах для случая слабого взаимодействия "малой системы" (например, одной частицы) с "большой системой".

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Bogolubov N.N. E17 - 10514

On the Stochastic Processes in the Dynamical Systems

The stochastic processes in the dynamical systems are considered for the case of weak interaction of a "small system" (e.g., one particle) with a "large one".

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1977

In our paper /1/ already published in 1939 we considered the question about the appearance of a stochastic process in a dynamical system, which is submitted to the influence of a "large" system.

For classical system this question was studied on the basis of the Liouville equation for the probability distribution in the phase space, and for quantum mechanical systems on the basis of the analogous equation for the von Neumann's statistical operators.

In the mentioned paper a method was elaborated which permits us to obtain, in the first approximation, the Fokker-Planck equations.

This method of course could not pretend to be on adequate mathematical foundation, and hence in the following paper /2/ a particular "model" example was considered in which the dynamical equations are of an exactly integrable type, what has made it possible to perform the analysis of the previously introduced approximation, in this case, on the fully rigorous mathematical basis.

For quantum mechanical systems the analogous results were also obtained /3/.

In my lectures, that were delivered at the Rockefeller University in autumn 1974, I presented a slightly modified version of the method of works /1/ and pointed to its connection with the theory of two-time Green functions.

The present article, which I earlier assumed to publish on the basis of these lectures, has been written now after getting acquainted with a number of important works on the theory of interactions of one particle with a large system, that has been developing during the last decade.

Therefore, I have found it suitable to introduce some essential changes in comparison with the original text of the mentioned lectures.

1.

Let us consider a "small system" S, for example, consisting of a single particle weakly interacting with a "large system" Σ .

We shall first treat this case in the framework of the classical mechanics.

Following the usual procedure in the classical statistical mechanics, we introduce a probability distribution function in the phase space of the total system $S+\Sigma$:

$$\mathcal{D}_t = \mathcal{D}_t(S, \Sigma) = \mathcal{D}_t(\Omega_S, \Omega_\Sigma), \tag{1.1}$$

where Ω_S , Ω_Σ denote the phase points of S and Σ in the phase spaces corresponding, respectively, to these systems.

We shall consider now the situation, when at the initial moment of time: $t=0$ the system Σ finds itself in the state of statistical equilibrium and at this moment the interaction between S and Σ is switched on.

We thus suppose that

$$\mathcal{D}_0(S, \Sigma) = f_0(S)\mathcal{D}(\Sigma), \tag{1.2}$$

where

$$\mathcal{D}(\Sigma) = \mathcal{D}_{eq}(\Sigma) = Z^{-1} e^{-\frac{H_\Sigma(\Omega_\Sigma)}{\theta}}$$

$$Z = \int d\Omega_\Sigma e^{-\frac{H_\Sigma(\Omega_\Sigma)}{\theta}}$$

represents the equilibrium distribution in the phase space of the system Σ.

Here $H_\Sigma = H_\Sigma(\Omega_\Sigma)$ is the energy of the system Σ.

As is well-known, the evolution of the probability distribution $\mathcal{D}_t$ is determined by the Liouville equation, which we shall write in the form:

$$\frac{\partial \mathcal{D}_t}{\partial t} = \Lambda \mathcal{D}_t \tag{1.3}$$

the normalization condition for $\mathcal{D}_t$ being:

$$\int \mathcal{D}_t \, d\Omega_S d\Omega_\Sigma = 1 .$$

The Liouville operator Λ acting on functions of $(\Omega_S, \Omega_\Sigma)$ can be defined by means of the Poisson brackets:

$$\Lambda \mathcal{D}_t = [H, \mathcal{D}_t], \tag{1.4}$$

where H is the total Hamiltonian of $S+\Sigma$.

We wish to point out that only the cases when Λ does not depend explicitly upon time t will be considered.

Usually the total Hamiltonian H is represented by the sum:

$$H = H_S^\circ + H_\Sigma + H_{int}$$

of proper Hamiltonian of S and Σ, supplied by a term describing the interaction between S and Σ. Correspondingly, the Liouville operator is taken in the form:

$$\mathcal{L} = \mathcal{L}_{S+\Sigma} = \mathcal{L}^{\circ}_{S} + \mathcal{L}_{\Sigma} + \mathcal{L}_{int} \,. \qquad (1.5)$$

The interaction part $\mathcal{L}_{int}$ of $\mathcal{L}$ will be treated in what follows as a weak perturbation as though it were proportional to a small parameter.

Let us indicate now some examples of S, Σ, $\mathcal{L}$.

Consider the case when S is a single particle and Σ is a system consisting of N identical particles so that:

$$\Omega_S = (\vec{r}_0, \vec{v}_0); \; \Omega_\Sigma = (\vec{r}_1, \vec{v}_1; \ldots \vec{r}_N, \vec{v}_N), \qquad (1.6)$$

where the vectors $\vec{r}$ and $\vec{v}$ denote the position and velocity of the corresponding particles.

As usual, all these particles are supposed to be inside a very large cube with a macroscopic volume V, and the ordinary cyclic boundary conditions are imposed.

We may take the following expressions for $\mathcal{L}^{\circ}_{S}$, $\mathcal{L}_{int}$:

$$\mathcal{L}^{\circ}_{S} = -\vec{v}_0 \frac{\partial}{\partial \vec{r}_0}, \qquad (1.7)$$

$$\mathcal{L}_{int} = \mathcal{L}^{(\Phi)}_{int} = \sum_{(1 \le j \le N)} \frac{\partial \Phi(\vec{r}_0 - \vec{r}_j)}{\partial \vec{r}_0} \left(\frac{1}{m} \frac{\partial}{\partial \vec{v}_0} - \frac{1}{M} \frac{\partial}{\partial \vec{v}_j} \right), \qquad (1.8)$$

where $\Phi(r)$ is a radial symmetric potential function proportional to a small parameter, m is the mass of the S particle and M denotes the mass of a particle of Σ.

We shall consider also the important special case when the interaction between the S particle and the particle of Σ can be defined as the interaction between the corresponding hard spheres.

Formally, the hard sphere interaction can be characterized by the special choice of $\Phi(r)$:

$$\Phi(r) = +\infty, \quad \text{if } r < a, \qquad \Phi(r) = 0, \quad \text{if } r \ge a, \qquad (1.9)$$

where "a" is the sum of radii of S and a particle of Σ or what is the same, "a" represents the distance between the centres of these particles at the moment of collision.

For such a potential function the expression (1.8) is evidently singular and not convenient for use.

It was found[4], however, that the dynamics of hard sphere interaction can be correctly described by the "integrated" Liouville operator of the form:

$$\mathcal{L}^{coll.}_{int.} = \sum_{(1 \le j \le N)} \bar{T}(0,j), \qquad (1.10)$$

where:

$$\bar{T}(0,1) = a^2 \int_{\vec{v}_{0,1} \cdot \vec{\sigma} > 0} (\vec{v}_{0,1} \cdot \vec{\sigma}) \{ \delta(\vec{r}_0 - \vec{r}_1 - a\vec{\sigma}) \times B_{v_0, v_1}(\sigma) - \delta(\vec{r}_0 - \vec{r}_1 + a\vec{\sigma}) \} d\vec{\sigma}. \qquad (1.11)$$

Here $\vec{v}_{0,1} = \vec{v}_0 - \vec{v}_1$, $\vec{\sigma}$ is the unit vector, $B_{v_0, v_1}(\sigma)$ is the operator acting on functions $F(\vec{v}_0, \vec{v}_1)$ and replacing their arguments by:

$$\vec{v}_0 \to \vec{v}^*_0 = \vec{v}_0 - \frac{2M}{M+m} \vec{\sigma} (\vec{v}_{0,1} \cdot \vec{\sigma}),$$

$$\vec{v}_1 \to \vec{v}_1^* = \vec{v}_1 + \frac{2M}{M+m}\vec{\sigma}(\vec{v}_{0,1}\cdot\vec{\sigma}). \qquad (1.12)$$

The possibility of replacing (1.8) by the "integrated" operator (1.10) is due entirely to the instantaneous duration of hard sphere collisions in the classical mechanics.

So, it should be noted that in analogous situations in quantum mechanics the replacement of the Poisson brackets:

$$[H_{int}, \mathfrak{D}]$$

by a collision type operator acting on $\mathfrak{D}$ may be permitted only as an approximation in the cases when the effective time of collision (which is here essentially positive) could be neglected. On the contrary, we make no approximation in the classical mechanics when we use $Л^{coll}_{int}$ instead of (1.8), but, of course, the unphysical overlapping configurations must be excluded so that $\mathfrak{D}$ must be equal to zero for such configurations.

We may also consider the case when, in addition to a hard core interaction, we have also the regular pair $(0,j)$ interaction described by a smooth function $\Phi(r)$, proportional to a small parameter, defined for $r \geq a$ and which we formally continue for $r < a$ by putting

$$\Phi'(r) = 0 \qquad \text{for} \quad r < a.$$

In this case:

$$Л_{int} = Л^{coll}_{int} + Л^{(\Phi)}_{int}. \qquad (1.13)$$

Let us notice now that in order that $Л^{coll}_{int}$ could be treated as a "small perturbation" we must suppose that the corresponding mean free path

$$\sim \frac{1}{\frac{N}{V}a^2}$$

is very large with respect to a:

$$\frac{N}{V}a^3 \ll 1. \qquad (1.14)$$

We wish to stress here that this condition (1.14) does not imply that the interaction between the particles of Σ is small.

Consider, for example, the model in which S represents a neutron interacting only with the nuclei of the Σ particles (these nuclei being also represented as hard spheres) and the system Σ is a van der Waals fluid consisting of hard spheres whose diameter a_Σ is by many orders larger than the effective diameter of their nuclei.

In this model

$$a_\Sigma \gg a.$$

Of course, the real problem of the diffusion of a neutron in a fluid must be treated on the basis of quantum mechanics.

But in some cases it can also be treated in the quasi-classical approximation. Then we need only to replace the $\bar{T}(0,j)$ operator from (1.11) by a similar collision operator calculated by solving the quantum mechanical two-body problem. It has a very simple form in the case when only s-scattering is to be taken into account.

Finally, we wish to point out that because all Σ particles are identical, we must take as $Л_\Sigma$ a Liouville operator symmetric with respect to phases of these particles.

The interaction part $\mathcal{L}_{int}$ (1.13) being symmetric in this sense, we see that the total Liouville operator $\mathcal{L}$ is also symmetric in this sense.

By noting that the initial function $\mathfrak{D}_0$ given by (1.2) is symmetric, we find that $\mathfrak{D}_t$ is a symmetric function of $(\vec{r}_1,\vec{v}_1);\ldots(\vec{r}_N,\vec{v}_N)$.

Let us return to the general equation (1.3) for the evolution of the probability distribution $\mathfrak{D}_t$ in the phase space.

It will be convenient to introduce the following notation:

$$(\overline{\mathfrak{U}})_S = \int \mathfrak{U}\, d\Omega_S; \qquad (\overline{\mathfrak{U}})_\Sigma = \int \mathfrak{U}\, d\Omega_\Sigma, \tag{1.15}$$

$$(\overline{\mathfrak{U}})_{S+\Sigma} = \int \mathfrak{U}\, d\Omega_S\, d\Omega_\Sigma.$$

Consider now a dynamical variable $A(S)$ which is related only to the system S:

$$A(S) = A(\Omega_S).$$

Its average value at the moment t will be:

$$\langle A(S)\rangle_t = (\overline{A(S)\mathfrak{D}_t(S,\Sigma)})_{S+\Sigma}$$

which yields:

$$\langle A(S)\rangle_t = (\overline{A(S) f_t(S)})_S = \int A(\Omega_S) f_t(\Omega_S)\, d\Omega_S, \tag{1.16}$$

where

$$f_t(S) = (\overline{\mathfrak{D}_t(S,\Sigma)})_\Sigma. \tag{1.17}$$

We thus see that the reduced distribution $f_t(S)$ represents the probability density in the phase space of S at the moment t.

It is clear also that in order to evaluate the average value of the dynamical variables $A(S)$ only the reduced probability distribution $f_t(S)$ is needed and not the complete distribution $\mathfrak{D}_t(S,\Sigma)$.

Proceed at present to outline a method for obtaining the approximate equation for $f_t(S)$ in a closed form.

Our starting point is the Liouville equation (1.3) written as follows

$$\frac{\partial \mathfrak{D}_t}{\partial t} = (\mathcal{L}_S^\circ + \mathcal{L}_\Sigma + \mathcal{L}_{int})\mathfrak{D}_t \tag{1.18}$$

with the initial condition (1.2).

Denote

$$\mathfrak{D}_t - f_t \mathfrak{D}(\Sigma) = \Delta_t \tag{1.19}$$

and remark that because of (1.17):

$$(\overline{\Delta_t})_\Sigma = 0. \tag{1.20}$$

By integrating (1.18) over Ω_Σ and observing that identically

$$(\overline{\mathcal{L}_\Sigma \mathfrak{D}})_\Sigma = 0$$

we get

$$\frac{\partial f_t}{\partial t} = \{\mathcal{L}_S^\circ + (\overline{\mathcal{L}_{int}\mathfrak{D}(\Sigma)})_\Sigma\} f_t + (\overline{\mathcal{L}_{int}\Delta_t})_\Sigma. \tag{1.21}$$

The relations (1.18), (1.19), (1.21) yield:

$$\frac{\partial \Delta_t}{\partial t} = \frac{\partial \mathfrak{D}_t}{\partial t} - \frac{\partial f_t}{\partial t}\mathfrak{D}(\Sigma) =$$

$$= (\mathcal{L}_S^\circ + \mathcal{L}_\Sigma + \mathcal{L}_{int}) f_t \mathfrak{D}(\Sigma) + (\mathcal{L}_S^\circ + \mathcal{L}_\Sigma + \mathcal{L}_{int})\Delta_t -$$

$$-\{(\text{Л}_S^\circ + (\overline{\text{Л}_{int}\,\mathcal{D}(\Sigma)})_\Sigma) f_t + (\overline{\text{Л}_{int}\,\Delta_t})_\Sigma\}\mathcal{D}(\Sigma).$$

By definition $\mathcal{D}(\Sigma)$ is the equilibrium distribution with respect to Л_Σ:

$$\text{Л}_\Sigma \mathcal{D}(\Sigma) = 0$$

and therefore:

$$\text{Л}_\Sigma f_t(S)\,\mathcal{D}(\Sigma) = f_t(S)\,\text{Л}_\Sigma \mathcal{D}(\Sigma) = 0.$$

Let us introduce the notation:

$$\text{Л}_S = \text{Л}_S^\circ + (\overline{\text{Л}_{int}\;\mathcal{D}(\Sigma)})_\Sigma$$

$$\Gamma = \text{Л}_{int} - (\overline{\text{Л}_{int}\,\mathcal{D}(\Sigma)})_\Sigma \tag{1.22}$$

and remark

$$\text{Л}_S + \Gamma = \text{Л}_S^\circ + \text{Л}_{int}$$

$$(\overline{\Gamma\Delta_t})_\Sigma = (\overline{\text{Л}_{int}\,\Delta_t})_\Sigma - (\overline{\text{Л}_{int}\;\mathcal{D}(\Sigma)})_\Sigma(\overline{\Delta_t})_\Sigma =$$

$$= (\overline{\text{Л}_{int}\;\Delta_t})_\Sigma\,.$$

We then obtain the following equation for Δ_t:

$$\frac{\partial\Delta_t}{\partial t} = (\text{Л}_S + \text{Л}_\Sigma)\Delta_t + \Gamma\Delta_t - (\overline{\Gamma\Delta_t})_\Sigma\;\mathcal{D}(\Sigma) + \Gamma f_t\;\mathcal{D}(\Sigma) \tag{1.23}$$

and the equation (1.21) can be rewritten in the form:

$$\frac{\partial f_t}{\partial t} = \text{Л}_S f_t + (\overline{\text{Л}_{int}\;\Delta_t})_\Sigma\,. \tag{1.24}$$

The initial condition (1.2) yields:

$$\Delta_t = 0 \qquad \text{for} \qquad t = 0. \tag{1.25}$$

When we consider the equation (1.23) with the initial condition (1.25), from the intuitive point of view it seems natural to admit that Δ_t is, roughly speaking, proportional to the strength of the interaction term Γ.

Therefore in this intuitive and somewhat naive approach the term in (1.23):

$$\Gamma\Delta_t - (\overline{\Gamma\Delta_t})_\Sigma\,\mathcal{D}(\Sigma)$$

should be treated as the term of the "second order of smallness".

By retaining in the exact equation (1.23) only the main term in interaction, we obtain the approximate equation:

$$\frac{\partial\Delta_t}{\partial t} = (\text{Л}_S + \text{Л}_\Sigma)\Delta_t + \Gamma f_t(S)\,\mathcal{D}(\Sigma) \tag{1.23*}$$

evidently with the same initial condition (1.25), which can formally be solved:

$$\Delta_t = \int_0^t e^{(\text{Л}_S+\text{Л}_\Sigma)(t-r)}\;\Gamma f_r(S)\mathcal{D}(\Sigma)\,dr.$$

By substituting this expression into (1.24) we get:

$$\frac{\partial f_t}{\partial t} = \text{Л}_S f_t + \int_0^t (\overline{\text{Л}_{int}\,e^{(\text{Л}_S+\text{Л}_\Sigma)(t-r)}\;\Gamma\mathcal{D}(\Sigma)})_\Sigma f_r\,dr$$

or

$$\frac{\partial f_t}{\partial t} = \text{Л}_S f_t + \overline{\int_0^t (\text{Л}_{int}\,e^{(\text{Л}_S+\text{Л}_\Sigma)(t-r)}\;\{[\text{Л}_{int} - (\overline{\text{Л}_{int}\;\mathcal{D}(\Sigma)})]\mathcal{D}(\Sigma)\})}_\Sigma f_r\,dr. \tag{1.26}$$

We thus have obtained an approximate non Markoffian kinetic equation for the reduced distribution $f_t(S)$ in the closed form — in the sense that it does not depend upon complete distribution for the total system $S+\Sigma$.

This equation was established in the framework of the classical mechanics. To make it valid also in the quantum mechanical treatment of the dynamical system $S+\Sigma$, only some obvious changes are needed.

First let us represent von Neumann's statistical operator in the matrix form:

$$\mathfrak{D}_t = \mathfrak{D}_t (X_S, X'_S; X_\Sigma, X'_\Sigma), \tag{1.27}$$

where X_S, X_Σ are the complete sets of values of the commuting variables characterizing respectively the states of the dynamical systems S and Σ. X'_S, X'_Σ are the sets of values of the same variables.

The Liouville operators Λ, Λ^0_S, Λ_Σ, Λ_{int} are to be regarded as operators acting on the expressions of the type (1.27) considered as the classical functions of variables X_S, X'_S; X_Σ, X'_Σ.

These Λ-operators may be defined by means of the quantum mechanical Poisson brackets:

$$[H, \mathfrak{D}] = \Lambda \mathfrak{D}$$

Further, the notation (1.15) must be trivially transformed as follows:

$$(\overline{\mathfrak{U}})_S = \underset{(S)}{Sp}\, \mathfrak{U} = \int \mathfrak{U}(X_S, X_S; X_\Sigma, X'_\Sigma) dX_S$$

$$(\overline{\mathfrak{U}})_\Sigma = \underset{(\Sigma)}{Sp}\, \mathfrak{U} = \int \mathfrak{U}(X_S, X'_S; X_\Sigma, X_\Sigma) dX_\Sigma$$

$$(\overline{\mathfrak{U}})_{S+\Sigma} = \underset{(S+\Sigma)}{Sp}\, \mathfrak{U} = \int \mathfrak{U}(X_S, X_S; X_\Sigma, X_\Sigma) dX_S\, dX_\Sigma .$$

In particular:

$$f_t(S) = f_t(X_S, X'_S) = \underset{(\Sigma)}{Sp}\, \mathfrak{D}_t .$$

In usual situations one can take for X_S, X_Σ the positions $\vec{r}$ and the spins of all particles concerned or, alternatively, their momenta $\vec{p}$ and spins.

The integration over X_S or X_Σ is to be understood as the integration over all continuous components of X and the summation over all their discrete components.

We may now literally repeat our reasoning, starting from the quantum mechanical Liouville equation, and obtain the approximate equation for the reduced statistical operator $f_t(S)$ in the same form as (1.26).

It can easily be seen that the method outlined here is a slightly modernized version of the method elaborated in our first paper [1] and also further developed by A.V.Shelest [5].

2.

We shall now proceed to consider the kinetic equation (1.26) for some specific examples of the dynamical systems S, Σ within the framework of the classical mechanics.

We first take the example mentioned at §1 when $(\Omega_S, \Omega_\Sigma)$, Λ^0_S, Λ_{int} are represented by formulae (1.6), (1.7), (1.8).

Our attention will be confined here to the case when the statistical equilibrium of the system Σ alone described by the Gibbs distribution $\mathfrak{D}(\Sigma)$ corresponds to a spatially homogeneous state.

Thus, the situation where the system Σ in the statistical equilibrium represents a crystal is to be excluded.

We further suppose that the interaction potential function , proportional to a small parameter, is regular.

The following Fourier representation:

$$\Phi(r) = \frac{1}{V} \sum_{(k)} e^{i\vec{k}\vec{r}} \nu(k) \qquad (2.1)$$

will be used, where:

$$\nu(k) = \int e^{-i\vec{k}\vec{r}} \Phi(r) d\vec{r}. \qquad (2.2)$$

The summation in (2.1) runs over the usual quasi-discrete spectrum of wave numbers $\vec{k}$, corresponding to the volume V:

$$\vec{k} = \left(\frac{2\pi n_1}{L}, \frac{2\pi n_2}{L}, \frac{2\pi n_3}{L}\right)$$

n_1 , n_2 , n_3 being integers and $L^3 = V$.

In view of the radial symmetry of $\Phi(r)$ the Fourier transform $\nu(k)$ is a real function invariant with respect to the reflection:

$$\nu(k) = \nu^*(k) = \nu(-k). \qquad (2.3)$$

Rewrite our kinetic equation in the form:

$$\frac{\partial f_t}{\partial t} = \text{Л}_S f_t + \int_0^t K(t-\tau) f_\tau \, d\tau , \qquad (2.4)$$

$$K(T) = \overline{\left(\text{Л}_{int} e^{(\text{Л}_S + \text{Л}_\Sigma)T} \left[\text{Л}_{int} - \overline{(\text{Л}_{int} \mathfrak{D}(\Sigma))}_\Sigma\right] \mathfrak{D}(\Sigma)\right)}_\Sigma . \qquad (2.5)$$

To investigate this equation, it will be convenient to establish a useful identity concerning the expressions of the type:

$$\overline{(\text{Л}_{int} F(S, \Sigma))}_\Sigma .$$

We have, in virtue of the definition (1.8):

$$\overline{(\text{Л}_{int} F(S,\Sigma))}_\Sigma = \sum_{(j)} \overline{\left(\frac{\partial \Phi(\vec{r}_0 - \vec{r}_j)}{\partial \vec{r}_0} \cdot \frac{1}{m} \frac{\partial}{\partial \vec{v}_0} F(S,\Sigma)\right)}_\Sigma -$$

$$- \sum_{(j)} \frac{1}{M} \overline{\left(\frac{\partial \Phi(\vec{r}_0 - \vec{r}_j)}{\partial \vec{r}_0} \cdot \frac{\partial}{\partial \vec{v}_j} F(S, \Sigma)\right)}_\Sigma .$$

But the second term in the right-hand side of this relation is identically zero because it contains

$$\frac{\partial}{\partial \vec{v}_j} F(S, \Sigma)$$

integrated over all the velocity space of v_j.

Therefore we obtain:

$$\overline{(\text{Л}_{int} F(S,\Sigma))}_\Sigma = \frac{1}{m} \frac{\partial}{\partial \vec{v}_0} \sum_{(j)} \overline{\left(\frac{\partial \Phi(\vec{r}_0 - \vec{r}_j)}{\partial \vec{r}_0} F(S,\Sigma)\right)}_\Sigma . \qquad (2.6)$$

Let us apply this identity to the case, when we take:

$$F(S, \Sigma) = \mathfrak{D}(\Sigma).$$

The substitution of the Fourier representation (2.1) in (2.6) yields:

$$\overline{(\text{Л}_{int} \mathfrak{D}(\Sigma))}_\Sigma = \frac{1}{m} \frac{\partial}{\partial \vec{v}_0} \cdot \frac{1}{V} \sum_{(k)} i\vec{k} e^{i\vec{k}\vec{r}_o} \nu(k) \overline{\left(\sum_{(j)} e^{-i\vec{k}\vec{r}_j} \mathfrak{D}(\Sigma)\right)}_\Sigma . \qquad (2.7)$$

Remark now that because of the spatially homogeneous character of the statistical equilibrium of the system Σ described by the Gibbs distribution $\mathfrak{D}(\Sigma)$ the expressions:

$$\overline{(e^{-i\vec{k}\vec{r}_j}\,\mathfrak{D}(\Sigma))_\Sigma}$$

must be invariant with respect to arbitrary space translations:

$\vec{r}_j \to \vec{r}_j + \vec{r}$.

Hence

$$\overline{(e^{-i\vec{k}\vec{r}_j}\,\mathfrak{D}(\Sigma))_\Sigma} = e^{-i\vec{k}\vec{r}}\,\overline{(e^{-i\vec{k}\vec{r}_j}\,\mathfrak{D}(\Sigma))_\Sigma}.$$

As $\vec{r}$ is an arbitrary space vector, we see:

$$\overline{(e^{-i\vec{k}\vec{r}_j}\,\mathfrak{D}(\Sigma))_\Sigma} = 0 \quad \text{if} \quad \vec{k} \neq 0$$

and in virtue of (2.7)

$$\overline{(\mathcal{J}_{int}\,\mathfrak{D}(\Sigma))_\Sigma} = 0. \tag{2.8}$$

Therefore from (1.22) it follows

$$\mathcal{J}_S = \mathcal{J}_S^{\circ}\ . \tag{2.9}$$

Let us further apply the identity (2.6) to expression (2.5).

By taking account of (2.8), (2.9) we get:

$$K(T) = \frac{1}{m}\,\frac{\partial}{\partial \vec{v}_0}\cdot\vec{Q}(T), \tag{2.10}$$

$$\vec{Q}(T) = \sum_{(j,j_1)} \overline{\left(\frac{\partial\Phi(\vec{r}_0-\vec{r}_j)}{\partial\vec{r}_0}\right) e^{(\mathcal{J}_S^{\circ}+\mathcal{J}_\Sigma)T}\,\frac{\partial\Phi(\vec{r}_0-\vec{r}_{j_1})}{\partial\vec{r}_{j_1}}\times}$$
$$\overline{\times\left(\frac{1}{m}\,\frac{\partial}{\partial\vec{v}_0}+\frac{\vec{v}_j}{\theta}\right)\mathfrak{D}(\Sigma))_\Sigma}\ . \tag{2.11}$$

Here the fundamental property of $\mathfrak{D}(\Sigma)$:

$$-\frac{1}{M}\,\frac{\partial}{\partial\vec{v}_j}\,\mathfrak{D}(\Sigma) = \frac{\vec{v}_j}{\theta}\,\mathfrak{D}(\Sigma) \tag{2.12}$$

also was used.

The substitution of (2.1) into (2.11) yields:

$$\vec{Q}(T) = -\frac{1}{V^2}\sum_{(k,k_1)}\sum_{(j,j_1)}\vec{k}\,\nu(k)\,\nu(k_1)\,\mathcal{E}(\vec{k},\vec{k}_1), \tag{2.13}$$

where:

$$\mathcal{E}(\vec{k},\vec{k}_1) = \overline{(e^{i\vec{k}\vec{r}_0}\,e^{-i\vec{k}\vec{r}_j}\,e^{(\mathcal{J}_S^{\circ}+\mathcal{J}_\Sigma)T}\,e^{i\vec{k}_1\vec{r}_0}\,e^{-i\vec{k}_1\vec{r}_{j_1}}\,\vec{k}_1\times}$$
$$\overline{\times\left(\frac{1}{m}\,\frac{\partial}{\partial\vec{v}_0}+\frac{\vec{v}_j}{\theta}\right)\mathfrak{D}(\Sigma))_\Sigma}\ .$$

But $\mathfrak{D}(\Sigma)$ is invariant with respect to the translations:

$\vec{r}_j \to \vec{r}_j + \vec{r}\ ;\ j = 1,\dots N,$

where $\vec{r}$ is an arbitrary space vector.

Therefore

$$\mathcal{E}(\vec{k},\vec{k}_1) = e^{-i(\vec{k}+\vec{k}_1)\vec{r}}\,\mathcal{E}(\vec{k},\vec{k}_1)$$

from which it follows:

$$\mathcal{E}(\vec{k},\vec{k}_1) = 0\ , \quad \text{if} \quad \vec{k}+\vec{k}_1 \neq 0.$$

We thus see that in the sum (2.13) only the terms with $\vec{k}_1 = -\vec{k}$ ought to be retained.

We further remark that $\mathcal{J}_S^{\circ}$ commutes with $\mathcal{J}_\Sigma$, $\vec{r}_j$ and $\mathcal{J}_\Sigma$ commutes with $\vec{r}_0$.

Hence, expression (2.13) may be rewritten in the form:

$$\vec{Q}(T) = \frac{1}{V^2}\sum_{(k)}\vec{k}\,\nu^2(k)\,e^{i\vec{k}\vec{r}_0}\,e^{\mathcal{J}_S^{\circ}T}\,e^{-i\vec{k}\vec{r}_0}\times \tag{2.14}$$

$$\overline{\times\left(\sum_{(j)} e^{-i\vec{k}\vec{r}_j}\,e^{\mathcal{J}_\Sigma T}\sum_{(j)} e^{i\vec{k}\vec{r}_j}\,\vec{k}\left(\frac{1}{m}\,\frac{\partial}{\partial\vec{v}_0}+\frac{1}{\theta}\,\vec{v}_j\right)\mathfrak{D}(\Sigma)\right)_\Sigma}\ .$$

By considering the movements in the isolated system Σ corresponding to the Liouville operator Л_Σ we get

$$e^{\text{Л}_\Sigma T}\sum_{(j)} e^{i\vec k\vec r_j}(\vec k\vec v_j) = \sum_{(j)} e^{i\vec k\vec r_j(-T)}(\vec k\vec v_j(-T)) =$$

$$= -\sum_{(j)} e^{i\vec k\vec r_j(-T)}\frac{d}{dT}(\vec k\vec r_j(-T)) = i\frac{d}{dT}\sum_{(j)} e^{i\vec k\vec r_j(-T)} = i\frac{d}{dT}e^{\text{Л}_\Sigma T}\times$$
$$\times\sum_{(j)} e^{i\vec k\vec r_j}$$

from which it follows, in view of (2.14):

$$\vec Q(T) = \frac{1}{V^2}\sum_{(k)}\vec k\nu^2(k)\, e^{i\vec k\vec r_0}\, e^{\text{Л}^{\circ}_S T}\, e^{-i\vec k\vec r_0}\times$$
$$\times\left\{U_k(T)\frac{1}{m}\left(\vec k\cdot\frac{\partial}{\partial\vec v_0}\right) + \frac{i}{\theta}\frac{\partial U_k(T)}{\partial T}\right\}, \tag{2.15}$$

where:

$$U_k(T) = \overline{\left(\sum_{(j)} e^{-i\vec k\vec r_j}\, e^{\text{Л}_\Sigma T}\sum_{(j)} e^{i\vec k\vec r_j}\,\mathfrak{D}(\Sigma)\right)_\Sigma} =$$

$$= N\overline{\left(e^{-i\vec k\vec r_1}\, e^{\text{Л}_\Sigma T}\sum_{(j)} e^{i\vec k\vec r_j}\,\mathfrak{D}(\Sigma)\right)_\Sigma} = NR_k(T) \tag{2.16}$$

$$R_k(T) = \overline{\left(e^{-i\vec k\vec r_1}\, e^{\text{Л}_\Sigma T}\sum_{(j)} e^{i\vec k\vec r_j}\,\mathfrak{D}(\Sigma)\right)_\Sigma}.$$

Introduce the average particle density:

$$n = \frac{N}{V} \tag{2.17}$$

and rewrite expression (2.15) with the help of (2.16):

$$\vec Q(T) = n\frac{1}{V}\sum_{(k)}\vec k\nu^2(k)\, e^{i\vec k\vec r_0}\, e^{\text{Л}^{\circ}_S T}\, e^{-i\vec k\vec r_0}\times$$
$$\times\left\{\frac{1}{m}R_k(T)\left(\vec k\frac{\partial}{\partial\vec v_0}\right) + \frac{i}{\theta}\frac{\partial R_k(T)}{\partial T}\right\}. \tag{2.18}$$

In this notation our kinetic equation (2.5), (2.10) takes the form:

$$\frac{\partial f_t(\vec r_0,\vec v_0)}{\partial t} = -\vec v_0\frac{\partial}{\partial\vec r_0}f_t(\vec r_0,\vec v_0) +$$
$$+\frac{1}{m}\frac{\partial}{\partial\vec v_0}\int_0^t Q(t-\tau)f_\tau(\vec r_0,\vec v_0)\,d\tau. \tag{2.19}$$

Consider the Fourier representation:

$$f_t(\vec r_0,\vec v_0) = \frac{1}{V}\sum_{(\ell)} e^{-i\vec\ell\vec r_0} f_\ell(t,\vec v_0) \tag{2.20}$$

and remark:

$$e^{i\vec k\vec r_0}\, e^{\text{Л}^{\circ}_S T}\, e^{-i(\vec k+\vec\ell)\vec r_0} = e^{-i\vec\ell\vec r_0}\, e^{i(\vec k+\vec\ell)\vec v_0 T}.$$

Then it is easy to see that equation (2.19) leads to the individual equations for each component $f_\ell(t,\vec v_0)$:

$$\frac{\partial f_\ell(t,\vec v_0)}{\partial t} = i(\vec\ell\cdot\vec v_0)f_\ell(t,\vec v_0) +$$
$$+\frac{1}{m}\frac{\partial}{\partial\vec v_0}\int_0^t \vec Q_\ell(t-\tau)f_\ell(\tau,\vec v_0)\,d\tau, \tag{2.21}$$

where

$$\vec Q_\ell(T) = n\frac{1}{V}\sum_{(k)}\vec k\nu^2(k)\, e^{i(\vec k+\vec\ell)\vec v_0 T}\left\{R_k(T)\frac{1}{m}\vec k\frac{\partial}{\partial\vec v_0} + \frac{i}{\theta}\frac{\partial R_k(T)}{\partial T}\right\} \tag{2.22}$$

Or, performing the usual limiting process of the statistical mechanics:

$$\vec{Q}_\ell(T) = \frac{n}{(2\pi)^3}\int \vec{k}\nu^2(k)\, e^{i(\vec{k}+\vec{\ell})\vec{v}_0 T} \times \tag{2.23}$$

$$\times\{R_k(T)\frac{1}{m}\vec{k}\frac{\partial}{\partial \vec{v}_0} + \frac{i}{\theta}\frac{\partial R_k(T)}{\partial T}\}d\vec{k}.$$

Equation (2.21) is convenient to make use of the Laplace transform:

$$\int_0^\infty e^{-zt} f_\ell(t,\vec{v}_0)dt = f_{\ell,z}(\vec{v}_0) \tag{2.24}$$

$$z = \epsilon - i\omega,\ \mathrm{Re}\, z = \epsilon > 0.$$

By performing the Laplace transform in both hand sides of equation (2.21) we get:

$$z f_{\ell,z}(\vec{v}_0) = i(\vec{\ell}\cdot\vec{v}_0) f_{\ell,z}(\vec{v}_0) +$$

$$+\frac{1}{m}\frac{\partial}{\partial \vec{v}_0}\int_0^\infty \vec{Q}_\ell(T)e^{-zT}dT\, f_{\ell,z}(\vec{v}_0) + f_\ell(0,\vec{v}_0) \tag{2.25}$$

$$\int_0^\infty \vec{Q}_\ell(T)e^{-zT}dT = \frac{n}{(2\pi)^3}\int \vec{k}\,\nu^2(k)\times \tag{2.26}$$

$$\times\{\int_0^\infty R_k(T)e^{[i(\vec{k}+\vec{\ell})\vec{v}_0 - z]T}dT\}\frac{1}{m}(\vec{k}\frac{\partial}{\partial \vec{v}_0})d\vec{k} +$$

$$+\frac{n}{(2\pi)^3}\int \vec{k}\nu^2(k)\{\frac{i}{\theta}\int_0^\infty e^{[i(\vec{k}+\vec{\ell})\vec{v}_0 - z]T}\frac{\partial R_k(T)}{\partial T}dT\}d\vec{k}.$$

But

$$\frac{i}{\theta}\int_0^\infty e^{[i(\vec{k}+\vec{\ell})\vec{v}_0 - z]T}\frac{\partial R_k(T)}{\partial T}dT = -\frac{i}{\theta}R_k(0) +$$

$$+\frac{1}{\theta}[(\vec{k}+\vec{\ell})\vec{v}_0 + iz]\int_0^\infty R_k(T)e^{[i(\vec{k}+\vec{\ell})\vec{v}_0 - z]T}dT$$

and on the other hand, from (2.16):

$$R_k(0) = \frac{1}{N}\overline{(\sum_{(j)} e^{-i\vec{k}\vec{r}_j}\sum_{(j)} e^{i\vec{k}\vec{r}_j}\,\mathfrak{D}(\Sigma))}_\Sigma$$

from which it follows:

$$R_k(0) = R_{-k}(0).$$

As the function $\nu(k)$ also has such a symmetry property, it is easy to see that:

$$\int \vec{k}\nu^2(k)R_k(0)d\vec{k} = 0.$$

Therefore equation (2.25) may be written in the form:

$$z f_{\ell,z}(\vec{v}_0) = i(\vec{\ell}\cdot\vec{v}_0) f_{\ell,z}(\vec{v}_0) +$$

$$+\frac{n}{m(2\pi)^3}\int(\vec{k}\frac{\partial}{\partial \vec{v}_0})\nu^2(k)\{\int_0^\infty R_k(T)e^{i[(\vec{k}+\vec{\ell})\vec{v}_0 - z]T}dT\}\times \tag{2.27}$$

$$\times(\frac{1}{m}\vec{k}\frac{\partial}{\partial \vec{v}_0} + \frac{(\vec{k}+\vec{\ell})\vec{v}_0 + iz}{\theta})d\vec{k}\, f_{\ell,z}(\vec{v}_0) + f_\ell(0,\vec{v}_0).$$

Remark that the integral term at the right-hand side of (2.27) containing $\nu^2(k)$ is formally proportional to the square of the small parameter. Considering the case when z, $\vec{\ell}$ are small, we may neglect them in this integral and obtain a simplified approximate equation:

$$zf_{\ell,z}(\vec{v}_0) = i(\vec{\ell}\cdot\vec{v}_0)f_{\ell,z}(\vec{v}_0) +$$

$$+\frac{n}{m(2\pi)^3}\int \nu^2(k)(\vec{k}\frac{\partial}{\partial \vec{v}_0})\int_0^\infty R_k(T)e^{i\vec{k}\vec{v}_0T}dT\vec{k}\times \tag{2.28}$$

$$\times(\frac{1}{m}\frac{\partial}{\partial\vec{v}_0}+\frac{\vec{v}_0}{\theta})d\vec{k}\, f_{\ell,z}(\vec{v}_0)+f_\ell(0,\vec{v}_0).$$

It must be stressed, however, that equation (2.27) does not contain the terms of higher degree in the interaction and that there is a possibility * that these terms may become singular in the neighbourhood of $z=0$, $\vec{\ell}=0$.

Therefore, equation (2.28) also cannot be considered as giving the true asymptotic behaviour of $f_{\ell,z}(\vec{v}_0)$ for $\vec{\ell}\to 0$, $z\to 0$.

On the other hand, it is interesting to point out that equation (2.28) can be formally obtained from the equation for the reduced probability distribution:

$$\frac{\partial f_t(\vec{r}_0,\vec{v}_0)}{\partial t} = -\vec{v}_0\frac{\partial}{\partial\vec{r}_0}f_t(\vec{r}_0,\vec{v}_0)+$$

$$+\frac{n}{m(2\pi)^3}\int\nu^2(k)(\vec{k}\frac{\partial}{\partial\vec{v}_0})\int_0^\infty R_k(T)e^{i\vec{k}\vec{v}_0T}dT\times \tag{2.29}$$

$$\times\vec{k}(\frac{1}{m}\frac{\partial}{\partial\vec{v}_0}+\frac{\vec{v}_0}{\theta})d\vec{k}\, f_t(\vec{r}_0,\vec{v}_0)$$

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* In fact there are strong indications for the plausible character of such a possibility.

by using the Fourier expansion (2.20) and the Laplace transform acting upon t.

So, these two equations (2.28), (2.29) are completely equivalent: one of them corresponds to (z,ℓ) representation and the other to $(t,\vec{r}_0)$ representation.

It is clear that (2.29) is the typical Fokker-Planck equation for the Markoffian stochastic process.

Evidently it possesses also spatially homogeneous solutions $f_t(\vec{v}_0)$ satisfying the kinetic equation:

$$\frac{\partial f_t(\vec{v}_0)}{\partial t} = \frac{n}{m(2\pi)^3}\int\nu^2(k)(\vec{k}\frac{\partial}{\partial\vec{v}_0})\int_0^\infty R_k(T)e^{i\vec{k}\vec{v}_0T}dT\times \tag{2.30}$$

$$\times\vec{k}(\frac{1}{m}\frac{\partial}{\partial\vec{v}_0}+\frac{\vec{v}_0}{\theta})d\vec{k}\, f_t(\vec{v}_0)$$

which shows that in the ordinary situations $f_t(\vec{v}_0)$ approaches the Maxwell velocity distribution with the increase of time.

We have already pointed out that for $\ell=0$ the correction terms to (2.27) or (2.28) may become singular for $z\to 0$.

Correspondingly, in t-representation equation (2.30) may not give the correct behaviour of asymptotical smallness of the difference:

$$f_t(\vec{v}_0)-f_{Max}(\vec{v}_0)$$

for sufficiently large values of t.

This question will be further discussed in §4.

Now we wish to establish some useful properties of the function $R_k(T)$.

Consider the equilibrium average for the system Σ

$$\langle \rho(t,\vec r)\rho(0,\vec r)\rangle_\Sigma = \overline{(\rho(t,\vec r)\rho(0,\vec r\,')\,\mathfrak{D}(\Sigma))}, \qquad (2.31)$$

where $\rho(t,\vec r)$ is the microscopic space density of the Σ particles

$$\rho(t,\vec r) = \sum_{(1\le j\le N)} \delta(\vec r-\vec r_j(t)).$$

Because the equilibrium average is invariant with respect to the time translation, expression (2.31) is equal to

$$\langle \rho(0,\vec r)\rho(-t,\vec r)\rangle_\Sigma = \langle \rho(0,\vec r)e^{\mathcal{J}_\Sigma t}\rho(0,\vec r\,')\rangle_\Sigma . \qquad (2.32)$$

Therefore, by applying the Fourier representation we get:

$$\langle \rho(t,\vec r)\rho(0,\vec r\,')\rangle_\Sigma =$$

$$= \frac{1}{V^2}\sum_{(k)} e^{i\vec k(\vec r-\vec r\,')}\,\overline{\left(\sum_{(j)} e^{-i\vec k\vec r_j}\, e^{\mathcal{J}_\Sigma t}\sum_{(j)} e^{i\vec k\vec r_j}\,\mathfrak{D}(\Sigma)\right)}_\Sigma =$$

$$= n^2 + \frac{1}{V^2}\sum_{(k\neq 0)} \overline{\left(\sum_{(j)} e^{-i\vec k\vec r_j}\, e^{\mathcal{J}_\Sigma t}\sum_{(j)} e^{i\vec k\vec r_j}\,\mathfrak{D}(\Sigma)\right)}_\Sigma$$

what yields, in view of (2.16):

$$\langle \rho(t,\vec r)\rho(0,\vec r\,')\rangle_\Sigma = n^2 + n\,\frac{1}{V}\sum_{(k\neq 0)} R_k(t)\,e^{i\vec k(\vec r-\vec r\,')} \qquad (2.33)$$

or in the limit of the statistical mechanics $V \to \infty$, $n = \text{const}$:

$$\langle \rho(t,\vec r)\rho(0,\vec r\,')\rangle_\Sigma = n^2 + \frac{n}{(2\pi)^3}\int R_k(t)\,e^{i\vec k(\vec r-\vec r\,')}\,d\vec k \qquad (2.34)$$

As the microscopic particle density is a real function, the left-hand side of the relation (2.34) will also be real and hence

$$R_k^*(t) = R_{-k}(t). \qquad (2.35)$$

Return now to the integral term in equation (2.29) and rewrite it in the form:

$$\frac{n}{(2\pi)^3 m}\int \nu^2(k)\left(\vec k\,\frac{\partial}{\partial \vec v_0}\right)\frac{1}{2}\left\{\int_0^\infty R_k(T)\,e^{i\vec k\vec v_0 T}dT + \int_0^\infty R_{-k}(T)\times\right.$$

$$\left.\times e^{-i\vec k\vec v_0 T}dT\right\}\vec k\left(\frac{1}{m}\,\frac{\partial}{\partial \vec v_0} + \frac{\vec v_0}{\theta}\right)d\vec k\, f_t(\vec r_0,\vec v_0).$$

But the relation (2.35) yields:

$$\frac{1}{2}\left\{\int_0^\infty R_k(T)\,e^{i\vec k\vec v_0 T}dT + \int_0^\infty R_{-k}(T)\,e^{-i\vec k\vec v_0 T}dT\right\} =$$

$$= \mathrm{Re}\int_0^T R_k(T)\,e^{i\vec k\vec v_0 T}dT.$$

Therefore equation (2.29) for the reduced probability distribution may be written as follows:

$$\frac{\partial f_t(\vec r_0,\vec v_0)}{\partial t} = -\vec v_0\frac{\partial}{\partial \vec r_0} f_t(\vec r_0,\vec v_0) +$$

$$(2.36)$$

$$+ \frac{n}{m(2\pi)^3}\int \nu^2(k)\left(\vec k\,\frac{\partial}{\partial \vec v_0}\right)F(\vec k\vec v_0)\,\vec k\left(\frac{1}{m}\,\frac{\partial}{\partial \vec v_0} + \frac{\vec v_0}{\theta}\right)d\vec k\, f_t(\vec r_0,\vec v_0),$$

where

$$F(\omega)=\mathrm{Re}\int_0^\infty R_k(t)e^{i\omega t}dt. \qquad (2.37)$$

We see that in order to make this equation completely definite, we need to determine the function (2.37).

In §3 we shall outline a method for actually calculating this function in some frequently considered situations.

Here we shall only note that in view of the equality between (2.31) and (2.32) it follows that:

$$R_k(-t)=R_{-k}(t)$$

what yields:

$$F(\omega)=\frac{1}{2}\int_{-\infty}^{\infty}R_k(t)e^{i\omega t}dt. \qquad (2.38)$$

Thus, because $F(\omega)$ is the Fourier transform of the equilibrium correlation average

$$R_k(t)=\frac{1}{n}\lim_{V\to\infty}\frac{1}{V}<\sum_{(j)}e^{-i\vec{k}\vec{r}_j}\,e^{\mathcal{H}\Sigma^t}\sum_{(j)}e^{i\vec{k}\vec{r}_j}>_\Sigma \qquad (2.39)$$

between two mutually complex conjugate dynamical variables, we have:

$$F(\omega)\geq 0. \qquad (2.40)$$

We shall proceed now to examine the next example when all conditions are the same as in the previous example with the only exception that here the hard sphere interaction will be considered instead of the regular interaction (1.8).

So we shall take

$$\mathcal{H}_{int}=\mathcal{H}_{int}^{coll}, \qquad (2.41)$$

where the expression of $\mathcal{H}_{int}^{coll}$ is given by formulae (1.10), (1.11).

In §1 we have already pointed out that the basic Liouville equation (1.18) gives the exact description of the dynamics when unphysical overlapping configurations are excluded. Of course, no overlapping configuration will appear at the later moments t if they were absent at the initial moment of time: $t=0$.

So we must impose the following condition: $\mathfrak{D}_0(S,\Sigma)=0$ (for $t=0$), if at least for one value of $j=1,\dots,N$

$$|\vec{r}_0-\vec{r}_j|<a. \qquad (2.42)$$

If this condition holds, it will be automatically satisfied by $\mathfrak{D}_t$ for $t>0$*.

We see now the difficulty which arises when we try to apply equation (1.26) for the present case (2.41). In fact, this equation was obtained with the help of the initial condition (1.2):

$$\mathfrak{D}_0(S,\Sigma)=f_0(S)\mathfrak{D}(\Sigma)$$

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* It is to be stressed that expression (1.11) for $\mathrm{T}(0,1)$ may be used only if we study the evolution of $\mathfrak{D}_t$ for $t>0$. If we wish to investigate this evolution in the inverse flow of time ($t<0$), another form for $\mathrm{T}(0,1)$ must be used. The direction of time in these operators is specified by our convention concerning the significance of $v, \overset{*}{v}$: either they are precollision and postcollision velocities or this order is to be reversed. For the clarification of this point see paper /4/.

and such a form for $\mathfrak{D}_0$ does not possess the property (2.42).

The probability of overlapping configurations is, therefore, not equal to zero.

We shall, nevertheless, make use of equation (1.26) taking into account that the probability of overlapping S with a Σ particle is proportional to a small quantity na^3 and assuming that the role of such overlapping will be negligible for the calculation of the lowest density contribution to the terms of this equation, especially, in the case when $\vec{\ell}$ in $f_{\vec{\ell}}(t, v_0)$ is sufficiently small and t is sufficiently large.

Further, in §4 another form for $\mathfrak{D}_0(S,\Sigma)$ will be introduced which automatically excludes the overlapping configurations, and in this way we shall present á posteriori justification of our present procedure.

Proceeding to concretize the considered approximate equation, let us first substitute (1.7), (1.10), (1.11) into the relation (1.22):

$$\mathcal{J}_S = \mathcal{J}_S^{\circ} + \sum_{(1\leq j\leq N)} \overline{(\bar{T}(0,j)\,\mathfrak{D}(\Sigma))_{\Sigma}} = \mathcal{J}_S^{\circ} + N\,\overline{(\bar{T}(0,1)\,\mathfrak{D}(\Sigma))_{\Sigma}}\,. \qquad (2.43)$$

Note that the equilibrium distribution $\mathfrak{D}(\Sigma)$ for the classical dynamical system Σ has the form:

$$\mathfrak{D}(\Sigma) = W(\vec{r}_1,\dots,\vec{r}_N) \prod_{(1\leq j\leq N)} \Phi_{\Sigma}(v_j), \qquad (2.44)$$

where

$$\Phi_{\Sigma}(v) = \left(\frac{M}{2\pi\theta}\right)^{3/2} e^{-\frac{Mv^2}{2\theta}}, \quad \int \Phi_{\Sigma}(v)\,d\vec{v} = 1 \qquad (2.45)$$

is the normalized Maxwell velocity distribution.

The normalization property of $\mathfrak{D}(\Sigma)$:

$$\int \mathfrak{D}(\Sigma)\,d\Omega_{\Sigma} = 1$$

yields:

$$\int W(\vec{r}_1,\dots,\vec{r}_N)\,d\vec{r}_1\dots d\vec{r}_N = 1\,. \qquad (2.46)$$

Consider the equilibrium average of the microscopic Σ-particle density at a point $\vec{r}$:

$$n = \langle\rho(\vec{r})\rangle_{\Sigma} = \sum_{(1\leq j\leq N)} \int \delta(\vec{r}-\vec{r}_j)\,\mathfrak{D}(\Sigma)\,d\Omega_{\Sigma} =$$

$$= N\int\delta(\vec{r}-\vec{r}_1)\,\mathfrak{D}(\Sigma)\,d\Omega_{\Sigma} = N\int\delta(\vec{r}-\vec{r}_1)\,W\,d\vec{r}_1\dots d\vec{r}_N\,.$$

By taking into account the spatial homogeneity, we see that this averaged density does not depend upon $\vec{r}$ and therefore:

$$N\int\delta(\vec{r}-\vec{r}_1)\,W\,d\vec{r}_1\dots d\vec{r}_N = n\,.$$

Due to this relation, formulae (1.11), (2.44), (2.45) yield:

$$N\,\overline{(\bar{T}(0,1)\,\mathfrak{D}(\Sigma))_{\Sigma}} =$$

$$= na^2 \int (\vec{v}_{0,1}\cdot\vec{\sigma})\,\theta(\vec{v}_{0,1}\cdot\vec{\sigma})\{B_{v_0,v_1}(\vec{\sigma})-1\}\,\Phi_{\Sigma}(v_1)\,d\vec{\sigma}\,d\vec{v}_1 \qquad (2.47)$$

$$\theta(r) = \begin{cases} 1, & \text{for } r > 0 \\ 0, & \text{for } r \leq 0. \end{cases}$$

We see that it is just the Lorentz-Boltzmann collision operator acting only on functions of $\vec{v}_0$:

$$N\overline{(\bar{T}(0,1)\,\mathfrak{T}(\Sigma))}_\Sigma\, f(S) =$$

$$= na^2 \int (\vec{v}_{0,1}\cdot\vec{\sigma})\,\theta(\vec{v}_{0,1}\cdot\vec{\sigma})\{B_{v_0 v_1}(\vec{\sigma})-1\}\Phi_\Sigma(v_1) f(\vec{r}_0,\vec{v}_0)\,d\vec{\sigma}\,d\vec{v}_1 \,.$$

It will be convenient to introduce the notation:

$$f(S) = \chi(S)\,\Phi_0(v_0) \tag{2.48}$$

$\Phi_0(v_0)$ being the normalized maxwellian for S:

$$\Phi_0(v_0) = \left(\frac{m}{2\pi\theta}\right)^{3/2} e^{-\frac{mv^2}{2\theta}} \,.$$

Then by observing that:

$$B_{v_0 v_1}(\vec{\sigma})\,\Phi_0(v_0)\Phi_\Sigma(v_1) = \left(\frac{m}{2\pi\theta}\right)^{3/2}\left(\frac{M}{2\pi\theta}\right)^{3/2} \times$$

$$\times \exp\left\{-\frac{m\overset{*}{v}{}^2}{2\theta} - \frac{M\overset{*}{v}{}_1^2}{2\theta}\right\} = \Phi_0(v_0)\Phi_\Sigma(v_1) \tag{2.49}$$

from (2.43) we get:

$$\mathcal{J}_S\, f(S) = \mathcal{J}_S\, \chi(S)\Phi_0(v_0) =$$

$$= \Phi_0(v_0)\left\{-\vec{v}_0 \frac{\partial\chi(\vec{r}_0,\vec{v}_0)}{\partial\vec{r}_0} + na^2 L_S\chi\right\} \tag{2.50}$$

$$L_S\chi = \int (\vec{v}_{0,1}\cdot\vec{\sigma})\,\theta(\vec{v}_{0,1}\cdot\vec{\sigma})\Phi_\Sigma(v_1)\{B_{v_0,v_1}(\vec{\sigma})-1\}\times$$

$$\times\chi(\vec{r}_0,\vec{v}_0)\,d\vec{\sigma}\,d\vec{v}_1 \,.$$

We now shall return to equation (1.26) and begin by putting in the form:

$$\frac{\partial f_t(\vec{r}_0,\vec{v}_0)}{\partial t} = \Phi_0(v_0)\left\{-\vec{v}_0\cdot\frac{\partial\chi_t(\vec{r}_0,\vec{v}_0)}{\partial\vec{r}} + na^2 L_S\chi\right\} + \tag{2.51}$$

$$+ \int_0^t K(t-\tau)\,\chi_\tau(\vec{r}_0,\vec{v}_0)\Phi_0(v_0)\,d\tau \,,$$

where

$$K(t) = \overline{\left(\sum_{(j)} \bar{T}(0,j)\, e^{(\mathcal{J}_S+\mathcal{J}_\Sigma)t} \sum_{(j)} (\bar{T}(0,j) - \overline{(\bar{T}(0,j)\,\mathfrak{D}(\Sigma))}_\Sigma)\,\mathfrak{D}(\Sigma)\right)}_\Sigma = \tag{2.52}$$

$$= N\overline{(\bar{T}(0,1)\, e^{\mathcal{J}_S t}\, e^{\mathcal{J}_\Sigma t} \sum_{(j)} (\bar{T}(0,j) - \overline{(\bar{T}(0,j)\,\mathfrak{D}(\Sigma))}_\Sigma)\,\mathfrak{D}(\Sigma))}_\Sigma \,.$$

Note here that $\mathcal{J}_S$ commutes with $\mathcal{J}_\Sigma$ and in general $\mathcal{J}_S$ commutes with the variables Ω_Σ while $\mathcal{J}_\Sigma$ is commuting with the variables Ω_S.

In order to simplify this expression (2.52) we shall use the Fourier representation:

$$\delta(\vec{r}-\vec{r}_j) = \frac{1}{V}\sum_{(k)} e^{i\vec{k}(\vec{r}-\vec{r}_j)}$$

and obtain:

$$\bar{T}(0,j) = \frac{1}{V}\sum_{(k)} e^{i\vec{k}(\vec{r}_0-\vec{r}_j)}\,\bar{T}_k(v_0,v_j) \tag{2.53}$$

with

$$\bar{T}_k(v_0, v_j) = a^2 \int (\vec{v}_{0,j} \cdot \vec{\sigma})\,\theta(\vec{v}_{0,j} \cdot \vec{\sigma})\{ e^{-ia\vec{k}\vec{\sigma}} B_{v_0 v_j}(\vec{\sigma}) - e^{ia\vec{k}\vec{\sigma}} \}\, d\vec{\sigma} \qquad (2.54)$$

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leqq 0 \end{cases}$$

On the other hand, from the identity (2.49) we find:

$$T_k(v_0, v_j)\,\chi(\vec{r}_0, \vec{v}_0)\,\Phi_0(v_0)\,\mathfrak{D}(\Sigma) =$$

$$= \{ T_k(v_0, v_j)\ \chi(\vec{r}_0, \vec{v}_0) \}\,\Phi_0(v_0)\,\mathfrak{D}(\Sigma).$$

We therefore may write:

$$K(t)\chi\Phi_0 = n \int \bar{T}(0,1)\, e^{\mathcal{H}_S t}\, Q(0,1)\, d\vec{r}_1 d\vec{v}_1 \,, \qquad (2.55)$$

where

$$Q(0,1) = \sum_{(k \neq 0)} \int e^{\mathcal{H}_\Sigma t} \sum_{(j)} e^{i\vec{k}(\vec{r}_0 - \vec{r}_j)} \{ \bar{T}_k(v_0, v_j)\chi \}\,\Phi_0(v_0) \times$$

$$\times \mathfrak{D}(\Sigma)\, d\vec{r}_2 d\vec{v}_2 \ldots d\vec{r}_N d\vec{v}_N + \int e^{\mathcal{H}_z t} \sum_{(j)} \{ \bar{T}_0(v_0, v_j)\chi - \qquad (2.56)$$

$$- \int \{ \bar{T}_0(v_0, v_j)\chi \}\,\Phi_\Sigma(v_j)\, d\vec{v}_j \}\,\Phi(v_0)\,\mathfrak{D}(\Sigma)\, d\vec{r}_2 d\vec{v}_2 \ldots d\vec{r}_N\, d\vec{v}_N.$$

Note that the first term in the right-hand side of (2.56) can be represented in the following way:

$$Q_1(\vec{r}_0, \vec{v}_0; \vec{r}_1, \vec{v}_1) = \sum_{(k \neq 0)} Q_1(k; \vec{r}_0, \vec{v}_0; \vec{r}_1, \vec{v}_1)\, e^{i\vec{k}\vec{r}_0}$$

$$Q_1(k; \vec{r}_0, \vec{v}_0; \vec{r}_1, \vec{v}_1) =$$

$$= \int e^{\mathcal{H}_\Sigma t} \sum_{(j)} e^{-i\vec{k}\vec{r}_j} \{ \bar{T}_k(v_0, v_j)\chi \}\,\Phi_0(v_0)\, W(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N) \times \qquad (2.57)$$

$$\times \prod_{(1 \leqq j \leqq N)} \Phi_\Sigma(v_j)\, d\vec{r}_2 d\vec{v}_2 \ldots d\vec{r}_N d\vec{v}_N \,.$$

Let $\vec{r}$ be an arbitrary space vector. Then by performing the change of integration variables:

$$\vec{r}_2 \to \vec{r}_2 + \vec{r} \ , \ldots \ \vec{r}_N \to \vec{r}_N + \vec{r}$$

we get:

$$Q_1(k; \vec{r}_0, \vec{v}_0; \vec{r}_1 + \vec{r}, \vec{v}_1) =$$

$$= e^{-i\vec{k}\vec{r}} \int e^{\mathcal{H}_\Sigma t} \sum_{(1 \leqq j \leqq N)} e^{-i\vec{k}\vec{r}_j} \{ \bar{T}_k(v_0, v_j)\chi \} \times$$

$$\times \Phi(v_0)\, W(\vec{r}_1 + \vec{r}, \vec{r}_2 + \vec{r}, \ldots \vec{r}_N + \vec{r}) \prod_{(1 \leqq j \leqq N)} \Phi_\Sigma(v_j)\, d\vec{r}_2 d\vec{v}_2 \ldots d\vec{r}_N d\vec{v}_N.$$

But because of the space homogeneity the function

$$W(\vec{r}_1 + \vec{r}, \ldots, \vec{r}_N + \vec{r})$$

is equal to

$$W(\vec{r}_1, \ldots \vec{r}_N).$$

We therefore obtain:

$$e^{i\vec{k}\vec{r}}\, Q_1(k; \vec{r}_0, \vec{v}_0, \vec{r}_1 + \vec{r}, \vec{v}_1) = Q_1(k; \vec{r}_0, \vec{v}_0; \vec{r}_1, \vec{v}_1).$$

For $\vec{r} = -\vec{r}_1$ this relation yields:

$$Q_1(k; \vec{r}_0, \vec{v}_0; \vec{r}_1, \vec{v}_1) = e^{-i\vec{k}\vec{r}_1}\, Q_1(k; \vec{r}_0, \vec{v}_0; \vec{v}_1) \qquad (2.58)$$

with

$$Q_1(k;\vec r_0,\vec v_0;\vec v_1)=Q_1(k;\vec r_0,\vec v_0;0,\vec v_1).$$

Considering the second term in (2.56) we find in the same way that it does not depend upon $\vec r_1$:

$$\int e^{\mathcal{H}_\Sigma t}\sum_{(1\leq j\leq N)}\tilde\chi(\vec r_0,\vec v_0;\vec v_j)\Phi_0(v_0)\mathfrak{D}(\Sigma)\,d\vec r_2 d\vec v_2\ldots d\vec r_N d\vec v_N = \qquad (2.59)$$

$$= Q_2(\vec r_0,\vec v_0;v_1),$$

where, for the abbreviation:

$$\tilde\chi(\vec r_0,\vec v_0;\vec v_j)=\bar T_0(v_0,v_j)\chi(\vec r_0,\vec v_0)- \qquad (2.60)$$

$$-\int\{\bar T_0(v_0,v_j)\chi(\vec r_0,\vec v_0)\}\Phi_\Sigma(v_j)d\vec v_j\ .$$

It is to be noted that the function $\tilde\chi$ satisfies the identity:

$$\int\tilde\chi(\vec r_0,\vec v_0,\vec v)\Phi_\Sigma(v)\,d\vec v=0. \qquad (2.61)$$

We now can sum up our results (2.55), (2.58), (2.59) and obtain:

$$K(t)\chi\Phi_0= n\sum_{(k\neq 0)}\int\bar T(0,1)e^{-i\vec k\vec r_1}\,e^{\mathcal{H}_S t}\,e^{i\vec k\vec r_0}\ \times$$

$$\times\,Q_1(k;\vec r_0,\vec v_0;\vec v_1)\,d\vec r_1\,d\vec v_1\ +$$

$$+\,n\int\bar T(0,1)e^{\mathcal{H}_S t}Q_2(\vec r_0,\vec v_0;\vec v_1)\,d\vec r_1 d\vec v_1\,.$$

On the other hand

$$\int\bar T(0,1)e^{-i\vec k\vec r_1}d\vec r_1=T_{-k}(v_0,v_1)e^{-i\vec k\vec r_0}$$

and therefore:

$$K(t)\chi\Phi_0= n\sum_{(k\neq 0)}\int T_{-k}(v_0,v_1)e^{-i\vec k\vec r_0}\,e^{\mathcal{H}_S t}\,e^{i\vec k\vec r_0}\times$$

$$\times\,Q_1(k;\vec r_0,\vec v_0;\vec v_1)d\vec v_1+ \qquad (2.62)$$

$$+\,n\int T_0(v_0,v_1)e^{\mathcal{H}_S t}Q_2(\vec r_0,\vec v_0;\vec v_1)\,d\vec v_1\,.$$

We now proceed to bring the expressions of Q_1, Q_2 to a more explicit form.

Consider the integral:

$$\int e^{\mathcal{H}_\Sigma t}\sum_{(1\leq j\leq N)}e^{-i\vec k\vec r_j}\delta(\vec v_j-\vec v)\mathfrak{D}(\Sigma)\,d\vec r_2 d\vec v_2\ldots d\vec r_N d\vec v_N.$$

By using our previous reasoning, we find that it depends upon $\vec r_1$ as $\exp\{-i\vec k\vec r_1\}$ and thus we may define a function $U_k(t,\vec v_1,\vec v)$ in the following way:

$$\int e^{\mathcal{H}_\Sigma t}\sum_{(1\leq j\leq N)}e^{-i\vec k\vec r_j}\,\delta(\vec v_j-\vec v)\mathfrak{D}(\Sigma)\,d\vec r_2 d\vec v_2\ldots d\vec r_N d\vec v_N= \qquad (2.63)$$

$$=e^{-i\vec k\vec r_1}\frac{1}{V}\Phi_\Sigma(v_1)U_k(t;\vec v_1,\vec v).$$

Of course, U_k depends on V.

This relation leads to the equality:

$$\int e^{\mathcal{H}_\Sigma t}\sum_{(1\leq j\leq N)}e^{-i\vec k\vec r_j}\phi(\vec v_j)\mathfrak{D}(\Sigma)\,d\vec r_2 d\vec v_2\ldots d\vec r_N d\vec v_N= \qquad (2.64)$$

$$=e^{-i\vec k\vec r_1}\,\frac{1}{V}\Phi_\Sigma(v_1)\int U_k(t;\vec v_1,\vec v_1')\phi(\vec v_1')d\vec v_1'\,.$$

It will be convenient to consider the expression

$$U_k(t,\vec{v}_1,\vec{v}_1')$$

as the matrix representation of an operator

$$U_k(t;1)$$

acting only on functions of $\vec{v}_1$ according to the formula

$$U_k(t;1)f(\vec{v}_1)=\int U_k(t;\vec{v}_1,\vec{v}_1')f(\vec{v}_1')d\vec{v}_1' . \qquad (2.65)$$

We thus can write:

$$\int e^{\Pi_\Sigma t}\sum_{(1\leq j\leq N)} e^{-i\vec{k}\vec{r}_j}\phi(\vec{v}_j)\mathfrak{D}(\Sigma)d\vec{r}_2d\vec{v}_2\dots d\vec{r}_Nd\vec{v}_N= \qquad (2.66)$$
$$=e^{-i\vec{k}\vec{r}_1}\frac{1}{V}\Phi_\Sigma(v_1)U_k(t;1)\phi(v_1).$$

In view of (2.58), (2.59) we now obtain:

$$Q_1(k;\vec{r}_0,\vec{v}_0;\vec{v}_1)=\Phi_0(v_0)\Phi_\Sigma(v_1)\frac{1}{V}U_k(t;1)\bar{T}_k(v_0,v_1)\chi(\vec{r}_0\vec{v}_0)$$

$$Q_2(\vec{r}_0,\vec{v}_0;\vec{v}_1)=\Phi_0(v_0)\Phi_\Sigma(v_1)\frac{1}{V}U_0(t;1)\tilde{\chi}(\vec{r}_0,\vec{v}_0;\vec{v}_1).$$

These equalities are to be substituted in the definition (2.62). We shall first transform the expression of the type:

$$e^{\Pi_S t}\Phi_0(v_0)h(\vec{r}_0,\vec{v}_0)$$

which figures in (2.62).

Owing to formula (2.50) we get

$$e^{\Pi_S t}\Phi_0(v_0)h(\vec{r}_0,\vec{v}_0)=$$
$$=\Phi_0(v_0)e^{(-\vec{v}_0\frac{\partial}{\partial\vec{r}_0}+na^2L_S)t}h(\vec{r}_0,\vec{v}_0).$$

We remark also that $\Phi_\Sigma(v_1)$ commutes with $e^{\Pi_S t}$ and

$$\bar{T}_{-k}(v_0,v_1)\Phi_0(v_1)\Phi_\Sigma(v_1)=\Phi_0(v_1)\Phi_\Sigma(v_1)\bar{T}_{-k}(v_0,v_1).$$

In such a way we finally obtain from (2.62):

$$K(t)\chi(S)\Phi_0(v_0)=$$
$$=\Phi_0(v_0)n\frac{1}{V}\sum_{(k\neq 0)}\int d\vec{v}_1\Phi_\Sigma(v_1)\bar{T}_{-k}(v_0,v_1)e^{-i\vec{k}\vec{r}_0}\times$$
$$\times e^{(-\vec{v}_0\frac{\partial}{\partial\vec{r}}+na^2L_S)t}e^{i\vec{k}\vec{r}_0}U_k(t,1)\bar{T}_k(v_0,v_1)\chi(\vec{r}_0,\vec{v}_0)+$$
$$+\Phi_0(v_0)n\frac{1}{V}\int d\vec{v}_1\Phi_\Sigma(v_1)\bar{T}_0(v_0,v_1)e^{(-\vec{v}_0\frac{\partial}{\partial\vec{r}}+na^2L_S)t}\times$$
$$\times U_0(t;1)\tilde{\chi}(\vec{r}_0,\vec{v}_0;\vec{v}_1)$$

and therefore equation (2.51) yields:

$$\frac{\partial\chi_t(\vec{r}_0,\vec{v}_0)}{\partial t}=\{-\vec{v}_0\frac{\partial}{\partial\vec{r}_0}+na^2L_S\}\chi_t(\vec{r}_0,\vec{v}_0)+$$

$$+n\frac{1}{V}\sum_{k\neq 0}\int_0^t d\tau\int d\vec{v}_1\Phi_\Sigma(v_1)\bar{T}_{-k}(v_0,v_1)e^{-i\vec{k}\vec{r}_0}\times$$
$$\times e^{(-\vec{v}_0\frac{\partial}{\partial\vec{r}_0}+na^2L_S)(t-\tau)}e^{i\vec{k}\vec{r}_0}U_k(t-\tau;1)\bar{T}_k(v_0,v_1)\chi_\tau(\vec{r}_0,\vec{v}_0)+$$

$$+n\frac{1}{V}\int_0^t d\tau \int d\vec{v}_1 \Phi_\Sigma(v_1)\bar{T}_0(v_0,v_1) e^{-(\vec{v}_0 \frac{\partial}{\partial \vec{r}_0} + n a^2 L_S)(t-\tau)} \times$$

$$\times U_0(t-\tau;1)\tilde{\chi}_\tau(\vec{r}_0, \vec{v}_0 \vec{v}_1) \quad . \tag{2.67}$$

$$f_t(\vec{r}_0, \vec{v}_0) = \Phi_0(v_0)\chi_t(\vec{r}_0, \vec{v}_0).$$

Note that

$$e^{-i\vec{k}\vec{r}_0} e^{(-\vec{v}_0 \frac{\partial}{\partial \vec{r}_0} + n a^2 L_S)(t-\tau)} e^{i(\vec{k}+\vec{\ell})\vec{r}_0} =$$

$$= e^{i\vec{\ell}\vec{r}_0} e^{(-i\vec{v}_0(\vec{k}+\vec{\ell})+na^2L_S)(t-\tau)} .$$

Then it is easy to see that by using the Fourier transform

$$\chi_t(\vec{r}_0, \vec{v}_0) = \frac{1}{V}\sum_{(\ell)} e^{i\vec{\ell}\vec{r}_0} \chi_\ell(t, \vec{v}_0) \tag{2.68}$$

we shall obtain from (2.67) the individual equations for each χ_ℓ:

$$\frac{\partial \chi_\ell(t,\vec{v}_0)}{\partial t} = \{-i\vec{\ell}\cdot\vec{v}_0 + na^2 L_S\}\chi_\ell(t,\vec{v}_0) +$$

$$+n\frac{1}{V}\sum_{(k\neq 0)}\int_0^t d\tau \int d\vec{v}_1 \Phi_\Sigma(v_1)\bar{T}_{-k}(v_0,v_1) e^{(-i\vec{v}_0(\vec{k}+\vec{\ell})+na^2 L_S)(t-\tau)} \times$$

$$\times U_k(t-\tau;1)\bar{T}_k(v_0,v_1)\chi_\ell(\tau,\vec{v}_0) +$$

$$+n\frac{1}{V}\int_0^t d\tau \int d\vec{v}_1 \Phi_\Sigma(v_1)\bar{T}_0(v_0,v_1) e^{(-i\vec{v}_0\vec{\ell}+na^2L_S)(t-\tau)} \times \tag{2.69}$$

$$\times U_0(t-\tau;1)\tilde{\chi}_\ell(\tau,\vec{v}_0,\vec{v}_1).$$

In particular, for $\ell=0$:

$$\frac{\partial \chi(t,\vec{v}_0)}{\partial t} = na^2 L_S \chi(t,\vec{v}_0) +$$

$$+n\frac{1}{V}\sum_{(k\neq 0)}\int_0^t d\tau \int d\vec{v}_1 \Phi_\Sigma(v_1)\bar{T}_{-k}(v_0,v_1) \times \tag{2.70}$$

$$\times e^{(-i\vec{v}_0\vec{k}+na^2L_S)(t-\tau)} U_k(t-\tau;1)\bar{T}_k(v_0,v_1)\chi(\tau,\vec{v}_0) +$$

$$+n\frac{1}{V}\int_0^t d\tau \int d\vec{v}_1 \Phi_\Sigma(v_1)\bar{T}_0(v_0,v_1) e^{na^2L_S(t-\tau)} U_0(t-\tau;1)\tilde{\chi}(\tau,\vec{v}_0,\vec{v}_1).$$

In these equations the kernels under the sign of the integral

$$\int_0^t \dots d\tau$$

are function of $t-\tau$ and therefore the method of the Laplace transform can be used.

We also see that for dealing with these equations the explicit expression for the operator

$$U_k(t;1)$$

which is defined only by the dynamics of the isolated system Σ, must be found.

This problem will be considered in the next section.

We now wish only to point out that by means of the U_k operator the functions $R_k(t)$ which appeared in the previous example may also be calculated.

In fact from (2.16) we get

$$R_k(T) = V e^{i\vec{k}\vec{r}_1} \int e^{\mathcal{L}_\Sigma T} \sum_{(j)} e^{-i\vec{k}\vec{r}_j} \mathfrak{D}(\Sigma) d\vec{v}_1 d\vec{r}_2 d\vec{v}_2 \ldots d\vec{r}_N d\vec{v}_N \quad (2.71)$$

and employing the definition (2.63), we find:

$$R_k(T) = \int \Phi_\Sigma(v_1) U_k(T; \vec{v}_1, \vec{v}_1') d\vec{v}_1 d\vec{v}_1' . \quad (2.72)$$

3.

In the present section we shall draw our attention to the study of the equilibrium correlation averages.

Let Σ be a dynamical system of the classical mechanics whose Gibbs canonical distribution will be denoted, as previously, by $\mathfrak{D}(\Sigma)$.

Consider a dynamical variable, a function of the phase point:

$$\mathfrak{U} = \mathfrak{U}(\Omega_\Sigma)$$

and denote its expression at the moment of time t by:

$$\mathfrak{U}(t) = \mathfrak{U}(\Omega_\Sigma(t)),$$

where $\Omega_\Sigma(t)$ is the solution of dynamical equations, which starts from Ω_Σ at the initial moment $t=0$:

$$\Omega_\Sigma(0) = \Omega_\Sigma$$

Note that for the general nonequilibrium

distribution $\mathfrak{D}_t(\Sigma)$ satisfying the Liouville equation:

$$\frac{\partial \mathfrak{D}_t}{\partial t} = \mathcal{L}_\Sigma \mathfrak{D}_t$$

$$\mathfrak{D}_t = \mathfrak{D}_0 \quad \text{for} \quad t=0$$

we have the well-known identity:

$$\langle \mathfrak{U} \rangle_t = \int \mathfrak{U}(t) \mathfrak{D}_0(\Sigma) d\Omega_\Sigma = \int \mathfrak{U}(\Omega_\Sigma) \mathfrak{D}_t(\Sigma) d\Omega_\Sigma . \quad (3.1)$$

We now proceed to investigate the correlation equilibrium averages of two dynamical variables:

$$\langle \mathfrak{U}(t) \mathfrak{B}(r) \rangle = \left(\overline{\mathfrak{U}(t) \mathfrak{B}(r) \mathfrak{D}(\Sigma)} \right)_\Sigma =$$

$$= \int \mathfrak{U}(t) \mathfrak{B}(r) \mathfrak{D}(\Sigma) d\Omega_\Sigma . \quad (3.2)$$

The invariance of such equilibrium averages with respect to the time translations gives:

$$\langle \mathfrak{U}(t) \mathfrak{B}(r) \rangle = \langle \mathfrak{U}(t-r) \mathfrak{B} \rangle .$$

Therefore the Fourier integral can be written in the form:

$$\langle \mathfrak{U}(t) \mathfrak{B}(r) \rangle = \int_{-\infty}^{\infty} J_{\mathfrak{U}\mathfrak{B}}(\omega) e^{-i\omega(t-r)} d\omega . \quad (3.3)$$

It is to be pointed out that just as in the quantum mechanical case, we here have the well-known inequality:

$$J_{\mathfrak{U}^*, \mathfrak{U}}(\omega) \geq 0. \quad (3.4)$$

In the quantum mechanical treatment of problems of the statistical mechanics a very important role is played by the method of two-time Green functions, defined by the relations:

$$G_{ret}(t-\tau) = \theta(t-\tau) < [\mathfrak{U}_t, \mathfrak{B}_\tau] >$$

$$G_{adv.}(t-\tau) = -\theta(\tau-t) < [\mathfrak{U}_t, \mathfrak{B}_\tau] > \qquad (3.5)$$

where $[\ldots,\ldots]$ denote the quantum Poisson brackets. N.Bogolubov jr. and B.Sadovnikov[6] have extended this method to the case of classical mechanics.

Their definition of two-time Green functions is the same (3.5) with the only difference that the Poisson brackets (3.5) are to be understood in the classical sense.

These authors introduced the function

$$\ll \mathfrak{U}, \mathfrak{B} \gg_\nu = \frac{1}{2\pi\theta} \int_{-\infty}^{\infty} J_{\mathfrak{U},\mathfrak{B}}(\omega) \frac{\omega'}{-\omega'+\nu} d\omega' \qquad (3.6)$$

which is regular in the complex plane of the variable ν with the exception of its real axis. The function (3.6) defines the frequency representation

$$\ll \mathfrak{U}, \mathfrak{B} \gg_\omega^{r,a} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{r,a}(t) e^{i\omega t} dt$$

of the retarded and advanced Green function by means of the relations:

$$\ll \mathfrak{U}, \mathfrak{B} \gg_\omega^r = \ll \mathfrak{U}, \mathfrak{B} \gg_{\omega+i0^+}$$

$$\ll \mathfrak{U}, \mathfrak{B} \gg_\omega^a = \ll \mathfrak{U}, \mathfrak{B} \gg_{\omega-i0^+} \qquad (3.7)$$

which yield:

$$J_{\mathfrak{U},\mathfrak{B}}(\omega) = i\frac{\theta}{\omega}\{\ll \mathfrak{U},\mathfrak{B} \gg_{\omega+i0^+} - \ll \mathfrak{U},\mathfrak{B} \gg_{\omega-i0^+}\}. \qquad (3.8)$$

It is to be noted that first the usual limiting process of the statistical mechanics: $V \to \infty$ must be performed and after that

the limiting process of approaching the real axis is to be carried out.

Let us now say a few words about the possibility of the effective determination of the Green functions.

One of the methods elaborated for this aim[6] can be briefly resumed as follows:

An infinitesimal explicitly time dependent term:

$$\delta H_t = e^{\epsilon t - i\omega t} \mathfrak{B}(\Omega_\Sigma)\delta\xi + e^{\epsilon t + i\omega t} \mathfrak{B}^*(\Omega_\Sigma)\delta\xi^* \qquad (3.9)$$
$$\epsilon > 0$$

is added to the Hamiltonian:

$$H_t = H_\Sigma + \delta H_t \quad .$$

Note that because of the sign of ϵ

$$\delta H_t \to 0, \qquad \text{when} \quad t \to -\infty .$$

We shall start with the corresponding Liouville equation:

$$\frac{\partial \mathfrak{D}_t}{\partial t} = \mathcal{H}_\Sigma + [\delta H_t, \mathfrak{D}_t]$$

with the initial condition at $t \to -\infty$:

$$\mathfrak{D}_{-\infty} = \mathfrak{D}(\Sigma).$$

In other words at $t = -\infty$ we have the statistical equilibrium situation and the infinitesimal perturbation (3.9) being adiabatically switched on.

Of course

$$\mathfrak{D}_t = \mathfrak{D}(\Sigma) + \delta\mathfrak{D}_t .$$

Then, if the one time average of a dynamical variable:

$$\mathfrak{U} = \mathfrak{U}(\Omega_\Sigma)$$

is considered, it was found that:

$$\langle \mathfrak{U} \rangle_t = \langle \mathfrak{U} \rangle_{eq.} + \delta \langle \mathfrak{U} \rangle_t$$

$$\delta\langle \mathfrak{U} \rangle_t = e^{-(\omega+i\epsilon)it}\, 2\pi \langle\langle \mathfrak{U}, \mathcal{B} \rangle\rangle_{\omega+i\epsilon}\, \delta\xi + \qquad (3.10)$$

$$+\, e^{-i(-\omega+i\epsilon)t}\, 2\pi \langle\langle \mathfrak{U}, \mathcal{B}^* \rangle\rangle_{-\omega+i\epsilon}\, \delta\xi^* .$$

We thus see that in order to obtain the expression of the Green function in the upper half plane of ν, it is sufficient to calculate the variation

$$\delta\langle \mathfrak{U} \rangle_t$$

of the one time average, induced by the infinitesimal perturbation term (3.9) in the Hamiltonian.

We further note that it follows from (3.9):

$$J_{\mathcal{B},\mathfrak{U}}(\omega) = J_{\mathfrak{U},\mathcal{B}}(-\omega)$$

what yields:

$$\langle\langle \mathfrak{U}, \mathcal{B} \rangle\rangle_{\omega-i\epsilon} = \langle\langle \mathcal{B}, \mathfrak{U} \rangle\rangle_{-\omega+i\epsilon} . \qquad (3.11)$$

Therefore the frequency representation of the Green function in the lower half plane can be obtained in the same way by inverting the roles of $\mathfrak{U}$ and $\mathcal{B}$.

This method is very fruitful especially when dealing with the so-called hydrodynamical approximation.

Here, however, we will adopt the other procedure connected with the Laplace transform method now widely used in the works concerning the problems of the statistical mechanics of classical systems.

By considering the expression of the statistical equilibrium distribution:

$$\mathfrak{D}(\Sigma) = Z^{-1} e^{-H_\Sigma(\Omega_\Sigma)/\theta}$$

we easily find:

$$[\mathfrak{U}(t); \mathfrak{D}(\Sigma)] = -\frac{1}{\theta}[\mathfrak{U}(t); H_\Sigma]\mathfrak{D}(\Sigma) = -\frac{1}{\theta}\frac{d\,\mathfrak{U}(t)}{dt}\mathfrak{D}(\Sigma).$$

Then the identity with the Poisson brackets:

$$[\mathfrak{U}(t); \mathcal{B}]\mathfrak{D}(\Sigma) + [\mathfrak{U}(t); \mathfrak{D}(\Sigma)]\mathcal{B} = [\mathfrak{U}(t); \mathcal{B}\mathfrak{D}(\Sigma)]$$

and the relation:

$$\overline{([\mathfrak{U}(t); \mathcal{B}\mathfrak{D}(\Sigma)])}_\Sigma = 0$$

yield:

$$\langle[\mathfrak{U}(t); \mathcal{B}]\rangle = \overline{([\mathfrak{U}(t); \mathcal{B}]\mathfrak{D}(\Sigma))}_\Sigma =$$

$$= \frac{1}{\theta}\frac{d}{dt}\langle \mathfrak{U}(t)\mathcal{B}\rangle .$$

On the other hand, the relations (3.5), (3.6) give:

$$\langle\langle \mathfrak{U}, \mathcal{B} \rangle\rangle_{\omega+i\epsilon} = \frac{1}{2\pi}\int_0^\infty e^{(-\epsilon+i\omega)t}\langle[\mathfrak{U}(t), \mathcal{B}]\rangle\, dt$$

from which it follows:

$$\langle\langle \mathfrak{U}, \mathcal{B} \rangle\rangle_{\omega+i\epsilon} = \frac{1}{2\pi\theta}\int_0^\infty e^{-zt}\frac{d}{dt}\langle \mathfrak{U}(t)\mathcal{B}\rangle\, dt, \qquad (3.12)$$

where

$$z = \epsilon - i\omega , \qquad (3.13)$$

or

$$\langle\langle \mathfrak{U}, \mathcal{B} \rangle\rangle_{\omega+i\epsilon} = \frac{1}{2\pi\theta}\{z\int_0^\infty e^{-zt}\langle \mathfrak{U}(t)\mathcal{B}\rangle dt - \langle \mathfrak{U}\mathcal{B}\rangle\} . \qquad (3.14)$$

Because of (3.11) we also obtain:

$$\langle\langle \mathfrak{U}, \mathcal{B} \rangle\rangle_{\omega-i\epsilon} = \frac{1}{2\pi\theta}\{\int_0^\infty z^* e^{-z^* t}\langle \mathfrak{U}(t)\mathcal{B}\rangle dt - \langle \mathfrak{U}\mathcal{B}\rangle\} . \qquad (3.15)$$

We, therefore, see that the Green functions on the upper and lower half plane are immediately determined by the Laplace transforms of the equilibrium correlation averages of the type:

$$< \mathfrak{U}_t \mathfrak{B} > , \; t \geq 0. \tag{3.16}$$

In order to reduce the problem of the determination of such correlation averages to the problem of calculation of the one time averages we may start with the standard Liouville equation:

$$\frac{\partial \mathfrak{D}_t}{\partial t} = \Lambda_\Sigma \mathfrak{D}_t \; ; \quad t \geq 0 \tag{3.17}$$

with the initial condition:

$$\mathfrak{D}_0 = \mathfrak{D}(\Sigma) + \mathfrak{B}(\Omega_\Sigma)\delta\xi \qquad \text{for } t=0 \tag{3.18}$$

expressing that the initial (for $t=0$) expression of $\mathfrak{D}_t$ is only infinitesimally different from the equilibrium distribution. In this situation:

$$\mathfrak{D}_t = \mathfrak{D}(\Sigma) + \delta\mathfrak{D}_t$$

and by making use of the relation (3.1) we get:

$$\delta < \mathfrak{U} >_t = \int \mathfrak{U}(t)\mathfrak{B}\mathfrak{D}(\Sigma)d\Omega_\Sigma \delta\xi = < \mathfrak{U}(t)\mathfrak{B} >_{eq.} \delta\xi = \tag{3.19}$$

$$= \int \mathfrak{U}\,\delta\mathfrak{D}_t \, d\Omega_\Sigma = \int \mathfrak{U} e^{\Lambda_\Sigma t} \mathfrak{B}\mathfrak{D}(\Sigma)d\Omega_\Sigma \delta\xi .$$

Note that within the present approach, we

have to deal only with the time independent Liouville operator Λ_Σ. The variation is now introduced not in Λ_Σ but in the initial expression of $\mathfrak{D}$.

To investigate a more specific situation we shall consider the case when, as in the previous section, the dynamical system Σ consists of N identical particles of mass M.

Let us further assume that the Liouville operator has the form:

$$\Lambda_\Sigma = \sum_{(1 \leq j \leq N)} \Lambda_j^{(0)} + \sum_{(1 \leq j_1 < j_2 \leq N)} \Lambda_{j_1, j_2} , \tag{3.20}$$

where

$$\Lambda_j^{(0)} = - \vec{v}_j \frac{\partial}{\partial \vec{r}_j} \tag{3.21}$$

and

$$\Lambda_{j_1, j_2} = \Lambda_{j_1, j_2}^{(\Phi_\Sigma)} , \tag{3.22}$$

or

$$\Lambda_{j_1, j_2} = \Lambda_{j_1, j_2}^{(coll)} ,$$

or

$$\Lambda_{j_1, j_2} = \Lambda_{j_1, j_2}^{(\Phi)} + \Lambda_{j_1, j_2}^{(coll)} \tag{3.23}$$

the notation here being the same as previously.

We shall now turn our attention upon the method of the reduced distribution functions in the form elaborated already in my monograph in 1946.

These reduced distributions are introduced as follows:

$$F_1(t;1) = F_1(t;\vec r_1,\vec v_1) = V\int \mathcal{D}_t\, d\vec r_2 d\vec v_2 \dots d\vec r_N d\vec v_N$$

$$F_2(t;1,2) = F_2(t;\vec r_1,\vec v_1;\vec r_2,\vec v_2) = V^2\left(1-\frac{1}{N}\right)\times \qquad (3.24)$$

$$\times \int \mathcal{D}_t\, d\vec r_3 d\vec v_3 \dots d\vec r_N d\vec v_N$$

. . . .

$$F_s(t;1,2,\dots,s) = F_s(t;\vec r_1,\vec v_1;\dots\vec r_s,\vec v_s) = V^s\left(1-\frac{1}{N}\right)\dots\left(1-\frac{s-1}{N}\right)\times$$

$$\times \int \mathcal{D}_t\, d\vec r_{s+1} d\vec v_{s+1} \dots d\vec r_N d\vec v_N\,.$$

Owing to the symmetry of $\mathcal{D}_t$ we see that F_s are symmetric functions of the phases (1),...(s): Because

$$\mathcal{D}_t = e^{\mathcal{J}_\Sigma t}\,\mathcal{D}_0$$

we may also write:

$$F_1(t;\vec r_1,\vec v_1) = V\int e^{\mathcal{J}_\Sigma t}\,\mathcal{D}_0\, d\vec r_2 d\vec v_2 \dots d\vec r_N d\vec v_N\,. \qquad (3.25)$$

. . . .

It is easy to see that the functions $F_1(t;1)$; $F_2(t;1,2)$;... give, respectively, the probability density to find one particle with the phase $(\vec r_1,\vec v_1)$, two particles with the phase $(\vec r_1,\vec v_1;\vec r_2,\vec v_2)$, and so on.

Consider the additive dynamical variable

$$\mathfrak{U} = \sum_{(1\le j\le N)} A(\vec r_j,\vec v_j). \qquad (3.26)$$

By starting with the definition (3.24) and utilizing the symmetry property, we find

$$\langle \mathfrak{U}\rangle_t = n\int A(\vec r_1,\vec v_1)\,F_1(t;\vec r_1,\vec v_1)\,d\vec r_1 d\vec v_1\,, \qquad (3.27)$$

or in a more abbreviated form:

$$\langle \mathfrak{U}\rangle_t = n\int A(1)\,F_1(t;1)\,d(1).$$

In the same way the average value of the binary dynamical variable can be expressed by means of $F_2(t;1,2)$, etc.

The Liouville equation yield the following hierarchy of equations:

$$\frac{\partial F_1(t;1)}{\partial t} = \mathcal{J}_1^{(0)} F_1(t;1) + n\int \mathcal{J}_{1,2} F_2(t;1,2)\,d(2)$$

$$\frac{\partial F_2(t;1,2)}{\partial t} = \left(\mathcal{J}_1^{(0)} + \mathcal{J}_2^{(0)} + \mathcal{J}_{1,2}\right) F_2(t;1,2) + \qquad (3.28)$$

$$+ n\int \left(\mathcal{J}_{1,3} + \mathcal{J}_{2,3}\right) F_3(t;1,2,3)\,d(3)$$

. . . .

$$\frac{\partial F_s(t;1,2,\dots s)}{\partial t} = \left(\sum_{(1\le j\le s)} \mathcal{J}_j^{(0)} + \sum_{(1\le j_1<j_2\le s)} \mathcal{J}_{j_1 j_2}\right) F_s(t;1,2,\dots s) +$$

$$+ n\int \sum_{(1\le j\le s)} \mathcal{J}_{j,s+1}\, F_{s+1}(t;1,2,\dots s,s+1)\,d(s+1)\,.$$

When dealing with the reduced distributions F_s it is usually assumed that for

$$V\to\infty\,,\quad \frac{N}{V} = n = \text{Const}$$

they have definite limits which also satisfy equations (3.28).

In case of the equilibrium distribution this assumption was rigorously proved /7/ for a vast class of physically admissible short range potential functions $\Phi_\Sigma(r)$, if the particle density is sufficiently small.

Under these conditions the analyticity of F_s as functions of n was also established /7/.

We may remark that the study of equilibrium F_s is greatly simplified by the fact of their factorization property:

$$F_{eq.}(1,\dots s)=f(\vec{r}_1,\dots\vec{r}_s)\prod_{(1\leq j\leq s)}\Phi_\Sigma(v_j). \qquad (3.29)$$

As far as I know the behaviour of nonequilibrium F_s was not investigated at the rigorous mathematical level.

Let us now consider the equations (3.28) in the limiting situation of infinite volume $V=\infty$. From the formal point of view we have here a system of linear equations for the reduced distribution functions F_s.

It is necessary, however, to point out that not all solutions of these equations are physically admissible.

Take, for example,

$$F_s(t;1,\dots s)$$

and distribute the indices $1,\dots s$ into ℓ groups $[j_1],\dots[j_\ell]$ containing, respectively, $s_1,\dots s_\ell$ numbers:

$$F_s(t;1,\dots s)=F_{s_1+\dots+s_\ell}(t;[j_1],\dots[j_\ell])$$

$$s=s_1+\dots+s_\ell$$

Suppose that the distances between the particles belonging to different groups tend to infinity. Then, from the physical point of view, it is natural to expect that the correlation between the sets $[j_1],\dots[j_\ell]$ of Σ-particles vanishes:

$$F_{s_1+\dots+s_\ell}(t;[j_1],\dots[j_\ell])-F_{s_1}(t;[j_1])\dots F_{s_\ell}(t;[j_\ell])\to 0 \qquad (3.30)$$

when

$$|\vec{r}_{j_p}-\vec{r}_{j_{p'}}|\to\infty;\ pp'=1,\dots\ell;\ j_p\in[j_p];\ j_{p'}\in[j_{p'}].$$

These relations expressing the general principle of correlation weakening /6/ can be considered as a kind of boundary conditions * imposed on F_s.

Of course, these "boundary conditions" are nonlinear. To make them linear /8/ we introduce the functions $G_s(t;1,\dots s)$ $(s=2,3,\dots)$ by putting:

$$F_2(t;1,2)=F_1(t;1)F_1(t;2)+G_2(t;1,2)$$

$$(3.31)$$

$$F_3(t;1,2,3)=F_1(t;1)F_1(t;2)F_1(t;3)+F_1(t;1)G_2(t;2,3)+$$

$$+F_1(t;2)G_2(t;1,3)+F_1(t;3)G_2(t;1,2)+G_3(t;1,2,3).$$

....

Then (3.30) yields the linear conditions:

$$G_2(t;1,2)\to 0, \quad \text{if } |\vec{r}_1-\vec{r}_2|\to\infty$$

$$(3.32)$$

$$G_3(t;1,2,3)\to 0, \quad \text{if } \max\{|\vec{r}_1-\vec{r}_2|,|\vec{r}_1-\vec{r}_3|,|\vec{r}_2-\vec{r}_3|\}\to\infty.$$

....

---

* For the mathematical treatment of equations (3.28), a lot of difficult questions arises, for example:

- In what sense the relations (3.30) are to be understood?
- What other conditions on F_s are to be taken into account?
- What initial conditions for $t=0$, are to be imposed on F_s, etc.?

By using the definition (3.31) we get the hierarchy of nonlinear equations for $F_1, G_2, G_3 \ldots$:

$$\frac{\partial F_1(t;1)}{\partial t} = \mathcal{L}_1^{(0)} F_1(t;1) + n \int \mathcal{L}_{1,2} \{F_1(t;1)F_1(t;2) + G_2(t;1,2)\} d(2)$$

$$\frac{\partial G_2(t;1,2)}{\partial t} = (\mathcal{L}_1^{(0)} + \mathcal{L}_2^{(0)} + \mathcal{L}_{1,2}) G_2(t;1,2) + \mathcal{L}_{1,2} F_1(t;1) \times \qquad (3.33)$$

$$\times F_1(t;2) + n \int \mathcal{L}_{1,3} \{ F_1(t;3) G_2(t;1,2) + F_1(t;1) G_2(t;2,3) +$$

$$+ G_3(t;1,2,3)\} d(3) + n \int \mathcal{L}_{2,3} \{ F_1(t;3) G_2(t;1,2) +$$

$$+ F_1(t;2) G_2(t;1,3) + G_3(t;1,2,3) \} d(3).$$

. . . .

Let us now return to the problem of equilibrium averages.

We shall have to deal with two additive dynamical variables:

$$\mathfrak{U} = \sum_{(1 \leq j \leq N)} A(j), \qquad \mathcal{B} = \sum_{(1 \leq j \leq N)} B(j)$$

for which:

$$\int B(1) F_1^{(eq)}(1) d(1) = 0 \qquad (3.34)$$

or, what is the same

$$\langle \mathcal{B} \rangle_{eq.} = 0. \qquad (3.34')$$

Consider the solution of the Liouville equation infinitely close to the Gibbs equilibrium distribution

$$\mathcal{D}_t = \mathcal{D}(\Sigma) + \delta \mathcal{D}_t \qquad (3.35)$$

starting from the initial expression:

$$\mathcal{D}_0 = \mathcal{D}(\Sigma) + \delta \mathcal{D}_0 \qquad (3.36)$$

$$\delta \mathcal{D}_0 = \sum_{(1 \leq j \leq N)} B(j) \delta \xi$$

and introduce the corresponding reduced distributions:

$$F_1^{(eq)}(1) + \delta F_1(t;1); \ldots F_s^{(eq)}(1, \ldots s) + \delta F_s(t;1, \ldots s); \ldots .$$

Then in accordance with (3.19):

$$\langle \mathfrak{U}(t) \mathcal{B} \rangle \delta \xi = n \int A(1) \delta F_1(t;1) d(1) \qquad (3.37)$$

and the relation (3.25) yields:

$$\delta F_1(t;1) = V \int e^{\mathcal{L}_\Sigma t} \delta \mathcal{D}_0 d(2) \ldots d(N). \qquad (3.38)$$

The variation of the relations (3.31) permits us to introduce $\delta G_2(t;1,2); \ldots \delta G_s(t;1,2,\ldots s); \ldots$

We further remark that the variation of the nonlinear equations (3.33) leads to linear equations for $\delta F_1(t;1); \delta G_2(t;1,2), \ldots$ $\ldots \delta G_s(t;1,2,\ldots s); \ldots$ with the coefficients depending upon equilibrium functions.

Let us now proceed to obtain the initial expressions for these variations.

So, from (3.35) we obtain:

$$1/\delta\xi \;\; \delta F_1(0;1) = B(1) F_1(1) + n\left(1 - \frac{1}{N}\right) \int B(3) F_2(1,3) d(3)$$

$$1/\delta\xi \;\; \delta F_2(0;1,2) = \{B(1) + B(2)\} F_2(1,2) + n\left(1 - \frac{2}{N}\right) \int B(3) F_3(1,2,3) d(3),$$

where to shorten the notation, we have omitted the index "eq" of $F_s(1,\ldots s)$.

In virtue of (3.34) it follows:

$$\int B(3) F_2(1,3)\, d(3) = \int B(3)\{F_2(1,3) - F_1(1) F_1(3)\} d(3) =$$

$$= \int B(3)\, G_2(1,3)\, d(3)$$

and thus:

$$\delta F_1(0;1) = \{B(1) F_1(1) + n(1-\frac{1}{N}) \int B(3)\, G_2(1,3)\, d(3)\}\, \delta\xi .$$

We also have:

$$\delta G_2(0;1,2) = \delta F_2(0;1,2) - F_1(1)\delta F_1(0;2) - F_1(2)\delta F_1(0;1) =$$

$$= \{B(1)+B(2)\} G_2(1,2) + n(1-\frac{1}{N}) \int B(3)\{F_3(1,2,3) - F_1(1)F_1(2)F_1(3) -$$

$$- F_1(1) G_2(2,3) - F_1(2) G_2(1,3) - F_1(3) G_2(1,2)\} d(3) -$$

$$- \frac{n}{N}\int B(3)\{F_3(1,2,3) - F_2(1,2) F_1(3)\} d(3).$$

Therefore, by neglecting terms of the order 1/N, we obtain:

$$\delta G_2(0;1,2) = \{(B(1)+B(2)) G_2(1,2) + n\int B(3)\, G_3(1,2,3) d(3)\}\delta\xi .$$

. . . .

As it was already pointed out we here consider only the case when the state of statistical equilibrium in Σ is spatially homogeneous.

Consequently:

$$F_1(1) = \Phi_\Sigma(v_1)$$

$$G_2(1,2) = g_2(\vec r_1 - \vec r_2)\Phi_\Sigma(v_1)\Phi_\Sigma(v_2)$$

$$G_3(1,2,3) = g_3(\vec r_1 - \vec r_3, \vec r_2 - \vec r_3)\Phi_\Sigma(v_1)\Phi_\Sigma(v_2)\Phi_\Sigma(v_3).$$

. . . .

We thus see that the condition (3.34) can be written in the form:

$$\int B(\vec r, \vec v)\, \Phi_\Sigma(\vec v)\, d\vec r\, d\vec v = 0. \tag{3.39}$$

We also have:

$$\delta F_1(0;1) = \Phi_\Sigma(v_1)\{B(\vec r_1, \vec v_1) + n\int g_2(\vec r_1 - \vec r_2) B(\vec r_2, \vec v_2)\Phi_\Sigma(v_2) \times$$

$$\times\, d\vec r_2\, d\vec v_2\}\delta\xi$$

(3.40)

$$\delta G_2(0;1,2) = \Phi_\Sigma(v_1)\Phi_\Sigma(v_2)\{(B(\vec r_1, \vec v_1) + B(\vec r_2, \vec v_2)) g_2(\vec r_1 - \vec r_2) +$$

$$+ n\int g_3(\vec r_1 - \vec r_3, \vec r_2 - \vec r_3) B(\vec r_3, \vec v_3)\, \Phi_\Sigma(v_3)\, d\vec r_3\, d\vec v_3\}\delta\xi .$$

. . . .

Consider now the special case when:

$$B(\vec r, \vec v) = B_k(\vec r, \vec v) = e^{-i\vec k\vec r}\phi(\vec v) \tag{3.40'}$$

and remark that for $k \neq 0$ the condition (3.39) is automatically verified and for $k = 0$ this condition requires:

$$\int \phi(\vec v)\Phi_\Sigma(v)\, dv = 0$$

(3.41)

$$\phi(\vec v) = B_0 .$$

Then

$$\delta F_1(0;1) = e^{-i\vec k\vec r_1}\Phi_\Sigma(v_1)\{\phi(\vec v_1) + n\int g(\vec r) e^{i\vec k\vec r} d\vec r \times \tag{3.42}$$

$$\times \int \phi(\vec v)\Phi_\Sigma(v)\, d\vec v\}\delta\xi$$

and in general:

$$\delta G_s(0;\vec r_1+\vec r,\vec v_1;\dots \vec r_s+\vec r,\vec v_s)=e^{-i\vec k\vec r}\,\delta G_s(0;\vec r_1,\vec v_1;\dots \vec r_s,\vec v_s)$$

As the linear equations, obtained from (3.33) for

$$\delta F_1(t;1);\dots \delta G_s(t;1,\dots s);\ \dots$$

are invariant with respect to the space translations, we, therefore, also get:

$$\delta F(t;1)=e^{-i\vec k\vec r_1}\,\Phi_k(t,\vec v_1)\delta\xi$$

$$\delta G_s(t;\vec r_1+\vec r,\vec v_1;\dots \vec r_s+\vec r,\vec v_s)=e^{-i\vec k\vec r}\,\delta G_s(t;\vec r_1,\vec v_1;,\dots \vec r_s,\vec v_s). \tag{3.43}$$

Here, of course, $\Phi_k(t,\vec v_1)$ as well as δG_s are linear functionals of $\phi(\vec v)$.

By using the relations (2.64), (3.36), (3.38) we obtain

$$\Phi_k(t,\vec v_1)=\Phi_\Sigma(v_1)\int U_k(t,\vec v_1,\vec v_1')\phi(\vec v_1')d\vec v_1', \tag{3.44}$$

where, for $k=0$ the condition (3.41) must be satisfied. From (2.72) it also follows:

$$R_k(t)=\int\Phi_k(t,\vec v_1)d\vec v_1 \quad \text{for } \phi(\vec v)=1,\ k\neq 0\ . \tag{3.45}$$

Take first the case when

$$Л_{1,2}=Л_{1,2}^{(\Phi_\Sigma)}\ ;\quad Л_{int}=Л_{int}^{(\Phi)} \tag{3.46}$$

We may then recall that to bring previously formulated approximate equations (2.29), (2.30) or the kinetic equation to the explicit form, we need the evaluation of $R_k(t)$ $(k\neq 0)$. For the case

$$Л_{1,2}=Л_{1,2}^{(coll.)}\ ;\quad Л_{int}=Л_{int}^{(coll)} \tag{3.47}$$

the corresponding approximate equations (2.69), (2.70) acquire the explicit form if we succeed in obtaining the expression of U_k.

We thus see that for both cases (3.46), (3.47) we need to evaluate $\Phi_k(t,v_1)$.

To attain this purpose, we shall restrict ourselves to the simplest approximation, in the nonlinear system of equations (3.39) we shall consider only the first of them and neglect here the correlation function $G_2(t;1,2)$. In such an approximation we have to deal only with one nonlinear equation:

$$\frac{\partial F_1(t;1)}{\partial t}=Л_1^{(0)}F_1(t;1)+n\int Л_{1,2}F_1(t;1)F_1(t;2)d(2). \tag{3.48}$$

It is evident that in the case (3.46) this equation turns into the well-known Vlasov equation:

$$\frac{\partial F_1(t;\vec r_1,\vec v_1)}{\partial t}=-\vec v_1\frac{\partial F_1(t;\vec r_1,\vec v_1)}{\partial \vec v_1}+ \tag{3.49}$$

$$+\frac{n}{M}\left\{\frac{\partial}{\partial\vec r_1}\int\Phi_\Sigma(\vec r_1-\vec r_2)\bar\rho(t;\vec r_2)d\vec r_2\right\}\frac{\partial F_1(t;\vec r_1,\vec v_1)}{\partial\vec v_1}$$

$$\bar\rho(t;\vec r)=\int F_1(t;\vec r,\vec v_1)d\vec v_1\ .$$

That kind of one-component Vlasov equation is used, for example, to describe a simplified model of electron plasma-namely of classic electron gas, consisting of negatively charged point particles in a uniform positively charged compensating background.

In this model

$$\Phi_\Sigma(r) = e^2/r. \tag{3.50}$$

Note that for the statistical equilibrium

$$\bar\rho_{eq.} = 1.$$

To take account of the external field created by the positive background, we must subtract its constant charge density from the charge density of electrons. This amounts to replace the expression (3.49) of the particle density by:

$$\bar\rho(t;\vec r) = \int F_1(t;\vec r, \vec v_1)d\vec v_1 - 1$$

In the state of the statistical equilibrium the total charge density is null and, therefore, the equation for the variation $\delta F_1(t;1)$ will be:

$$\frac{\partial \delta F_1(t;\vec r_1,\vec v_1)}{\partial t} = -\vec v_1 \frac{\partial \delta F_1(t;\vec r_1, \vec v_1)}{\partial t} + \tag{3.51}$$
$$+\frac{n}{M}\frac{\partial}{\partial \vec r_1}\int \Phi_\Sigma(\vec r_1 - \vec r_2)\delta\bar\rho(t;\vec r_2)d\vec r_2 \frac{\partial \Phi_\Sigma(v_1)}{\partial \vec v_1}.$$

As we consider here the case when $\phi(\vec v) = 1$ and as for consistency of approximation we must drop from (3.42) the term containing the correlation function $g(r)$, we obtain:

$$\delta F_1(0;\vec r_1,\vec v_1) = e^{-i\vec k\vec r_1}\Phi_\Sigma(v_1)\delta\xi.$$

Then the relations (3.43), (3.45) yield:

$$\frac{\partial \Phi_k(t;\vec v)}{\partial t} = i(\vec k\vec v)\{\Phi_k(t;\vec v) + \frac{4\pi e^2 n}{\theta k^2}R_k(t)\Phi_\Sigma(\vec v)\} \tag{3.52}$$

$$\Phi_k(0;\vec v) = \Phi_\Sigma(v).$$

To solve this equation let us introduce the Laplace transforms:

$$\int_0^\infty \Phi_k(t;\vec v)e^{-zt}dt = \tilde\Phi_k(z;\vec v)$$
$$\mathrm{Re}\, z > 0 \tag{3.53}$$
$$\int_0^\infty R_k(t)e^{-zt}dt = \int_0^\infty \tilde\Phi_k(z,\vec v)d\vec v$$

which bring the equation (3.52) to the form:

$$(z - i(\vec k\vec v))\tilde\Phi_k(z;\vec v) = i\vec k\vec v \frac{4\pi e^2 n}{\theta k^2}\tilde R_k(z) + \Phi_\Sigma(v)$$

from which it follows

$$\tilde\Phi_k(z;v) = \frac{\Phi_\Sigma(v)}{z - i(\vec k\vec v)} + \frac{i\vec k\vec v}{z-i(\vec k\vec v)}\frac{4\pi e^2 n}{\theta k^2}\tilde R_k(z).$$

Therefore, in virtue of (3.53)

$$\int_0^\infty R_k(t)e^{-zt}dt = \frac{\int \frac{\Phi_\Sigma(v)}{z - i(\vec k\vec v)}d\vec v}{1 - \frac{4\pi e^2 n}{\theta k^2}\int \frac{i\vec k\vec v}{z - i(\vec k\vec v)}\Phi_\Sigma(\vec v)d\vec v},$$

or:

$$\int_0^\infty R_k(t)e^{-zt}dt = \frac{\int \frac{\Phi_\Sigma(v)}{z - i(\vec k\vec v)}d\vec v}{1 + \frac{4\pi e^2 n}{\theta k^2} - \frac{4\pi e^2 n}{\theta k^2} z\int \frac{\Phi_\Sigma(v)}{z - i\vec k\vec v}d\vec v}. \tag{3.54}$$

$$\mathrm{Re}\, z > 0.$$

We see that just the left-hand side of (3.54) really enters into equations (2.27), (2.36).

We can now give a more explicit form to the integral

$$\int \frac{\Phi_\Sigma(v)}{z - i\vec{k}\vec{v}}\, d\vec{v} \qquad (3.55)$$

by remarking that $\Phi_\Sigma(v)$ is here the normalized maxwellian velocity distribution.

To this end let us choose the direction of the vector $\vec{k}$ as the z-axis in the integration space of (3.55).

We then obtain:

$$\int \frac{\Phi_\Sigma(v)}{z - i\vec{k}\vec{v}}\, d\vec{v} = \left(\frac{M}{2\pi\theta}\right)^{1/2} \int_{-\infty}^{\infty} \frac{e^{-\frac{Mu^2}{2\theta}}}{z - iku}\, du .$$

Here

$$\frac{1}{z - iku} = \int_0^\infty e^{-\tau(z-iku)}\, d\tau, \quad \mathrm{Re}\, z > 0.$$

The integration over "u" yields:

$$\left(\frac{M}{2\pi\theta}\right)^{1/2} \int_{-\infty}^{\infty} e^{i\tau ku - \frac{Mu^2}{2\theta}}\, du = e^{-\tau^2 k^2 u_{eq}^2}$$

$$u_{eq} = \sqrt{\frac{\theta}{2M}}$$

from which it follows:

$$\int \frac{\Phi_\Sigma(v)}{z - i\vec{k}\vec{v}}\, d\vec{v} = \int_0^\infty e^{-\tau z - u_{eq}^2 k^2 \tau^2}\, d\tau = \frac{1}{ku_{eq}} \int_0^\infty e^{-\tau \frac{z}{ku_{eq}}}\, e^{-\tau^2}\, d\tau$$

and in particular:

$$\lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \int \frac{\Phi_\Sigma(v)}{\epsilon - i\omega - i\vec{k}\vec{v}}\, d\vec{v} = \frac{1}{ku_{eq}} \int_0^\infty e^{-\tau^2} \left\{ \cos\frac{\omega\tau}{ku_{eq}} + i \sin\frac{\omega\tau}{ku_{eq}} \right\} d\tau =$$

$$= \frac{1}{ku_{eq}} \left\{ \frac{\sqrt{\pi}}{2} e^{-\frac{\omega^2}{4k^2u_{eq}^2}} + i \int_0^\infty e^{-\tau^2} \sin\frac{\omega\tau}{ku_{eq}}\, d\tau \right\}.$$

Therefore the relations (2.37), (3.54) give:

$$F(\vec{k}\vec{v}_0) = \mathrm{Re} \frac{\frac{1}{ku_{eq}} \left\{ \frac{\sqrt{\pi}}{2} e^{-\frac{(\vec{\sigma}\cdot\vec{v}_0)^2}{u_{eq}^2}} + i \int_0^\infty e^{-\tau^2} \sin\left(\frac{\omega\tau}{ku_{eq}}\right) d\tau \right\}}{1 + \frac{4\pi e^2 n}{\theta k^2}\left(1 - \frac{\vec{\sigma}\cdot\vec{v}_0}{u_{eq}} \int_0^\infty e^{-\tau^2} \sin\left(\tau \frac{\vec{\sigma}\cdot\vec{v}_0}{u_{eq}}\right) d\tau\right) + \frac{4\pi e^2 n}{\theta k^2}\, i\, \frac{\vec{\sigma}\cdot\vec{v}_0}{2} \sqrt{\pi}\, e^{-\frac{(\vec{\sigma}\cdot\vec{v}_0)^2}{4u_{eq}^2}}} . \qquad (3.56)$$

Consider now equation (2.36) for the case when S is a point particle with the charge Ze interacting with the Σ particles only via the Coulomb interaction. Then

$$\nu(k) = \frac{4\pi Z e^2}{k^2} . \qquad (3.57)$$

By substituting (3.56), (3.57) into (2.36) we obtain the kinetic equation of the Markofian type.

In a more simplified approximation the analogous kinetic equation was previously found by S.V.Temko /9/. The generalization to the quantum case was considered in the paper /10/ by Yu.L.Klimontovich and S.V.Temko.

It is clear that the main use of the mentioned equation was directed to describe the movement of a charged particle in the classical electron plasma.

We wish, however, to point out that all our equations were derived from the general approximate equation (1.26) which itself was obtained under the assumption that the interaction between S and Σ was small.

But if we suppose that e^2 can really be treated as a small parameter, we ought to drop from (3.56) the denominator, because all its terms but 1 are proportional to e^2 and $\nu^2(k)$ already contains the square of this parameter.

We then obtain a simplified expression:

$$F(\vec{k}\cdot\vec{v}_0) = \frac{1}{ku_{eq}} \frac{\sqrt{\pi}}{2} e^{-\frac{(\vec{\sigma}\cdot\vec{v}_0)^2}{4u_{eq}^2}}$$

which is proportional to $1/k$.

In the considered equation (2.36)

$$d\vec{k} = k^2 dk d\vec{\sigma}$$

so the integral over k would be

$$\int_0^\infty \frac{1}{k^4} k^2 \frac{1}{k} k^2 dk = \int_0^\infty \frac{dk}{k}.$$

We see that it diverges logarithmically both for small and large values of k.

In the language of quantum field theory we have here both "infrared" and "ultraviolet" divergence.

It is easy to see the physical origin of these divergences in the considered case of the Coulomb interaction.

Note first that the potential energy of interaction between S and a Σ-particle will be small relative to their mean kinetic energy when:

$$\frac{1}{r} \ll \frac{\theta}{|Z|e^2}.$$

Therefore, a correctly evaluated contribution of $\vec{k}$ space to the integral comes only from the region where:

$$k \ll k_{max} = \frac{\theta}{|Z|e^2} \quad . \tag{3.58}$$

On the other hand, it is necessary to take into consideration the effect of screening of a charge in the plasma at large distances characterized by the Debye length.

Neglecting of such a screening effect causes the divergence for small k.

When we take the full form (3.56), denominator being included, we see that for small k the function $F(\vec{k}\cdot\vec{v}_0)$ is of the order k so that the "infrared" singularity is eliminated.

But, for $k \to \infty$

$$F(\vec{k}\cdot\vec{v}_0) \sim \frac{1}{ku_{eq}} e^{-\frac{(\vec{\sigma}\cdot\vec{v}_0)^2}{4u_{eq}^2}}$$

and the logarithmic divergence remains for large k.

Therefore, in order to make the integral term in the right-hand side of equation (2.36) convergent, we may use a cut off procedure by integrating over k in the interval $(0, k_{max})$ instead of $(0, +\infty)$.

To elaborate a self-consistent way of approximation not needing cut off procedure, introduced ad hoc, we must refine our approach by separating, for example, from the short range part of the Coulomb interaction, a special collision type Liouville operator.

We shall not consider here this problem and proceed to the study of the case (3.47).

Equation (3.48) now turns into the Boltzmann-Enskog equation for the hard sphere interaction:

$$\frac{\partial F_1(t;\vec{r}_1,\vec{v}_1)}{\partial t} = -\vec{v}_1 \frac{\partial F_1(t;\vec{r}_1,\vec{v}_1)}{\partial \vec{v}_1} + \tag{3.59}$$

$$+ n a_0^2 \int \vec{v}_{1,2}\cdot\vec{\sigma}\,\theta(\vec{v}_{1,2}\cdot\vec{\sigma})\{\delta(\vec{r}_1-\vec{r}_2-a_0\vec{\sigma})b_{v_1,v_2}(\vec{\sigma})-$$

$$-\delta(\vec{r}_1-\vec{r}_2+a_0\vec{\sigma})\}F_1(t;\vec{r}_1,\vec{v}_1)F_1(t;\vec{r}_2,\vec{v}_2)d\vec{\sigma}d\vec{r}_2 d\vec{v}_2 .$$

Here $b_{v_1,v_2}(\vec{\sigma})$ represents the operator acting on functions $f(\vec{v}_1,\vec{v}_2)$ by changing their arguments into:

$$\vec{v}_1 \to \vec{v}_1^* = \vec{v}_1 - \vec{\sigma}(\vec{v}_{1,2}\cdot\vec{\sigma});\quad \vec{v}_2 \to \vec{v}_2^* = \vec{v}_2 + \vec{\sigma}(\vec{v}_{1,2}\cdot\vec{\sigma}). \tag{3.60}$$

Vector $\vec{\sigma}$ is a unit vector and

$$a_0 = a_\Sigma$$

is the diameter of hard spheres characterizing the Σ-particle interaction.

It is to be pointed out that when

$$\mathcal{H}_{1,2} = \mathcal{H}_{1,2}^{(\Phi_\Sigma)}$$

and $\Phi_\Sigma(r)$ corresponds to a short range repulsion, we can obtain for $F_1(t;1)$ the kinetic equation containing a collision type operator by making use of the second equation of system (3.33) and neglecting there the term proportional to the particle density.

Here, however, we shall deal only with the simplest case of the Boltzmann-Enskog equation (3.59) for hard sphere dynamics.

The corresponding generalization of the following considerations does not lead to any essential difficulty.

By taking variation of equation (3.59) in the infinitesimal neighbourhood of the equilibrium solution, we obtain for $\delta F_1(t;1)$ the following equation:

$$\frac{\partial \delta F_1(t;\vec{r}_1,\vec{v}_1)}{\partial t} = -\vec{v}_1 \frac{\partial}{\partial \vec{r}_1}\delta F_1(t;\vec{r}_1,\vec{v}_1) + n a_0^2 \int (\vec{v}_{1,2}\cdot\vec{\sigma})\theta(\vec{v}_{1,2}\cdot\vec{\sigma})\times$$

$$\times\{\delta(\vec{r}_1-\vec{r}_2-a_0\vec{\sigma})b_{v_1,v_2}(\vec{\sigma})-\delta(\vec{r}_1-\vec{r}_2+a_0\vec{\sigma})\}(\Phi_\Sigma(v_1)\delta F_1(t;\vec{r}_2,\vec{v}_2)+$$

$$+\Phi_\Sigma(v_2)\delta F_1(t;\vec{r}_1,\vec{v}_1))d\vec{\sigma}d\vec{r}_2 d\vec{v}_2 . \tag{3.61}$$

As it was previously noted, the initial condition is given by formula (3.42).

For the consistency of our low density approximation we ought to retain only the first term and hence:

$$\delta F_1(0;\vec{r}_1,\vec{v}_1) = e^{-i\vec{k}\vec{r}_1}\phi(\vec{v}_1)\Phi_\Sigma(v_1).$$

From (3.43) further we have:

$$\delta F_1(t;\vec{r}_1,\vec{v}_1) = e^{-i\vec{k}\vec{r}_1}\,\Phi_k(t;\vec{v}_1)\delta\xi .$$

Therefore by putting here:

$$\Phi_k(t;\vec{v}_1) = \Phi_\Sigma(v_1)X_k(t;\vec{v}_1) \tag{3.62}$$

we may bring equation (3.61) to the form:

$$\frac{\partial X_k(t;\vec{v}_1)}{\partial t} = i\vec{k}\,\vec{v}_1 X_k(t;\vec{v}_1) + n a_0^2 L_k(\vec{v}_1)X_k(t;\vec{v}_1) \tag{3.63}$$

$$X_k(0;\vec{v}_1) = \phi(\vec{v}_1), \tag{3.64}$$

where $L_k(\vec v_1)$ is the operator acting on functions $f(\vec v_1)$ as follows:

$$L_k(\vec v_1)f(\vec v_1)=\int(\vec v_{1,2}\cdot\vec\sigma)\theta(\vec v_{1,2}\cdot\vec\sigma)\{e^{ia_0\vec k\cdot\vec\sigma}f(\vec v_2^*)-e^{-ia_0\vec k\cdot\vec\sigma}f(\vec v_2)+$$

$$+f(\vec v_1^*)-f(\vec v_1)\}\Phi_0(v_0)\Phi_\Sigma(v_1)d\vec\sigma d\vec v_1 .$$

To solve (3.63), let us introduce the Laplace transform:

$$\int_0^\infty e^{-zt}X_k(t;\vec v)dt=\tilde X_k(z;v)$$

(3.65)

$$\int_0^\infty e^{-zt}\Phi_k(t;\vec v)dt=\Phi_\Sigma(v)\tilde X_k(z;v)$$

which brings the equation with the initial condition (3.64) to the form:

$$(z-i\vec k\vec v_1)\tilde X_k(z;v_1)=na_0^2L_k(v_1)\tilde X_k(z;v_1)+\phi(v_1).$$

Therefore:

$$\tilde X_k(z;v_1)=\{z-i\vec k\cdot\vec v_1-na_0^2L_k(v_1)\}^{-1}\phi(v_1). \quad (3.66)$$

By making use of (3.44), (3.65), (3.66) we get:

$$\int_0^\infty e^{-tz}U_k(t,\vec v_1)dt=\{z-i\vec k\cdot\vec v_1-na_0^2L_k(v_1)\}^{-1} \quad (3.67)$$

Here it must be remembered that this operator relation was obtained by using the initial condition (3.42) and therefore (3.67) holds always for $k\neq 0$, and for $k=0$ it remains valid only if applied to a function $f(v_1)$ satisfying the condition (3.41).

Recall further that each of equations (2.69), (2.70) has only one term containing $U_0(t-\tau;1)$. This operator is applied to an expression $\bar\chi$ which as a function of v_1 satisfies the condition (3.41), in virtue of (2.61). We may also add that the mentioned terms are proportional to $1/V$. Let us now investigate, for instance, equation (2.70). By using the Laplace transform and taking the limit $V\to\infty$ the following equation is obtained:

$$(z-na^2L_S(\vec v_0))\bar\chi(z;\vec v_0)=\chi(\vec v_0)+$$

(3.68)

$$+\frac{n}{(2\pi)^3}\int d\vec k\int d\vec v_1\Phi_\Sigma(v_1)\bar T_{-k}(v_0,v_1)W_k(z;1)\bar T_k(v_0,v_1)\times$$

$$\times\bar\chi(z;\vec v_0)$$

$$\chi(\vec v_0)=\chi(0,\vec v_0),$$

where

$$L_S(\vec v_0)f(\vec v_0)=\int(\vec v_{0,1}\cdot\vec\sigma)\theta(\vec v_{0,1}\cdot\vec\sigma)\Phi_\Sigma(v_1)\{B_{v_0,v_1}(\vec\sigma)-1\}\times$$

$$\times d\vec\sigma d\vec v_1 f(\vec v_0) \quad (3.69)$$

$$\bar\chi(z;\vec v_0)=\int_0^\infty e^{-zt}\chi(t;\vec v_0)dt, \quad \mathrm{Re}\,z>0$$

$$W_k(z;1)=\int_0^\infty e^{-\{z+i\vec v_0\vec k-na^2L_S(\vec v_0)\}t}U_k(t,\vec v_1)dt.$$

As the operators

$$i\vec{v}_0\vec{k} - na^2 L_S(\vec{v}_0), \quad i\vec{k}\vec{v}_1 + na_0^2 L_k(\vec{v}_1)$$

are acting on functions with different arguments, they do commute and therefore the relation (3.67) gives:

$$W_k(z;1) = \{z + i\vec{v}_0 \cdot \vec{k} - na^2 L_S(\vec{v}_0) - i\vec{v}_1 \cdot \vec{k} - na_0^2 L_k(\vec{v}_1)\}^{-1} \quad (3.70)$$

We may thus bring equation (3.68) to the form:

$$\{z - na^2 L_S(\vec{v}_0) - R(z;\vec{v}_0)\}\bar{\chi}(z;\vec{v}_0) = \chi(\vec{v}_0), \quad (3.71)$$

where

$$R(z;\vec{v}_0) = \frac{n}{(2\pi)^3}\int d\vec{k} \int d\vec{v}_1 \Phi_\Sigma(v_1)\bar{T}_{-k}(v_0,v_1)\{z + i(\vec{v}_0 - \vec{v}_1)\vec{k} - \quad (3.72)$$

$$- na^2 L_S(\vec{v}_0) - na_0^2 L_k(\vec{v}_1)\}^{-1}\bar{T}_k(v_0,v_1).$$

Consider now a function $F(\vec{v}_0)$. By recalling our reasoning of §1, which led us to formulae (1.15), (1.16), (1.17) we find that:

$$\int F(\vec{v}_0)\Phi_0(\vec{v}_0)\chi(t;\vec{v}_0)d\vec{v}_0 = \int F(\vec{v}_0) f(t;\vec{v}_0)d\vec{v}_0 =$$

$$= \frac{1}{V}\int F(\vec{v}_0) f(t;\vec{v}_0)d\vec{r}_0 d\vec{v}_0 = \frac{1}{V}\int F\{\vec{v}_0(t)\}\mathcal{D}_0(S,\Sigma)d\Omega_S d\Omega_\Sigma =$$

$$= \int F\{\vec{v}_0(t)\}\chi(\vec{v}_0)\mathcal{D}_{ae}(S,\Sigma)\, d\Omega_S d\Omega_\Sigma,$$

where

$$\mathcal{D}_{ae}(S,\Sigma) = \frac{1}{V}\Phi_0(v_0)\mathcal{D}_{eq}(\Sigma)$$

$$\int \mathcal{D}_{ae}(S,\Sigma)\,d\Omega_S\, d\Omega_\Sigma = 1.$$

We thus see that the expression:

$$\langle F(\vec{v}_0(t))\chi(\vec{v}_0)\rangle_{ae} = \int \Phi_0(v_0)F(\vec{v}_0)\chi(t;\vec{v}_0)d\vec{v}_0$$

represents the two time correlation average taken over the "approximate equilibrium" probability distribution $\mathcal{D}_{ae}(S,\Sigma)$ which differs from the exact equilibrium distribution $\mathcal{D}_{eq}(S,\Sigma)$ for the total system $S+\Sigma$ by neglecting correlations between S and Σ particles.

But we must point out that only the case when the probability of collision between S and Σ particles is small:

$$na^3 \ll 1$$

is considered here and in such a situation the corresponding correlations may be neglected while computing the main term. So, in this approximation we may write:

$$\langle F(\vec{v}_0(t))\chi(\vec{v}_0)\rangle_{eq} = \int F(\vec{v}_0)\Phi_0(v_0)\chi(t;\vec{v}_0)d\vec{v}_0. \quad (3.73)$$

Let us take, for example:

$$F(\vec{v}_0) = \chi(\vec{v}_0) = v_{0,x}$$

Then in the adopted approximation:

$$\int_0^\infty e^{-zt}\langle v_{0,x}(t)\, v_{0,x}\rangle dt = \int \Phi_0(v_0)v_{0,x}\bar{\chi}(z;\vec{v}_0)\,d\vec{v}_0, \quad (3.74)$$

where $\bar{\chi}(z;\vec{v}_0)$ is defined by equation (3.71) in which $\chi(\vec{v}_0) = v_{0,x}$.

The validity of the approximation (3.73), (3.74) will be discussed in §4 where the initial condition for $\mathfrak{D}_t(S, \Sigma)$ will be taken in the form:

$$\mathfrak{D}_0(S,\Sigma) = \chi(S)\, \mathfrak{D}_{eq}(S, \Sigma) \quad (3.75)$$

instead of (1.2).

Let us now stress that equation (3.71) is quite analogous to that found in the paper/11/ by J.R.Dorfmann and E.G.D.Cohen for the low density case and therefore can be treated by the procedure elaborated by these authors. In their paper $M=m$, $a_0=a$, so that the particle S can be considered as a "tagged" particle of one large system Σ, but this circumstance is not relevant for the validity of their approach and it can be repeated almost literally for equation (3.71).

Therefore, we shall not discuss this point further.

It is to be stressed that this equation follows from (2.69), (2.70) which were obtained without any assumption about the smallness of interaction in the Σ system.

Of course, to reduce (2.69), (2.70) to an explicit form the expression of the operator $U_k(t;1)$ is needed.

But such an expression can be found not only by using the Boltzmann-Enskog equation for hard spheres. The application of other more sophisticated kinetic equations is quite possible.

We may also use the so-called hydrodynamic approximation (which is independent of the assumption about the smallness of interactions in Σ) to find the explicit expression of the operator $U_k(t;1)$ in the region:

$$k \ll \frac{1}{\ell_\Sigma}, \quad t \gg t_\Sigma, \quad (3.76)$$

where ℓ_Σ, t_Σ denote, respectively, the mean free path and the mean free time for Σ.

As it can easily be shown, just this region is relevant for the long time behaviour of the correlation averages, e.g., of the type (3.73).

4.

We shall continue here to study the interaction of the particle S with the large system Σ under the same conditions as in §§1,2 with the only exception that instead of (1.2) we choose the initial expression of $\mathfrak{D}_t(S, \Sigma)$ in the form:

$$\mathfrak{D}_0(S, \Sigma) = h(S)\, \mathfrak{D}_{eq}(S, \Sigma),$$

where $\mathfrak{D}_{eq}(S, \Sigma)$ is the distribution function corresponding to the overall statistical equilibrium of the total system $S+\Sigma$.

In the considered situation:

$$\mathfrak{D}_{eq}(S, \Sigma) = W(\vec{r}_0, \vec{r}_1 \ldots \vec{r}_N)\, \Phi_0(v_0) \prod_{(1 \leq j \leq N)} \Phi_\Sigma(v_j) \quad (4.1)$$

and the normalization

$$\overline{(\mathfrak{D}_{eq}(S, \Sigma))}_{S+\Sigma} = 1.$$

Therefore

$$\int_V \cdots \int_V W(\vec{r}_0, \vec{r}_1, \ldots \vec{r}_N)\, d\vec{r}_0\, d\vec{r}_1 \ldots d\vec{r}_N = 1.$$

Since W is translationally invariant, this relation yields:

$$\int_V \ldots \int_V W(\vec{r}_0, \vec{r}_1, \ldots \vec{r}_N) d\vec{r}_1 \ldots d\vec{r}_N = \frac{1}{V}. \tag{4.2}$$

We thus obtain:

$$\overline{(\mathfrak{D}_0(S, \Sigma))}_\Sigma = h(S)\overline{(\mathfrak{D}_{eq}(S, \Sigma))}_\Sigma = h(S)\frac{1}{V}\Phi_0(v_0). \tag{4.3}$$

Let us note that in the case of the previously considered initial condition (1.2)

$$\overline{(\mathfrak{D}_0(S, \Sigma))}_\Sigma = f(S) = \chi(S)\Phi_0(v_0). \tag{4.4}$$

Therefore in order to preserve this previously adopted normalization, we take in (4.3) $h(S) = V\chi(S)$.

Then the initial value of $\mathfrak{D}_t(S,\Sigma)$

$$\mathfrak{D}_0(S,\Sigma) = V\chi(S)\mathfrak{D}_{eq}(S,\Sigma) \tag{4.5}$$

will satisfy the same relation (4.4) as in the case (1.2).

Starting from (4.5) the time evolution of $\mathfrak{D}_t(S,\Sigma)$ is defined by the Liouville equation:

$$\frac{\partial \mathfrak{D}_t}{\partial t} = (\mathcal{L}_S^{(0)} + \mathcal{L}_\Sigma + \mathcal{L}_{int})\mathfrak{D}_t . \tag{1.18}$$

We now introduce the function $\chi_t(S)$:

$$\overline{(\mathfrak{D}_t)}_\Sigma = \chi_t(S)\Phi_0(v_0) = f_t(S) \tag{4.6}$$

and note that it can be used to compute the equilibrium correlation averages of the type:

$$< F(\Omega_S(t))\chi(\Omega_S) >_{eq} .$$

Really, it is easy to see:

$$V < F(\Omega_S(t))\chi(\Omega_S)>_{eq} = \overline{(F(\Omega_S(t))V\chi(S)\mathfrak{D}_{eq}(S,\Sigma))}_{S+\Sigma} =$$

$$= \overline{(F(\Omega_S(t))\mathfrak{D}_0(S,\Sigma))}_{S+\Sigma} = \overline{(F(\Omega_S)\mathfrak{D}_t(S,\Sigma))}_{S+\Sigma} =$$

$$= \overline{(F(S)\overline{(\mathfrak{D}_t(S,\Sigma))}_\Sigma)}_S$$

and thus:

$$V < F(\Omega_S(t))\chi(\Omega_S) >_{eq} = \overline{(F(S)\chi_t(S))}_S = \int F(\vec{r}_0, \vec{v}_0)\chi(t;\vec{r}_0,\vec{v}_0)\Phi_0(v_0)d\vec{r}_0 d\vec{v}_0 . \tag{4.7}$$

Noting that

$$<f(S)>_{eq} = \overline{(f(S)\mathfrak{D}_{eq}(S,\Sigma))}_{S+\Sigma} = \frac{1}{V}\overline{(f(S)\Phi_0(v_0))}_S$$

we also may write:

$$\frac{\langle F(\Omega_S(t))\ \chi(\Omega_S)\rangle_{eq}}{\{\langle|F(\Omega_S)|^2\rangle_{eq}\langle|\chi(\Omega_S)|^2\rangle_{eq}\}^{1/2}}= \tag{4.8}$$

$$=\frac{\int F(\vec r_0,\vec v_0)\chi(t;\vec r_0,\vec v_0)\Phi_0(v_0)\,d\vec r_0 d\vec v_0}{\{\int|F(\vec r_0,\vec v_0)|^2\Phi_0(v_0)\,d\vec r_0 d\vec v_0\int|\chi(\vec r_0,\vec v_0)|^2\Phi_0(v_0)\,d\vec r_0 d\vec v_0\}^{1/2}}$$

and this expression is clearly independent of the normalization of $\chi(S)$.

We now proceed to use the method outlined in §1 to obtain an approximate equation for $\chi_t(S)$.

Denote:

$$\mathcal{D}_t - V\chi_t(S)\,\mathcal{D}_{eq}(S,\Sigma) = \Delta_t . \tag{4.9}$$

Then because of (4.3), (4.5), (4.6):

$$(\overline{\Delta_t})_\Sigma = 0, \quad \Delta_0 = 0 . \tag{4.10}$$

By integrating (1.18) over Ω_Σ and using the identity:

$$(\overline{Л_\Sigma F(S,\Sigma)})_\Sigma = 0 \tag{4.11}$$

we get:

$$\frac{\partial\chi_t(S)}{\partial t}\Phi_0(v_0) = Л_S^\circ\chi_t(S)\Phi_0(v_0) +$$

$$+V(\overline{Л_{int}\chi_t(S)\mathcal{D}_{eq}(S,\Sigma)})_\Sigma + (\overline{Л_{int}\ \Delta_t})_\Sigma$$

and thus:

$$\frac{\partial\chi_t(S)}{\partial t} = Л_S^\circ\chi_t(S) + V\frac{1}{\Phi_0(v_0)}(\overline{Л_{int}\chi_t(S)\,\mathcal{D}_{eq}(S,\Sigma)})_\Sigma + \tag{4.12}$$

$$+\frac{1}{\Phi_0(v_0)}(\overline{Л_{int}\ \Delta_t})_\Sigma$$

since

$$Л_S^\circ\chi_t(S)\Phi_0(v_0) = \Phi_0(v_0)Л_S^\circ\chi_t(S).$$

Let us now introduce the operator $Л_S^{(1)}$ acting only on the functions f(S) of the phase Ω_S:

$$Л_S^{(1)} f(S) = V(\overline{Л_{int} f(S)\Phi_0^{-1}(v_0)\mathcal{D}_{eq}(S,\Sigma)})_\Sigma . \tag{4.13}$$

Then (4.12) yields:

$$\frac{\partial\chi_t(S)}{\partial t} = Л_S^\circ\chi_t(S) + \frac{1}{\Phi_0(v_0)}Л_S^{(1)}\chi_t(S)\Phi_0(v_0) + \frac{1}{\Phi_0(v_0)}(\overline{Л_{int}\Delta_t})_\Sigma . \tag{4.14}$$

From (1.18), (4.9), (4.15) we get:

$$\frac{\partial\Delta_t}{\partial t} = (Л_S^\circ + Л_\Sigma + Л_{int})\Delta_t + V(Л_S^\circ + Л_\Sigma + Л_{int})\chi_t(S)\mathcal{D}_{eq}(S,\Sigma) -$$

$$-V\{Л_S^\circ\chi_t(S) + \frac{1}{\Phi_0(v_0)}Л_S^{(1)}\chi_t(S)\Phi_0(v_0) + \tag{4.15}$$

$$+\frac{1}{\Phi_0(v_0)}(\overline{Л_{int}\ \Delta_t})_\Sigma\}\mathcal{D}_{eq}(S,\Sigma)$$

$$\Delta_0 = 0.$$

It is easy to see that:

$$(\mathcal{L}^{\circ}_{S} + \mathcal{L}_{\Sigma})\chi_{t}(S)\mathcal{D}_{eq}(S,\Sigma) = \{\mathcal{L}^{\circ}_{S}\chi_{t}(S)\}\mathcal{D}_{eq}(S,\Sigma) +$$

$$+\chi_{t}(S)(\mathcal{L}^{\circ}_{S} + \mathcal{L}_{\Sigma})\mathcal{D}_{eq}(S,\Sigma).$$

But

$$(\mathcal{L}^{\circ}_{S} + \mathcal{L}_{\Sigma} + \mathcal{L}_{int})\mathcal{D}_{eq}(S,\Sigma) = 0$$

and therefore

$$(\mathcal{L}^{\circ}_{S} + \mathcal{L}_{\Sigma})\chi_{t}(S)\mathcal{D}_{eq}(S,\Sigma) = \{\mathcal{L}^{\circ}_{S}\chi_{t}(S)\}\mathcal{D}_{eq}(S,\Sigma) -$$

$$-\chi_{t}(S)\mathcal{L}_{int}\mathcal{D}_{eq}(S,\Sigma).$$

From (4.15) it now follows:

$$\frac{\partial\Delta_t}{\partial t} = (\mathcal{L}^{\circ}_{S} + \mathcal{L}_{\Sigma} + \mathcal{L}_{int})\Delta_t - \frac{V}{\Phi_0(v_0)}\{\overline{(\mathcal{L}_{int}\Delta_t)}_{\Sigma}\}\mathcal{D}_{eq}(S,\Sigma) +$$

$$+V\{\mathcal{L}_{int}\chi_t(S)\mathcal{D}_{eq}(S,\Sigma) - \chi_t(S)\mathcal{L}_{int}\mathcal{D}_{eq}(S,\Sigma)\} -$$

$$-\frac{V}{\Phi_0(v_0)}\{\mathcal{L}^{(1)}_{S}\chi_t(S)\Phi_0(v_0)\}\mathcal{D}_{eq}(S,\Sigma),$$

or

$$\frac{\partial\Delta_t}{\partial t} = (\mathcal{L}_{S} + \mathcal{L}_{\Sigma} + \Gamma)\Delta_t - \frac{V}{\Phi_0(v_0)}\{\overline{(\Gamma\Delta_t)}_{\Sigma}\}\mathcal{D}_{eq}(S,\Sigma) +$$

$$+V(\mathcal{L}_{int}\chi_t(S) - \chi_t(S)\mathcal{L}_{int})\mathcal{D}_{eq}(S,\Sigma) -$$

$$-V\mathcal{D}_{eq}(S,\Sigma)\{\frac{1}{\Phi_0(v_0)}\mathcal{L}^{(1)}_{S}\chi_t(S)\Phi_0(v_0)\} \qquad (4.16)$$

$$\Delta_0 = 0,$$

where

$$\Gamma = \mathcal{L}_{int} - \mathcal{L}^{(1)}_{S} \qquad (4.17)$$

and

$$\mathcal{L}_{S} = \mathcal{L}^{(0)}_{S} + \mathcal{L}^{(1)}_{S}. \qquad (4.18)$$

Let us consider the case when:

$$\mathcal{L}_{int} = \sum_{(1\leq j\leq N)} \mathcal{L}(0,j). \qquad (4.19)$$

Here $\mathcal{L}(0,j)$ represents the Liouville operator corresponding to the interaction between S and j-th particle of Σ.

For example:

$$\mathcal{L}^{(coll)}_{int} = \sum_{(1\leq j\leq N)} \bar{T}(0,j). \qquad (1.10)$$

Consider the expression:

$$V\overline{(\mathcal{L}(0,j)\mathcal{D}_{eq}(S,\Sigma)f(S))}_{\Sigma}. \qquad (4.20)$$

Note that (4.1) gives:

$$V\overline{(\mathcal{L}(0,j)\mathcal{D}_{eq}(S,\Sigma)f(S))}_{\Sigma} = \qquad (4.21)$$

$$= V\int \mathcal{L}(0,j)F_{S,\Sigma}(0,j)f(S)\Phi_0(v_0)\Phi_{\Sigma}(v_j)\,d\vec{r}_j\,d\vec{v}_j,$$

where

$$F_{S,\Sigma}(0,j)=\int\limits_V\cdots\int\limits_V \delta(\vec r_0-\vec r\,'_0)\delta(\vec r_j-\vec r_j)W(\vec r\,'_0,\vec r\,'_1\ldots\vec r\,'_N)\,d\vec r\,'_0 d\vec r\,'_1\ldots d\vec r\,'_N .$$

Taking into account the symmetry of the function

$$W(\vec r_0,\vec r\,'_1,\ldots\vec r\,'_N)$$

with respect to the variables $\vec r\,'_1,\ldots\vec r\,'_N$, we see that,

$$F_{S,\Sigma}(0,j)=\int\limits_V\cdots\int\limits_V \delta(\vec r_0-\vec r\,'_0)\delta(\vec r_j-\vec r\,'_1)W(\vec r\,'_0,\vec r\,'_1\ldots\vec r\,'_N)\,d\vec r\,'_0\, d\vec r\,'_1\ldots d\vec r\,'_N= \qquad (4.22)$$

$$=\int\limits_V\cdots\int\limits_V W(\vec r_0,\vec r_j\,\vec r\,'_2\ldots\vec r\,'_N)\,d\vec r\,'_2\ldots d\vec r\,'_N .$$

Let us introduce the reduced space correlation function with the usual normalization:

$$w(\vec r_0,\vec r_1)=V^2\int\limits_V\cdots\int\limits_V W(\vec r_0,\vec r_1,\vec r_2,\ldots\vec r_N)\,d\vec r_2\ldots d\vec r_N . \qquad (4.23)$$

From the translational invariance and isotropy it follows that this function has a radially symmetric form:

$$w(\vec r_0,\vec r_1)=w(|\vec r_0-\vec r_1|).$$

The limiting expression (for $V\to\infty$) of $w(r)$ possesses the property of correlation weakening:

$$w(r)\to 1$$
$$r\to\infty .$$

If the interaction between S and Σ is completely absent, this function would be equal to 1.

In the considered case of small interaction $w(r)$ is close to unity with the possible exception of the range of strong repulsion forces.

Returning to (4.22), (4.23) from (4.21) we get:

$$\overline{V(\mathcal{J}(0,j)\mathfrak{D}_{eq}(S,\Sigma)f(S))}_\Sigma =$$

$$=\frac{1}{V}\int \tilde{\mathcal{J}}(0,j)f(S)\,\Phi_0(v_0)\Phi_\Sigma(v_j)\,d\vec r_j\,d\vec v_j = \qquad (4.24)$$

$$=\overline{V(\tilde{\mathcal{J}}(0,j)f(S)\frac{\Phi_0(v_0)}{V}\mathfrak{D}_{eq}(\Sigma))}_\Sigma ,$$

where

$$\tilde{\mathcal{J}}(0,j)=\mathcal{J}(0,j)\,w(|\vec r_0-\vec r_j|) \qquad (4.25)$$

and therefore:

$$\overline{V(\mathcal{J}_{int}f(S)\mathfrak{D}_{eq}(S,\Sigma))}_\Sigma=\overline{V(\tilde{\mathcal{J}}_{int}f(S)\frac{\Phi_0(v_0)}{V}\mathfrak{D}_{eq}(\Sigma))}_\Sigma . \qquad (4.26)$$

Here

$$\tilde{\mathcal{J}}_{int}=\sum_{(1\leq j\leq N)}\tilde{\mathcal{J}}(0,j) . \qquad (4.27)$$

We thus may formulate a kind of prescription: if we replace $\mathfrak{D}_{eq}(S,\Sigma)$ by its approximation completely ignoring the correlation between S and Σ:

$$\mathcal{D}_{eq}(S,\Sigma) \to \frac{\Phi_0(v_0)}{V}\,\mathcal{D}_{eq}(\Sigma) \qquad (4.28)$$

then the "renormalization" of the interaction, that is the replacement:

$$\mathit{Л}_{int} \to \tilde{\mathit{Л}}_{int}\,, \qquad (4.29)$$

serves to correct the effect of correlation neglecting (4.28).

At least this prescription comes true when applied to the construction of the operator $\mathit{Л}_S^{(1)}$.

From (4.24) we also notice that all these expressions for $j = 1, \dots N$ are mutually equal and, therefore, by taking into account the definition of $\mathit{Л}_S^{(1)}$ we obtain:

$$\mathit{Л}_S^{(1)} f(S) = \frac{n}{\Phi_0(v_0)} \int \tilde{\mathit{Л}}(0,1)\,\Phi_\Sigma(v_1)\, f(S)\, d\vec{r}_1\, d\vec{v}_1\,. \qquad (4.30)$$

We now turn to the evaluation of the correction term in the right-hand side of (4.12)

$$\frac{1}{\Phi_0(v_0)}\,\overline{(\mathit{Л}_{int}\,\Delta_t)}_\Sigma\,. \qquad (4.31)$$

For this purpose let us go back to (4.17), (4.18). In order to extract from (4.17) the approximate expression of Δ_t which could be used in (4.31), we neglect in (4.17) the "terms of the second order of smallness", Δ_t itself being considered as having a first order of smallness.

In such an approach we first drop out from (4.17) the terms containing $\Gamma\Delta_t$.

Further the zeroth order approximation for $\mathcal{D}_{eq}(S,\Sigma)$, namely (4.28), is used.

In order to correct somehow the accepted way of approximation, we may try to apply here the previously formulated prescription and replace

$$\mathit{Л}_{int} \to \tilde{\mathit{Л}}_{int} \qquad (4.32)$$

in (4.17), (4.31).

We thus obtain the following approximate equations:

$$\frac{\partial \Delta_t^{(a)}}{\partial t} = (\mathit{Л}_S + \mathit{Л}_\Sigma)\,\Delta_t^{(a)} +$$

$$+(\tilde{\mathit{Л}}_{int}\chi_t(S) - \chi_t(S)\tilde{\mathit{Л}}_{int})\,\Phi_0(v_0)\,\mathcal{D}_{eq}(\Sigma) - \mathcal{D}_{eq}(\Sigma)\,\times \qquad (4.33)$$

$$\times\{\mathit{Л}_S^{(1)}\chi_t(S)\,\Phi_0(v_0)\}$$

$$\Delta_t^{(a)} = 0, \quad \text{for} \quad t = 0$$

and, from (4.14), since $\chi_t(S)\Phi_0(v) = f_t(S)$ we get:

$$\frac{\partial f_t(S)}{\partial t} = \mathit{Л}_S f_t(S) + \overline{(\mathit{Л}_{int}\,\Delta_t^{(a)})}_\Sigma\,. \qquad (4.34)$$

We must emphasize, however, that the accepted procedure for accounting $S-\Sigma$ correlation does not formally possess inner consistency.

In fact we retain here only some correction terms while the other, formally of the same order of smallness, are neglected.

Nevertheless, from the intuitive physical standpoint this procedure may be justified in the same way as that used by Enskog in his theory of dense hard sphere gases.

For example the correlation function w(r) becomes vanishing in the region of strong repulsive forces.

So its introduction via the replacement (4.32) serves to restore the smallness of probability of finding $|\vec{r}_0 - \vec{r}_j|$ within this region.

Going back to (4.33) we easily obtain:

$$\Delta_t^{(a)} = \int_0^t e^{(Л_S + Л_\Sigma)(t-\tau)} \big((\tilde{Л}_{int}\chi_\tau(S) - \chi_\tau(S)\tilde{Л}_{int})\times$$

$$\times\, \Phi_0(v_0)\mathfrak{D}_{eq}(\Sigma) - \qquad (4.35)$$

$$-\mathfrak{D}_{eq}(\Sigma)\{Л_S^{(1)}\chi_\tau(S)\Phi_0(v_0)\}\big)d\tau .$$

On the other hand,

$$\{Л_S^{(1)}\chi_\tau(S)\Phi_0(v_0)\} = \overline{(\tilde{Л}_{int}\chi_\tau(S)\Phi_0(v_0)\mathfrak{D}_{eq}(\Sigma))}_\Sigma$$

$$\overline{(\tilde{Л}_{int}\Phi_0(v_0)\mathfrak{D}_{eq}(\Sigma))}_\Sigma = V\,\overline{(Л_{int}\mathfrak{D}_{eq}(S,\Sigma))}_\Sigma =$$

$$= -V\,\overline{([Л_S^\circ + Л_\Sigma]\mathfrak{D}_{eq}(S,\Sigma))}_\Sigma =$$

$$= -Л_S^\circ\Phi_0(v_0) - V\,\overline{(Л_\Sigma\mathfrak{D}_{eq}(S,\Sigma))}_\Sigma = 0 .$$

Therefore (4.35) gives:

$$\Delta_t^{(a)} = \int_0^t e^{(Л_S + Л_\Sigma)(t-\tau)} \{(\tilde{Л}_{int}\chi_\tau(S) - \chi_\tau(S)\tilde{Л}_{int})\Phi_0(v_0)\times$$

$$\times\,\mathfrak{D}_{eq}(\Sigma) - \qquad (4.36)$$

$$-\mathfrak{D}_{eq}(\Sigma)\overline{((\tilde{Л}_{int}\chi_\tau(S) - \chi_\tau(S)\tilde{Л}_{int})\Phi_0(v_0)\mathfrak{D}_{eq}(\Sigma))}_\Sigma\}d\tau .$$

Since this function is symmetric with respect to the particles 1,2,... N of Σ from (4.19), (4.34) we get:

$$\frac{\partial f_t(S)}{\partial t} = (Л_S^\circ + Л_S^{(1)})f_t(S) + N\,\overline{(\tilde{Л}(0,1)\Delta_t^{(a)})}_\Sigma . \qquad (4.37)$$

The substitution of (4.36) into (4.37) leads us to the approximate equation for $\chi_t(S)$ in a closed form.

We now proceed to disentangle this equation in the case of hard sphere collision interaction (1.10).

First note that in virtue of (1.11):

$$\tilde{Л}(0,j) = w(a)\bar{T}(0,j)$$

from which it follows that:

$$Л_S^{(1)} = w(a)na^2\mathcal{L}_S , \qquad (4.38)$$

where the operator $\mathcal{L}_S$ is defined in (4.44).

Let us further notice that:

$$\tilde{Л}_{int} = w(a)Л_{int} = w(a)\sum_{(1\leq j\leq N)}\bar{T}(0,j) \qquad (4.39)$$

and consider the expression:

$$\bar{T}(0,j)\chi(S)\Phi_0(v_0)\mathcal{D}_{eq}(\Sigma)-\chi(S)\bar{T}(0,j)\Phi_0(v_0)\mathcal{D}_{eq}(\Sigma)=$$

$$=a^2\int\theta(\vec{v}_{0,j}\cdot\vec{\sigma})\vec{v}_{0,j}\cdot\vec{\sigma}\{\delta(\vec{r}_0-\vec{r}_j-a\vec{\sigma})B_{v_0,v_j}(\vec{\sigma})\chi(S)\times$$

$$\times\Phi_0(v_0)\mathcal{D}_{eq}(\Sigma)-\delta(\vec{r}_0-\vec{r}_j-a\vec{\sigma})\chi(S)B_{v_0,v_j}(\vec{\sigma})\Phi_0(v_0)\mathcal{D}_{eq}(\Sigma)\}d\vec{\sigma}.$$

Since identically:

$$B_{v_0,v_j}(\vec{\sigma})\,\Phi_0(v_0)\Phi_\Sigma(v_j)=\Phi_0(v_0)\Phi_\Sigma(v_j)$$

it follows that:

$$B_{v_0,v_j}(\vec{\sigma})\Phi_0(v_0)\mathcal{D}_{eq}(\Sigma)=\Phi_0(v_0)\mathcal{D}_{eq}(\Sigma).$$

Thus:

$$\bar{T}(0,j)\chi(S)\Phi_0(v_0)\mathcal{D}_{eq}(\Sigma)-\chi(S)\bar{T}(0,j)\Phi_0(v_0)\mathcal{D}_{eq}(\Sigma)= \tag{4.40}$$

$$=T(0,j)\chi(S)\Phi_0(v_0)\mathcal{D}_{eq}(\Sigma)=T(0,j)f(S)\mathcal{D}_{eq}(\Sigma),$$

where the operator $T(0,j)$ is defined by the relation:

$$T(0,1)=a^2\int\theta(\vec{v}_{0,1}\cdot\vec{\sigma})\vec{v}_{0,1}\cdot\vec{\sigma}\delta(\vec{r}_0-\vec{r}_1-a\vec{\sigma})\times \tag{4.41}$$
$$\times\{B_{v_0,v_1}(\vec{\sigma})-1\}d\vec{\sigma}.$$

Taking into account (4.36), (4.39), (4.40), we can write (4.37) in the form:

$$\frac{\partial f_t(S)}{\partial t}=\left(-\vec{v}_0\frac{\partial}{\partial\vec{r}_0}+na^2w(a)\mathcal{L}_S\right)f_t(S)+ \tag{4.42}$$
$$+w^2(a)\int_0^t K(t-\tau)f_\tau(S)d\tau,$$

where $K(t)$ is the operator, acting on functions $f(S)$, defined by the relation:

$$K(t)=$$

$$=\overline{N(\bar{T}(0,1)e^{(\mathcal{M}_S+\mathcal{M}_\Sigma)t}\sum_{(1\le j\le N)}[T(0,j)-\overline{(T(0,j)\mathcal{D}_{eq}(\Sigma))}_\Sigma]\mathcal{D}_{eq}(\Sigma))}_\Sigma \tag{4.43}$$

$$\mathcal{M}_S=-\vec{v}_0\frac{\partial}{\partial\vec{r}_0}+\mathcal{M}_S^{(1)}$$

and where:

$$\mathcal{M}_S^{(1)}=nw(a)\int\bar{T}(0,1)\Phi_\Sigma(v_1)\,d\vec{r}_1d\vec{v}_1=na^2w(a)\mathcal{L}_S \tag{4.44}$$

$$\mathcal{L}_S=\int\theta(\vec{v}_{0,1}\cdot\vec{\sigma})(\vec{v}_{0,1}\cdot\vec{\sigma})\{B_{v_0,v_1}(\vec{\sigma})-1\}\Phi_\Sigma(v_1)d\vec{v}_1.$$

We now wish to note the connection between the operators $\mathcal{L}_S, L_S$ from (2.50), (4.44):

$$L_S=\int\theta(\vec{v}_{0,1}\cdot\vec{\sigma})(\vec{v}_{0,1}\cdot\vec{\sigma})\Phi_\Sigma(v_1)\{B_{v_0,v_1}(\vec{\sigma})-1\}d\vec{v}_1. \tag{4.45}$$

Since *

$$\mathcal{L}_S\,\Phi_0(v_0)h(S)=\Phi_0(v_0)L_S h(S)$$

it follows that:

$$(-\vec{v}_0\frac{\partial}{\partial\vec{r}_0}+na^2w(a)\mathcal{L}_S)\Phi_0(v_0)h(S)=$$

$$=\Phi_0(v_0)(-\vec{v}_0\frac{\partial}{\partial\vec{r}_0}+na^2w(a)L_S)h(S)$$

what leads us to the identity:

$$e^{t(-\vec{v}_0\frac{\partial}{\partial\vec{r}_0}+na^2w(a)\mathcal{L}_S)}\Phi_0(v_0)h(S)=$$

$$=\Phi_0(v_0)e^{t(-\vec{v}_0\frac{\partial}{\partial\vec{r}_0}+na^2w(a)L_S)}h(S).\qquad(4.46)$$

Going back to (4.42), (4.43) we see that the present equation is essentially the same as previous ones (2.51), (2.52), the only difference, apart from the Enskog factor $w(a)$, is entirely due to the appearance of the operator $T(0,1)$ in the right-hand side of (4.43) instead of $\bar{T}(0,1)$ figuring in (2.52). Therefore, we can apply the procedure used in §§2,3.

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* Because

$$B_{v_0,v_1}(\vec{\sigma})\Phi_0(v_0)\Phi_\Sigma(v_1)h(S)=\Phi_0(v_0)\Phi_\Sigma(v_1)B_{v_0,v_1}(\vec{\sigma})h(S)$$

Thus we obtain:

$$\frac{\partial\chi_t(\vec{r}_0,\vec{v}_0)}{\partial t}=(-\vec{v}_0\frac{\partial}{\partial\vec{r}_0}+na^2w(a)L_S)\chi_t(\vec{r}_0,\vec{v}_0)+\qquad(4.47)$$

$$+w^2(a)\int_0^t Q(t-\tau)\chi_\tau(\vec{r}_0,\vec{v}_0)d\tau,$$

where:

$$Q(t)=\frac{n}{(2\pi)^3}\int d\vec{k}\int d\vec{v}_1\,\Phi_\Sigma(v_1)\bar{T}_{-k}(v_0,v_1)\times\qquad(4.48)$$

$$\times e^{-i\vec{k}\vec{r}_0}e^{-(\vec{v}_0\frac{\partial}{\partial\vec{r}_0}+na^2w(a)L_S)t}e^{i\vec{k}\vec{r}_0}U_k(t,1)T_k(v_0,v_1)$$

and where

$$T_k(v_0,v_1)=a^2\int(\vec{v}_{0,1}\cdot\vec{\sigma})\theta(\vec{v}_{0,1}\cdot\vec{\sigma})e^{-ia\vec{k}\vec{\sigma}}(B_{v_0,v_1}(\vec{\sigma})-1)d\vec{\sigma}\qquad(4.49)$$

$$\bar{T}_{-k}(v_0,v_1)=a^2\int(\vec{v}_{0,1}\cdot\vec{\sigma})\theta(\vec{v}_{0,1}\cdot\vec{\sigma})(e^{ia\vec{k}\vec{\sigma}}B_{v_0,v_1}(\vec{\sigma})-e^{-ia\vec{k}\vec{\sigma}})d\vec{\sigma}.$$

Here the operator $U_k(t;1)$ can be defined just as in §3, namely by considering the infinitesimal variations of the reduced distribution functions for the system Σ

in the case (3.40′). Then:

$$\delta F_1 (t;1) = e^{-i\vec{k}\vec{r}_1} \Phi_k (t, v_1) \delta\xi \tag{3.43}$$

and

$$\Phi_k (t, \vec{v}_1) = \Phi_\Sigma (v_1) U_k (t;1) \phi (\vec{v}_1) =$$

$$= \Phi_\Sigma (v_1) \int U_k (t; \vec{v}_1, \vec{v}_1') \phi (\vec{v}_1') d\vec{v}_1' . \tag{3.44}$$

It is interesting to note that had we introduced another operator $U_k' (t;1)$ by putting:

$$U_k' (t; \vec{v}_1, \vec{v}_1') \Phi_\Sigma (v_1') = \Phi_\Sigma (v_1) U_k (t; \vec{v}_1, \vec{v}_1') \tag{4.50}$$

then (4.47) could be written in the form:

$$\frac{\partial f_t (\vec{r}_0, \vec{v}_0)}{\partial t} = \left(-\vec{v}_0 \frac{\partial}{\partial \vec{r}_0} + n a^2 w(a) \mathfrak{L}_S\right) f_t (\vec{r}_0, \vec{v}_0) + \tag{4.51}$$

$$+ w^2(a) \int_0^t Q'(t-\tau)\, f_\tau (\vec{r}_0, \vec{v}_0) d\tau$$

with:

$$Q'(t) = \frac{n}{(2\pi)^3} \int d\vec{k} \int d\vec{v}_1\, \bar{T}_{-k} (v_0, v_1) e^{-i\vec{k}\vec{r}_0} \times$$

$$\times e^{\left(-\vec{v}_0 \frac{\partial}{\partial \vec{r}_0} + n a^2 w(a) \mathfrak{L}_S\right) t} e^{i\vec{k}\vec{r}_0} \times \tag{4.52}$$

$$\times U_k' (t;1) \Phi_\Sigma (v_1) T_k (v_0, v_1).$$

The equivalence of two representation (4.47), (4.51) is transparent due to (4.46).

It is also easy to see that the operators:

$$e^{-i\vec{k}\vec{r}_0}\, e^{-\left(\vec{v}_0 \frac{\partial}{\partial \vec{r}_0} + n a^2 w(a) L_S\right) t}\, e^{i\vec{k}\vec{r}_0}$$

and

$$U_k(t;1)$$

commute since they act on functions of different variables, namely on $h(S)$ and $F(\vec{v}_1)$.

Consider now another identity:

$$e^{-i\vec{k}\vec{r}_0}\, e^{\left(-\vec{v}_0 \frac{\partial}{\partial \vec{r}_0} + n a^2 w(a) L_S\right) t}\, e^{i(\vec{k} + \vec{\ell})\vec{r}_0} =$$

$$= e^{i\vec{\ell}\vec{r}_0}\, e^{\left(-i\vec{v}_0 (\vec{k} + \vec{\ell}) + n a^2 w(a) L_S\right) t}$$

from which it follows that (4.47) has solutions of the type:

$$\chi_t (\vec{r}_0, \vec{v}_0) = e^{i\vec{\ell}\vec{r}_0} \chi_\ell (t, \vec{v}_0), \tag{4.53}$$

where χ_ℓ satisfies the equation:

$$\frac{\partial \chi_\ell (t, v_0)}{\partial t} = \left(-i\vec{\ell}\vec{v}_0 + n a^2 w(a) L_S\right) \chi_\ell (t, \vec{v}_0) + \tag{4.54}$$

$$+ w^2(a) \int_0^t Q_\ell (t-\tau) \chi_\ell (\tau, \vec{v}_0) d\tau$$

with

$$Q_\ell(t)=\frac{n}{(2\pi)^3}\int d\vec{k}\int d\vec{v}_1\Phi_\Sigma(v_1)\bar{T}_{-k}(v_0,v_1)\times \quad (4.55)$$

$$\times U_k(t;1)e^{(-i\vec{v}_0(\vec{k}+\vec{\ell})+na^2w(a)L_S)t}T_k(v_0,v_1).$$

In particular for $\ell=0$ we have:

$$\frac{\partial\chi(t,\vec{v}_0)}{\partial t}=na^2w(a)L_S\chi(t,\vec{v}_0)+w^2(a)\int_0^1 Q_0(t-\tau)\chi(\tau,\vec{v}_0)d\tau, \quad (4.56)$$

where:

$$Q_0(t)=\frac{n}{(2\pi)^3}\int d\vec{k}\int d\vec{v}_1\Phi_\Sigma(v_1)\bar{T}_{-k}(v_0,v_1)U_k(t;1)\times \quad (4.57)$$

$$\times e^{(-i\vec{v}_0\vec{k}+na^2w(a)L_S)t}T_k(v_0,v_1).$$

In the case of arbitrary initial expression:

$$\chi_0(\vec{r}_0,\vec{v}_0)$$

we can use the Fourier representation and deal with each Fourier component by making use of (4.54).

Proceed now to obtain the hydrodynamical approximation for $U_k(t;1)$. We start with the local equilibrium distribution:

$$F_1^{(hyd.)}(t,\vec{r},\vec{v})=\rho/n\left(\frac{M}{2\pi\theta}\right)^{3/2}e^{-\frac{M(\vec{v}-\vec{u})^2}{2\theta}}\ ;\ \theta=k_BT, \quad (4.58)$$

where the quantities:

$$\rho=\rho(t,\vec{r}),\quad T=T(t,\vec{r}),\quad u=\vec{u}(t,\vec{r})$$

represent, respectively, the local particle density,temperature and velocity vector. These functions must be very slowly varying over the distances of an order of the mean free path ℓ_Σ and over the time intervals of an order to guarantee the smallness of the correction term in the right-hand side of (4.58).

Here, all we need is to consider the situation where the local equilibrium is only infinitesimally different from the overall equilibrium state:

$$\begin{aligned} &\rho(t,\vec{r})=n+\delta\rho(t,\vec{r}) \\ &T(t,\vec{r})=T+\delta T(t,\vec{r}) \\ &\vec{u}(t,\vec{r})=\delta\vec{u}(t,\vec{r}) \\ &n,T=\mathrm{Const} \end{aligned} \quad (4.59)$$

$\delta\rho$, δT, $\delta\vec{u}$ being infinitesimally small.

In such a case the main term of $\delta F_1^{(hyd.)}$ obtained by the substitution of (4.59) into (4.58) can be written in the form:

$$\delta F_1^{(hyd.)}(t,\vec{r},\vec{v})=\Phi_\Sigma(v)\left\{\frac{\delta\rho(t,\vec{r})}{n}+\frac{Mv^2-3\theta}{2\theta}\frac{\delta T(t,\vec{r})}{T}+\frac{M(\vec{v}\delta\vec{u}(t,\vec{r}))}{\theta}\right\} \quad (4.60)$$

where $\delta\rho$, δT, $\delta\vec{u}$ satisfy the well-known linearized Navier Stokes equations. The correction terms to the right-hand side of (4.60) are, roughly speaking, proportional to the gradients $\ell_\Sigma\frac{\partial}{\partial\vec{r}}$, $t_\Sigma\frac{\partial}{\partial t}$ of the variations $\delta\rho,\delta T,\delta\vec{u}$.

Due to the linearity, we may consider here the complex values for these variations because, separately, the real and the imaginary parts of them satisfy the mentioned equations.

Put:

$$\delta\rho(t,\vec{r}) = e^{-i\vec{k}\vec{r}}\, n\sigma_k(t)\delta\xi;\ \delta T(t,\vec{r}) = e^{-i\vec{k}\vec{r}}\,\tau_k(t)\delta\xi;$$

$$\delta\vec{u}(t,\vec{r}) = e^{-i\vec{k}\vec{r}}\,\vec{\Psi}_k(t)\delta\xi .$$

Then:

$$\delta F_1^{(hyd.)}(t,\vec{r},\vec{v}) = \Phi_\Sigma(v)e^{-i\vec{k}\vec{r}}\{\sigma_k(t) + \frac{Mv^2-3\theta}{2\theta}\frac{\tau_k(t)}{T} + \frac{M(\vec{v}\vec{\Psi}_k(t))}{\theta}\}\delta\xi \qquad (4.61)$$

where in virtue of the linearized Navier Stokes equations we have:

$$\frac{1}{k}\frac{\partial\sigma_k}{\partial t} = i(\vec{c}\cdot\vec{\Psi}_k)$$

$$\frac{1}{k}\frac{\partial\vec{\Psi}_k}{\partial t} = i\frac{c_0^2}{\gamma}\vec{c}\sigma_k - \nu k\vec{\Psi}_k - k(D_\ell-\nu)\vec{c}(\vec{e}\vec{\Psi}_k) + \frac{c_0^2\alpha}{\gamma}i\vec{c}\tau_k \qquad (4.62)$$

$$\frac{1}{k}\frac{\partial\tau_k}{\partial t} = i\frac{\gamma-1}{\alpha}\vec{c}\cdot\vec{\Psi}_k - \gamma D_T k\tau_k; \qquad \vec{e} = \frac{\vec{k}}{k},$$

where c_0 is the long wavelength sound velocity, $\gamma = C_p/C_v$ is the ratio of the specific heats per particle at constant pressure and density, respectively, $\alpha = \frac{\partial p}{\partial T}(n\frac{\partial p}{\partial n})^{-1}$ is the termal expansion coefficient, $p = p(n,T)$ is the equilibrium pressure, ν is the kinematic viscosity, D_T is the termal diffusivity, $D_\ell = \frac{4}{3}\nu + \zeta(nM)^{-1}$ and ζ is the bulk viscosity.

As is well-known (4.62) have solutions corresponding to five modes - two shear

modes, one heat mode and two sound modes.

The time dependence for these modes is given, respectively, by the exponentially decreasing functions:

$$e^{-\nu k^2 t} \qquad \text{(shear, or viscosity modes)}$$

$$e^{-D_T k^2 t} \qquad \text{(heat mode)} \qquad (4.63)$$

$$e^{-(\pm ic_0 k + \frac{1}{2}\Gamma_S k^2)t} \qquad \text{(sound modes)},$$

where

$$\Gamma_S = D_\ell + (\gamma-1)D_T .$$

Therefore any solution of (4.62) and hence, the expression appearing in brackets in the right-hand side of (4.61), considered as functions of t are linear combinations from (4.63).

Notice also that ν, D_T, Γ_S are of an order of $\ell_\Sigma^2 t_\Sigma^{-1}$. We thus see that these functions change very slowly with t/t_Σ when k is sufficiently small

$$k\ell_\Sigma \ll 1,\quad kc_0 t_\Sigma \ll 1. \qquad (4.64)$$

Let us turn again to the variations of reduced distribution functions around the overall statistical equilibrium in case (3.40′). We first consider (3.43), (3.44) and make the following statement:

For sufficiently small k (4.64) the function $\Phi_k(t,\vec{v})$ rapidly approaches towards the expression:

$$\Phi_\Sigma(v)\{\sigma_k(t) + \frac{Mv^2-3\theta}{2\theta}\frac{\tau_k(t)}{T} + \frac{M}{\theta}(\vec{v}\cdot\vec{\Psi}_k(t)) + \qquad (4.65)$$

$$+ \text{correction term}\}$$

so, that starting from a certain relaxation time $t_{rel.} \gg t_\Sigma$ $\Phi_k(t,\vec{v})$ practically coincides with (4.65) and the hydrodynamic regime becomes established.

Here the correction term contains a factor k and depends upon time as a linear combination of functions of the type (4.63).

Due to (3.41) this statement leads us to the conclusion that asymptotically:

$$\int U_k(t,\vec{v},\vec{v}')\,\phi(\vec{v}')\,d\vec{v}' =$$

$$= \sigma_k(t) + \frac{Mv^2 - 3\theta}{2\theta}\,\frac{\tau_k(t)}{T} + \frac{M}{\theta}(\vec{v}\cdot\vec{\Psi}_k(t)) + \text{correction term} \tag{4.66}$$

for

$$t > t_{rel.} \gg t_\Sigma \qquad k \ll \frac{1}{\ell_\Sigma},\ \frac{1}{c_0 t_\Sigma}.$$

It is to be emphasized that in the situation where a kinetic equation, for example the Boltzmann-Enskog equation or the Enskog equation for dense gases, can be used, the mentioned statement can be formally deduced.

Really, when we have such a kinetic equation and observe that Φ_k is proportional to δF_1, all we need is to examine the corresponding linearized equation verified by Φ_k. From this linearized kinetic equation not only the validity of the announced statement follows. It is also possible to deduce the linearized Navier Stokes equation and effectively compute their coefficients. In fact such a program was realized since the classical work of Chapman and Enskog.

But it is to be stressed that when the kinetic equation method fails, as for example in the case of fluids, our statement is only the usually accepted assumption and the coefficients in the Navier Stokes equations must be determined by experiment.

Before going to compute the main term in (4.66), we shall make one rather an obvious remark concerning the integrals of the type:

$$\int_0^{k_{max}} e^{-\xi k^2 t}(1 + a_1 k + a_2 k^2 + \dots)k^2 dk, \quad \xi > 0 \tag{4.67}$$

which enter into the expression of $Q_0(t)$.

By a change of variables:

$$k = \frac{q}{\sqrt{\xi t}}$$

(4.67) reduces to

$$\frac{1}{(\xi t)^{3/2}} \int_0^{\sqrt{\xi t}\,k_{max}} e^{-q^2}\left(1 + a_1\frac{q}{\sqrt{\xi t}} + a_2\frac{q^2}{\xi t} + \dots\right)q^2 dq.$$

So, for large t, we asymptotically obtain

$$\frac{1}{(\xi t)^{3/2}} \int_0^{\infty} e^{-q^2} q^2 dq = \frac{\sqrt{\pi}}{4(\xi t)^{3/2}} \tag{4.67'}$$

and, we see that the correction terms $a_1 k + a_2 k^2 \dots$ in (4.67) do not contribute to this result.

The same situation also arizes in the case of more complicated integrals with which we have to deal when considering $Q_\ell(t)$.

For this reason we need to compute only the main terms of the coefficients appearing in (4.66) with the functions (4.63) and neglect there the terms of the order $O(k)$.

Let us now proceed to obtain the explicit form for the right-hand side of (4.66). We first notice that here it is supposed that $\sigma_k(t)$, $r_k(t)$, $\vec{\Psi}_k(t)$ satisfy equations (4.62) but it has not yet been specified which initial values $\sigma_k(0)$, $r_k(0)$, $\vec{\Psi}_k(0)$ are to be chosen.

Because of (4.66) we only know that these initial values are the linear functionals of $\phi(\vec{v})$.

In order to solve this problem and to determine these linear functionals, we shall apply the ideas of the paper /12/ by M.H.Ernst, E.H.Hauge, J.M.J. van Leeuwen.

Consider the variations of particle density, momentum density and energy density.

We have:

$$\delta\rho(t,\vec{r}) = n\int \delta F_1(t,\vec{r},\vec{v})d\vec{v} = e^{-i\vec{k}\vec{r}}\, n\int \Phi_k(t,\vec{v})d\vec{v}$$

$$\delta\vec{j}(t,\vec{r}) = nM\int \vec{v}\,\delta F_1(t,\vec{r},\vec{v})\,d\vec{v} = e^{-i\vec{k}\vec{r}}\, nM\int \vec{v}\Phi_k(t,\vec{v})d\vec{v}$$

$$\delta E(t,\vec{r}) = n\frac{M}{2}\int v^2\delta F_1(t,\vec{r},\vec{v})\,d\vec{v} + \frac{n^2}{2}\int \Phi(\vec{r}-\vec{r}')\delta f_2(t,\vec{r},\vec{r}')d\vec{r}',$$

where:

$$\delta f_2(t,\vec{r},\vec{r}') = \delta\int F_2(t,\vec{r},\vec{v},\vec{r}',\vec{v}')d\vec{v},d\vec{v}'. \qquad (4.68)$$

Remind that we here consider the case (3.40′).

So, the variations of every reduced distributed function has the form:

$$\delta F_s(t,\vec{r}_1,\vec{v}_1,\ldots\vec{r}_s,\vec{v}_s) = e^{-i\vec{k}\vec{r}_1}\,\Phi_k^{(s)}(t,\vec{r}_1,\vec{v}_1,\ldots\vec{r}_s,\vec{v}_s)\delta\xi, \qquad (4.69)$$

where $\Phi_k^{(s)}$ are invariant with respect to the space translations. We thus may write

$$\delta f_2(t,\vec{r},\vec{r}') = e^{-i\vec{k}\vec{r}_1}\,\bar{\Phi}_k^{(2)}(t,\vec{r}-\vec{r}')\delta\xi \qquad (4.70)'$$

$$\bar{\Phi}_k^{(2)}(t,\vec{r}_1-\vec{r}_2) = \int \Phi_k^{(2)}(t,\vec{r}_1,\vec{v}_1,\vec{r}_2,\vec{v}_2)d\vec{v}_1 d\vec{v}_2.$$

Hence:

$$\delta\rho(t,\vec{r}) = e^{-i\vec{k}\vec{r}}\, n\int \Phi_k(t,\vec{v})d\vec{v}\delta\xi$$

$$\delta\vec{j}(t,\vec{r}) = e^{-i\vec{k}\vec{r}}\, nM\int \vec{v}\,\Phi_k(t,\vec{v})d\vec{v}\delta\xi \qquad (4.71)$$

$$\delta E(t,\vec{r}) = e^{-i\vec{k}\vec{r}}\left\{\frac{nM}{2}\int v^2\Phi_k(t,\vec{v})d\vec{v} + \frac{n^2}{2}\int \Phi(\vec{r}-\vec{r}')\bar{\Phi}_k^{(2)}(t,\vec{r}-\vec{r}')d\vec{r}'\right\}\delta\xi.$$

Note, that in the limiting case $k \to 0$ we should have the space homogeneity and the variations (4.71) of particles, momentum and energy would be the exact integrals of motion.

In the considered case of sufficiently small k, we can examine the time derivative $\partial/\partial t$ of (4.71).

By using the hierarchy of equations for δF_s and by taking into account (4.69), it is possible to see that these derivatives are proportional to k.

Therefore the quantities (4.71) are so to say "quasi-integrals", i.e., they are practically conserved on the larger time interval, the smaller k would be.

Let us fix certain time $t_0 \geq t_{rel.}$ when the transition to the hydrodynamic regime is already achieved.

We then may find such k_0 that up to the terms of the order $O(k)$:

$$\int \Phi_k(t_0,\vec{v})\,d\vec{v} = \int \Phi_k(0,\vec{v})\,d\vec{v}$$

$$\int \vec{v}\Phi_k(t_0,\vec{v})\,d\vec{v} = \int \vec{v}\,\Phi_k(0,\vec{v})\,d\vec{v} \qquad (4.72)$$

$$\frac{nM}{2}\int v^2\Phi_k(t_0,\vec{v})\,d\vec{v} + \frac{n^2}{2}\int\Phi(\vec{r}-\vec{r}')\bar{\Phi}_k^{(2)}(t_0,\vec{r}-\vec{r}')\,d\vec{r}' =$$

$$= \frac{nM}{2}\int v^2\Phi_k(0,\vec{v})\,d\vec{v} + \frac{n^2}{2}\int\Phi(\vec{r}-\vec{r}')\bar{\Phi}_k^{(2)}(0,\vec{r}-\vec{r}')\,d\vec{r}'$$

for $k \leq k_0$.

On the other hand, since at the moment t_0 the hydrodynamic regime is established, we have:

$$\delta\rho(t_0,\vec{r}) = e^{-i\vec{k}\vec{r}}\, n\sigma_k(t_0)\,\delta\xi$$

$$\delta\vec{j}(t_0,\vec{r}) = e^{-i\vec{k}\vec{r}}\, nM\vec{\Psi}_k(t_0)\,\delta\xi \qquad (4.73)$$

$$\delta E(t_0,\vec{r}) = \frac{\partial\epsilon(n,T)}{\partial n}\,\delta\rho(t_0,\vec{r}) + \frac{\partial\epsilon(n,T)}{\partial T}\,\delta T(t_0,\vec{r}) =$$

$$= e^{-i\vec{k}\vec{r}}\left\{n\,\frac{\partial\epsilon(n,T)}{\partial n}\,\sigma_k(t_0) + \frac{\partial\epsilon(n,T)}{\partial T}\,\tau_k(t_0)\right\}\delta\xi\,,$$

where $\epsilon(n,T)$ is the equilibrium energy density.

We further notice that because $\sigma_k(t)$, $\tau_k(t)$, $\vec{\Psi}_k(t)$ are linear combinations of the functions (4.63) we may write asymptotically:

$$\sigma_k(t_0) = \sigma_k(0),\quad \tau_k(t_0) = \tau_k(0),\quad \vec{\Psi}_k(t_0) = \vec{\Psi}_k(0) \qquad (4.74)$$

for

$$k \ll \frac{1}{ct_0},\ \frac{1}{\sqrt{D_T t_0}},\ \frac{1}{\sqrt{\Gamma_S t_0}},\ \frac{1}{\sqrt{\nu t_0}}\,.$$

Therefore, due to (4.72) and to the asymptotic equality of expressions (4.71), (4.73) at the moment t_0, for sufficiently small k, we get up to the terms $O(k)$:

$$\sigma_k(0) = \int \Phi_k(0,\vec{v})\,d\vec{v}$$

$$\vec{\Psi}_k(0) = \int \vec{v}\,\Phi_k(0,\vec{v})\,d\vec{v} \qquad (4.75)$$

$$n\,\frac{\partial\epsilon(n,T)}{\partial n}\,\sigma_k(0) + \frac{\partial\epsilon(n,T)}{\partial T}\,\tau_k(0) =$$

$$= \frac{nM}{2}\int v^2\,\Phi_k(0,\vec{v})\,d\vec{v} + \frac{n^2}{2}\int\Phi(\vec{r}-\vec{r}')\bar{\Phi}_k^{(2)}(0,\vec{r}-\vec{r}')\,d\vec{r}'$$

for $k \leq k_1$,

where

$$k_1 \leq k_0;\quad k_1 \ll \frac{1}{ct_0},\ \frac{1}{\sqrt{D_T t_0}},\ \frac{1}{\sqrt{\Gamma_S t_0}},\ \frac{1}{\sqrt{\nu t_0}}.$$

In (4.75):

$$\frac{\partial\epsilon(n,T)}{\partial n} = nC_v\,, \qquad (4.76)$$

where C_v is the heat capacity per particle at constant density.

Let us now make some comments concerning $\epsilon(n,T)$. We have:

$$\epsilon(n,T) = \frac{3\theta}{2}\,n + \frac{n^2}{2}\int\Phi(r)\,f_2^{(eq)}(r)\,d\vec{r}, \qquad (4.77)$$

where

$$f_2^{(eq)}(\vec{r}_1-\vec{r}_2)=\int F_2^{(eq)}(1,2)\,d\vec{v}_1\,d\vec{v}_2$$

is the second reduced space distribution function for the statistical equilibrium.

Of course $f_2^{(eq)}$ depends upon n and T. It will be useful to introduce the chemical potential:

$$\mu=\mu(n,T);\quad n=n(\mu,T).$$

Then by using the equilibrium fluctuation properties, we find:

$$\theta/n\left(\frac{\partial n}{\partial\mu}\right)_T=1+n\int g_2(\vec{r})\,d\vec{r};\quad g_2(\vec{r})=f_2^{(eq)}(\vec{r})-1$$

$$\theta/n\left(\frac{\partial}{\partial\mu}n^2 f_2^{(eq)}(\vec{r}_1-\vec{r}_2)\right)_T=2nf_2^{(eq)}(\vec{r}_1-\vec{r}_2)+ \tag{4.78}$$

$$+n^2\int[f_3^{(eq)}(\vec{r}_1-\vec{r}_2,\vec{r}_1-\vec{r}_3)-f_2^{(eq)}(\vec{r}_1-\vec{r}_2)]\,d\vec{r}_3$$

and thus:

$$n\frac{\partial\epsilon(n,T)}{\partial n}=\frac{3\theta}{2}n+\frac{n^3}{2}\int\Phi(\vec{r}_1-\vec{r}_3)[2f_2^{(eq)}(\vec{r}_1-\vec{r}_2)+$$

$$+n\int\{f_3^{(eq)}(\vec{r}_1-\vec{r}_2,\vec{r}_1-\vec{r}_3)-f_2^{(eq)}(\vec{r}_1-\vec{r}_2)\}d\vec{r}_3]\,d\vec{r}_2\left(\theta\frac{\partial n}{\partial\mu}\right)_T^{-1}.$$

We now can present the 3rd equation from (4.75) in the form:

$$C_v\,\tau_k(0)=\int\frac{Mv^2-3\theta}{2}\Phi_k(0,\vec{v})\,d\vec{v}+\frac{n}{2}\int\Phi(\vec{r}_1-\vec{r}_2)\{\bar{\Phi}_k^{(2)}(0,\vec{r}_1-\vec{r}_2)-$$

$$-[2f_2^{(eq)}(\vec{r}_1-\vec{r}_2)+n\int(f_3^{(eq)}(\vec{r}_1-\vec{r}_2,\vec{r}_1-\vec{r}_3)-f_2^{(eq)}(\vec{r}_1-\vec{r}_2)\,d\vec{r}_3]\times \tag{4.79}$$

$$\times\sigma_k(0)\,n\left(\theta\frac{\partial n}{\partial\mu}\right)_T^{-1}\}\,d\vec{r}_2\,.$$

In order to find the expressions for $\Phi_k(0,\vec{v})$; $\bar{\Phi}_k^{(2)}(0,\vec{r}_1-\vec{r}_2)$ we shall make use of our previous results (§3).

So, from (3.42), (3.43) we get:

$$\Phi_k(0,\vec{v})=\Phi_\Sigma(v)\{\phi(\vec{v})+n\int g_2(\vec{r})e^{i\vec{k}\vec{r}}\,d\vec{r}\int\phi(\vec{v}')\Phi_\Sigma(\vec{v}')\,d\vec{v}'\} \tag{4.80}$$

Therefore:

$$\int\frac{Mv^2-3\theta}{2}\Phi_k(0,\vec{v})\,d\vec{v}=\int\frac{Mv^2-3\theta}{2}\Phi_\Sigma(v)\phi(\vec{v})\,d\vec{v} \tag{4.81}$$

and (4.75) yields:

$$\sigma_k(0)=(1+n\int g_2(\vec{r})e^{i\vec{k}\vec{r}}\,d\vec{r})\int\phi(\vec{v})\Phi_\Sigma(v)\,d\vec{v}$$
$$\vec{\Psi}_k(0)=\int\vec{v}\,\Phi_\Sigma(v)\phi(\vec{v})\,d\vec{v}. \tag{4.82}$$

Note that the equilibrium correlation function $g_2(\vec{r})$ practically vanishes when r becomes much larger than the correlation length.

If the equilibrium state of Σ is not close to a critical point, what we here tacitly assume, then this length is of an order of the range a_Σ of interparticle forces. In case of fluids ℓ_Σ is of an order of a_Σ, in case of gases $a_\Sigma \ll \ell_\Sigma$.

Anyway since $k \ll \frac{1}{\ell_\Sigma}$ we see that the following asymptotic equality:

$$\int g_2(\vec r)e^{i\vec k\vec r}\,d\vec r = \int g_2(\vec r)\,d\vec r$$

holds up to the terms of the order $O(k^2)$.

Therefore (4.82) yields, in the accepted approximation:

$$\sigma_k(0) = \theta/n\left(\frac{\partial n}{\partial \mu}\right)_T \int \phi(\vec v)\Phi_\Sigma(v)\,d\vec v \tag{4.83}$$

To obtain an expression for $\bar\Phi_k^{(2)}(0,\vec r_1-\vec r_2)$, we shall start from (3.40), (3.40′). These formulae give:

$$\delta F_2(0;1,2) = \Phi_\Sigma(v_1)\Phi_\Sigma(v_2)\{(e^{-i\vec k\vec r_1}\phi(\vec v_1)+e^{-i\vec k\vec r_2}\phi(\vec v_2))f_2(\vec r_1-\vec r_2)+$$

$$+n\int[f_3(\vec r_1-\vec r_2,\vec r_1-\vec r_3)-f_2(\vec r_1-\vec r_2)]e^{-i\vec k\vec r_3}\,d\vec r_3\int\phi(\vec v)\Phi_\Sigma(\vec v)d\vec v\}\delta\xi.$$

Due to (4.70), it follows that:

$$\bar\Phi_k^{(2)}(0,\vec r_1-\vec r_2) = \{(1+e^{i\vec k(\vec r_1-\vec r_2)})f_2(\vec r_1-\vec r_2)+$$

$$+n\int[f_3(\vec r_1-\vec r_2,\vec r_1-\vec r_3)-f_2(\vec r_1-\vec r_2)]e^{i\vec k(\vec r_1-\vec r_3)}\,d\vec r_3\}\times \tag{4.84}$$

$$\times\int\phi(\vec v)\Phi_\Sigma(\vec v)\,d\vec v.$$

This expression is needed here only to calculate the integral

$$\frac{n}{2}\int\Phi(\vec r_1-\vec r_2)\vec\Phi_k^{(2)}(0,\vec r_1-\vec r_2)\,d\vec r_2.$$

Therefore the relevant distances $|\vec r_1-\vec r_2|$ are of an order of the effective radius a_Σ of interparticle forces, and we may replace in (4.84) the factor $1+e^{i\vec k(\vec r_1-\vec r_2)}$ by 2.

Furthermore when $|\vec r_1-\vec r_3| \gg a_\Sigma$ and, hence, also $|\vec r_2-\vec r_3|\gg a_\Sigma$ the form:

$$f_3^{(eq)}(\vec r_1-\vec r_2,\vec r_1-\vec r_3)-f_2^{(eq)}(\vec r_1-\vec r_2)$$

characterizing the correlation between particles at $\vec r_3$ and those at $\vec r_1$, $\vec r_2$ is practically zero.

So, in our approximation:

$$\frac{n}{2}\int\Phi(\vec r_1-\vec r_2)\bar\Phi_k^{(2)}(0,\vec r_1-\vec r_2)\,d\vec r_2 =$$

$$=\frac{n}{2}\int\Phi(\vec r_1-\vec r_2)\{2f_2^{(eq)}(\vec r_1-\vec r_2)+n\int[f_3^{(eq)}(\vec r_1-\vec r_2,\vec r_1-\vec r_3)- \tag{4.85}$$

$$-f_2^{(eq)}(\vec r_1-\vec r_2)]d\vec r_3\}d\vec r_2\int\phi(\vec v)\Phi_\Sigma(\vec v)\,d\vec v.$$

But due to (4.83)

$$n\sigma_k(0)\left(\theta\frac{\partial n}{\partial\mu}\right)_T^{-1} = \int\phi(\vec v)\Phi_\Sigma(\vec v)\,d\vec v$$

and thus the second term in the right-hand side of (4.79) is equal to zero. Note, also, that

$$\left(\frac{\partial n}{\partial\mu}\right)_T = n\left(\frac{\partial p}{\partial n}\right)_T^{-1}.$$

Summing up our results (4.79), (4.82), (4.83), (4.85) we finally can write down the adequate initial values computed up to the order $O(k)$:

$$\sigma_k(0) = \theta\left(\frac{\partial p}{\partial n}\right)_T^{-1}\int\Phi_\Sigma(v')\phi(\vec v')\,d\vec v'$$

$$r_k(0) = C_v^{-1} \int \Phi_\Sigma(v') \frac{Mv'^2 - 3\theta}{2} \phi(\vec{v}')d\vec{v}'$$
$$\vec{\Psi}_k(0) = \int \Phi_\Sigma(v')\vec{v}'\phi(\vec{v}')d\vec{v}'. \qquad (4.86)$$

It is now possible to calculate the corresponding solutions of (4.62). The unit vector $\vec{e} = \frac{\vec{k}}{k}$ enters into these equations.

Let us also introduce two other unit vectors $\vec{e}_1$, $\vec{e}_2$ in such a way that three vectors $\vec{e}$, $\vec{e}_1$, $\vec{e}_2$ were mutually orthogonal. Then:

$$\vec{\Psi}_k = \vec{e}_1(\vec{e}_1\vec{\Psi}_k) + \vec{e}_2(\vec{e}_2\vec{\Psi}_k) + \vec{e}(\vec{e}\vec{\Psi}_k) \qquad (4.87)$$

and from (4.62) it follows:

$$\frac{d}{dt}(\vec{e}_j\vec{\Psi}_k(t)) = -\nu k^2(\vec{e}_j\,\vec{\Psi}_k(t)), \quad j=1,2.$$

Hence:

$$(\vec{e}_j\,\vec{\Psi}_k(t)) = e^{-\nu k^2 t}(\vec{e}_j\,\vec{\Psi}_k(0)) =$$
$$= e^{-\nu k^2 t}\int \Phi_\Sigma(v')(\vec{e}_j\cdot\vec{v}')\phi(\vec{v}')d\vec{v}'. \qquad (4.88)$$
$$j=1,2.$$

It remains to determine three functions:

$$\sigma_k(t),\ s_k(t) = (\vec{e}\,\vec{\Psi}_k(t)),\ r_k(t). \qquad (4.89)$$

Note that from (4.62) it follows

$$1/k\,\frac{\partial\sigma_k}{\partial t} = is_k$$

$$1/k\,\frac{\partial s_k}{\partial t} = i\frac{c_0^2}{\gamma}\sigma_k - D_\ell k s_k + i\frac{c_0^2 a}{\gamma} r_k$$
$$1/k\,\frac{\partial r_k}{\partial t} = i\frac{\gamma-1}{a}s_k - \gamma D_T k r_k. \qquad (4.90)$$

In order to solve these equations we shall define three independent combinations A_H, $A_\pm$ from (4.89) in such a way that (4.90) yield:

$$\frac{\partial A(t)}{\partial t} = -\Omega A(t)$$

and

$$A(t) = e^{-\Omega t}A(0).$$

We shall calculate Ω so that the term proportional to k^2 would be included since just this term is responsible for the damping of (4.89).

On the other hand, the coefficients of the linear forms A_H, $A_\pm$ are to be calculated by neglecting the terms of the order $O(k)$ since the initial values of (4.89) themselves were calculated only up to this order of smallness:

Then we can obtain:

$$A_H(t) = \gamma^{-1}((\gamma-1)\sigma_k(t) - a r_k(t)); \quad \Omega_H = D_T k^2$$
$$A_\pm(t) = \frac{1}{2}\gamma^{-1}(\sigma_k(t) + a r_k(t)) \mp \frac{1}{2}c_0^{-1}s_k(t); \qquad (4.91)$$
$$\Omega_\pm = \pm ic_0 k + \frac{1}{2}\Gamma_S k^2$$

and inversely:

$$\sigma_k(t) = A_H(t) + A_+(t) + A_-(t)$$

$$\tau_k(t) = -\alpha^{-1} A_H(t) + (\gamma-1)\alpha^{-1}(A_+(t) + A_-(t))$$

$$s_k(t) = c_0 (A_-(t) - A_+(t)).$$

Therefore

$$\sigma_k(t) = e^{-\Omega_H t} A_H(0) + e^{-\Omega_+ t} A_+(0) + e^{-\Omega_- t} A_-(0)$$

$$\tau_k(t) = -\alpha^{-1} e^{-\Omega_H t} A_H(0) + (\gamma-1)\alpha^{-1} e^{-\Omega_+ t} A_+(0) + \qquad (4.92)$$

$$+ (\gamma-1)\alpha^{-1} e^{-\Omega_- t} A_-(0)$$

$$(\vec{e}\vec{\Psi}_k(t)) = s_k(t) = c_0 A_-(0) e^{-\Omega_- t} - c_0 A_+(0) e^{-\Omega_+ t},$$

Due to (4.86), (4.91) we see that here:

$$A_H(0) = \int \{(1-\gamma^{-1})\theta \left(\frac{\partial p}{\partial n}\right)_T^{-1} - \gamma^{-1} \alpha\, C_v^{-1} \frac{Mv'^2 - 3\theta}{2}\} \Phi_\Sigma(v')\phi(\vec{v}')d\vec{v}'$$

$$A_\pm(0) = \int \{ \frac{1}{2}\gamma^{-1}\theta \left(\frac{\partial p}{\partial n}\right)_T^{-1} + \frac{1}{2}(\gamma C_v)^{-1} \alpha \frac{Mv'^2 - 3\theta}{2} \mp \qquad (4.93)$$

$$- \frac{1}{2} c_0^{-1} (\vec{e}\vec{v}')\} \Phi_\Sigma(v')\phi(\vec{v}')d\vec{v}'.$$

By inserting (4.87), (4.88), (4.92) into (4.66), we get:

$$\int U_k(t,\vec{v},\vec{v}')\phi(\vec{v}')d\vec{v}' =$$

$$= e^{-\nu k^2 t} \frac{M}{\theta} (\vec{v}_1\vec{e}_1) \int (\vec{v}_1'\vec{e}_1) \Phi_\Sigma(v_1')\phi(\vec{v}_1')d\vec{v}_1' +$$

$$+ e^{-\nu k^2 t} \frac{M}{\theta} (\vec{v}_1\vec{e}_2) \int (\vec{v}_1'\vec{e}_2) \Phi_\Sigma(v_1')\phi(\vec{v}_1')d\vec{v}_1' +$$

$$+ e^{-\Omega_H t} \{1 - \frac{Mv^2 - 3\theta}{2\theta}(\alpha T)^{-1}\} A_H(0) +$$

$$+ e^{-\Omega_+ t} \{1 + \frac{Mv^2 - 3\theta}{2\theta}(\alpha T)^{-1}(\gamma-1) - \frac{M}{\theta} c_0 (\vec{v}\cdot\vec{e})\} A_+(0) +$$

$$+ e^{-\Omega_- t} \{1 + \frac{Mv^2 - 3\theta}{2\theta}(\alpha T)^{-1}(\gamma-1) + \frac{M}{\theta} c_0 (\vec{v}\cdot\vec{e})\} A_-(0). \qquad (4.94)$$

In order to shorten the notations let us put:

$$\theta_1^{(L)}(\vec{e},\vec{v}) = \theta_1^{(R)}(\vec{e},\vec{v}) = \sqrt{\frac{M}{\theta}} (\vec{e}_1\vec{v});$$

$$\theta_2^{(L)}(\vec{e},\vec{v}) = \theta_2^{(R)}(\vec{e},\vec{v}) = \sqrt{\frac{M}{\theta}} (\vec{e}_2\vec{v});$$

$$\omega_1(k) = \omega_2(k) = \nu k^2$$

$$\theta_3^{(L)}(\vec{e},\vec{v}) = \left(\frac{Mv^2 - 3\theta}{2\theta} - \alpha T\right)\left(\frac{k_B}{C_p}\right)^{1/2}$$

$$\theta_3^{(L)}(\vec{e},\vec{v}) = \left(\frac{Mv^2 - 3\theta}{2\theta} - (\gamma-1)\frac{nC_v}{\left(\frac{\partial p}{\partial T}\right)_n}\right)\left(\frac{k_B}{C_p}\right)^{1/2}$$

$$\omega_3(k) = \Omega_H = D_T k^2$$

$$\theta_{\binom{4}{5}}^{(L)}(\vec{e},\vec{v}) = \left(1 + \frac{Mv^2 - 3\theta}{2\theta}(\alpha T)^{-1}(\gamma-1) \mp \frac{M}{\theta} c_0 (\vec{v}\vec{e})\right)\left(\frac{1}{2}\right)^{1/2}$$

$$\theta^{(R)}_{\substack{4\\(5)}}(\vec{e},\vec{v})=(\theta\gamma^{-1}(\frac{\partial p}{\partial n})^{-1}+(\gamma C_v)^{-1}a\frac{Mv^2-3\theta}{2}\mp\frac{1}{c_0}(\vec{v}\vec{e}))(\frac{1}{2})^{1/2} \tag{4.95}$$

$$\omega_4(k)=\Omega_+=ic_0k+\frac{1}{2}\Gamma_S k^2,\quad \omega_5(k)=\Omega_-=-ic_0k+\frac{1}{2}\Gamma_S k^2;$$

where k_B is the Boltzmann constant.

Then (4.93), (4.94) yield:

$$\int U_k(t,\vec{v},\vec{v}')\phi(\vec{v}')d\vec{v}' = \sum_{(1\leq j\leq 5)}\theta_j^{(L)}(\vec{e},\vec{v})e^{-\omega_j(k)t}\int\theta_j^{(R)}(\vec{e},\vec{v}')\Phi_\Sigma(v')\phi(\vec{v}')d\vec{v}'. \tag{4.96}$$

$$t>t_{rel}\ ;\ k<k_1$$

It is to be emphasized that in cases when the kinetic equations of Boltzmann type or of Enskog type* are used, the same result (4.96) could be obtained.

Strictly speaking not the complete non linear kinetic equations are needed but only their linearized versions.

These linearized equations lead us to the form (4.96) when the terms proportional to k^2 are calculated for $\omega_j(k)$, while in calculating the coefficients $\theta_j^{(L)}, \theta_j^{(R)}$ the terms of the order k are neglected.

Of course in such an approach the equilibrium and transport (ν, D_T, Γ_S) coefficients would have the values corresponding to the approximation on which the kinetic equation is founded.

We now can use (4.96) for reducing our equations (4.54), (4.56) to an explicit form.

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* For moderately dense hard sphere gases.

Consider first the expression

$Q_\ell(t)\chi(\vec{v}_0)$

and note that it contains the operator:

$$e^{(-i\vec{v}_0\vec{\lambda}+na^2w(a)L_S)t}\ ;\quad \vec{\lambda}=\vec{k}+\vec{\ell}$$

acting on functions of $\vec{v}_0$.

Introduce the scalar product for such functions:

$$(g,h)=\int\Phi_0(v_0)g(\vec{v}_0)h(\vec{v}_0)\,d\vec{v}_0 \tag{4.97}$$

the corresponding Hilbertian scalar product being:

$$(g,h)_H=(g^*,h). \tag{4.98}$$

Due to its definition (4.45) the operator

$$na^2w(a)L_S \tag{4.99}$$

is symmetric and hermitian:

$$(g,L_S h)=(L_S g,h)$$

$$(g,L_S h)_H=(L_S g,h)_H.$$

It is also well-known that its spectrum consists of a negative part and of nondegenerate zero eigenvalue, corresponding to the normalized eigenfunction $\phi(v)=1$:

$$L_S\cdot 1=0.$$

The gap between negative part and zero point for (4.99) is of the order t_0^{-1}, where

$$t_0=\frac{(m/\pi\theta)^{1/2}}{4na^2w(a)} \tag{4.100}$$

represents the Enskog approximation for the mean free time for S.

Of course the eigenfunctions $\Psi(\vec{v})$ of (4.99) corresponding to its negative eigenvalues are orthogonal to 1:

$$\int \Phi_0(v)\Psi(\vec{v})d\vec{v}=0. \qquad (4.101)$$

The operator

$$E_\lambda = -i\vec{v}\vec{\lambda} + na^2 w(a) L_S$$

is evidently not hermitian but it conserves the symmetry property:

$$(g,E_\lambda h) = (E_\lambda g,h).$$

Consider the eigenfunction Ψ_λ

$$E_\lambda \Psi_\lambda(\vec{v}) = -\omega_0(\lambda)\Psi_\lambda(\vec{v})$$

for which

$$\omega_0(\lambda) \to 0, \quad \text{when } \lambda\to 0.$$

By using ordinary perturbation theory we easily find:

$$\Psi_\lambda(\vec{v}) = 1+\frac{1}{na^2 w(a)} L_S^{-1}(\vec{\lambda},\vec{v})+O(\lambda^2)$$

$$\omega_0(\lambda) = D_0\lambda^2 + O(\lambda^2) \qquad (4.102)$$

$$D_0 = -\int\Phi_0(v)v_x L_S^{-1} v_x d\vec{v}(na^2 w(a))^{-1}.$$

Notice here that the functions

$$v_x, v_y, v_z$$

belong to the class (4.101) where the inverse operator L_S^{-1} is well defined.

In the first Enskog approximation:

$$D_0 = \frac{3}{8na^2 w(a)}\left(\frac{m}{\pi\theta}\right)^{-1/2}. \qquad (4.103)$$

By neglecting for $t \gg t_0$ the fast decaying exponentials caused by the negative part of the spectrum of (4.99) we could write:

$$e^{E_\lambda t}\chi(\vec{v}) = e^{-\omega_0(\lambda)t}\Psi_\lambda(\vec{v})\int\Phi_0(v)\Psi_\lambda(\vec{v})\chi(\vec{v})d\vec{v}.$$

We must recall, however, that the gap between the zero point and negative part of the spectrum of (4.99) is of the order t_0^{-1}. So, for the validity of this asymptotic relation, it is necessary that

$$D\lambda^2 \ll t_0^{-1},$$

or

$$\lambda \ll \ell_0^{-1} = \left(\frac{3}{2}\right)^{1/2} 4na^2 w(a). \qquad (4.104)$$

Thus keeping to the adopted approximation scheme we shall neglect the terms of the order $O(\lambda)$ in Ψ_λ and the terms of the order higher than $O(\lambda^2)$ in $\omega_0(\lambda)$ and put:

$$\Psi_\lambda(\vec{v}) = 1$$

$$\omega_0(\lambda) = D_0\lambda^2. \qquad (4.105)$$

Proceeding in such a way we obtain:

$$e^{(-i\vec{v}_0\vec{\lambda}+na^2 w(a)L_S)t}\chi(\vec{v}_0) = e^{-\omega_0(\lambda)t}\int\Phi_0(v)\chi(\vec{v})d\vec{v}, \qquad (4.106)$$

when $t \gg t_0$.

Before inserting this result into (4.55), it is useful to note that (4.55) contains the operators:

$$\bar{T}_k, T_k \qquad (4.107)$$

whose dependence upon k is determined by the factors

$$e^{\pm iak(\vec{e}\cdot\vec{\sigma})}$$

But

$$ka \ll \frac{a}{\ell_0} \ll 1$$

and therefore for the inner consistency of approximations used we must replace (4.107) by

$$T_0 = \bar{T}_0.$$

On the other hand the integration over $\vec{k}$ in (4.55) clearly needs a cutoff:

$$k < k_{max}\,; \quad \text{where } k_{max} < k_1\,; \quad k_{max} \ll \ell_0^{-1} \qquad (4.108)$$

since we are studying here only that part of $Q_\ell(t)$ which decreases slower than any exponential e^{-t/t_f}, with a fixed t_f and since all our approximation scheme is strongly dependent upon this condition (see, e.g., (4.96), (4.104)).

We now proceed to insert our results into (4.55). First, from (4.106) it follows:

$$e^{(-i\vec{v}_0(\vec{k}+\vec{\ell})+na^2 w(a)L_S)t}\, T_k(v_0,v_1)\chi(\vec{v}_0) =$$

$$= e^{-t\omega_0(\vec{k}+\vec{\ell})}\int d\vec{v}_0'\,\Phi_0(v_0')T_0(v_0',v_1)\chi(\vec{v}_0').$$

Here the right-hand side is a function of $\vec{v}_1$.

Hence, by making use of (4.96)

$$U(t;1)\,e^{(-i\vec{v}_0(\vec{k}+\vec{\ell})+na^2 w(a)L_S)t}\, T_k(v_0,v_1)\chi(\vec{v}_0) =$$

$$= \sum_{(1\leq j\leq 5)} e^{-(\omega_j(k)+\omega_0(\vec{k}+\vec{\ell}))t}\,\theta_j^{(L)}(\vec{e},\vec{v}_1)\int d\vec{v}_0'd\vec{v}_1'\,\Phi_0(v_0')\Phi_\Sigma(v_1')\theta_j^{(R)}(\vec{e},\vec{v}_1')\times$$
$$\times\, T_0(v_0',v_1')\chi(\vec{v}_0').$$

From (4.55), it now follows:

$$Q_\ell(t)\chi(\vec{v}_0) = \frac{n}{(2\pi)^3}\int_{|k|<k_{max}} d\vec{k}\sum_{(1\leq j\leq 5)} e^{-(\omega_j(k)+\omega_0(\vec{k}+\vec{\ell}))t}\{\int d\vec{v}_1\Phi_\Sigma(v_1)\times$$

$$\times T_0(v_0,v_1)\theta_j^{(L)}(\vec{e},\vec{v}_1)\}\{\int d\vec{v}_0'd\vec{v}_1'\Phi_0(v_0')\Phi_\Sigma(v_1')\theta_j^{(R)}(\vec{e},\vec{v}_1')T_0(v_0',v_1')\chi(\vec{v}_0')\}.$$

By noting that the functions:

$$g(\vec{v}_0,\vec{v}_1) = \begin{cases} mv_0^2 + Mv_1^2 \\ m\vec{v}_0 + M\vec{v}_1 \\ \text{Const} \end{cases}$$

are collision invariants it is easy to see that:

$$\int d\vec{v}_1\,\Phi_\Sigma(v_1)T_0(v_0,v_1)\theta_j^{(L)}(\vec{e},\vec{v}_1) =$$

$$= -\int d\vec{v}_1\Phi_\Sigma(v_1)T_0(v_0,v_1)\,\Psi_j^{(R)}(\vec{e},\vec{v}_0) = -a^2L_S\Psi_j^{(L)}(\vec{e},\vec{v}_0)$$

$$\int d\vec{v}_0'd\vec{v}_1'\Phi_0(v_0')\Phi_\Sigma(v_1')\,\theta_j^{(R)}(\vec{e},\vec{v}_1')\,T_0(v_0',v_1')\chi(\vec{v}_0') =$$

$$= -\int d\vec{v}_0'd\vec{v}_1'\,\Phi_0(v_0')\Phi_\Sigma(v_1')\Psi_j^{(R)}(\vec{e},\vec{v}_0')T_0(v_0',v_1')\chi(\vec{v}_0') =$$

$$= -a^2\int d\vec{v}_0'\,\Phi_0(v_0')\Psi_j^{(R)}(\vec{e},\vec{v}_0')\,L_S\chi(\vec{v}_0'),$$

where*:

$$\Psi_j^{(R)}(\vec{e},\vec{v})=\Psi_j^{(L)}(\vec{e},\vec{v})=\frac{m}{(M\theta)^{1/2}}(\vec{e}_j\vec{v}), \quad j=1,2$$

$$\Psi_3^{(R)}(\vec{e},\vec{v})=\Psi_3^{(L)}(\vec{e},\vec{v})=\left(\frac{mv^2-3\theta}{2\theta}\right)\left(\frac{k_B}{C_p}\right)^{1/2} \tag{4.109}$$

$$\Psi_{\binom{4}{5}}^{(L)}(\vec{e},\vec{v})=\left(\frac{1}{2}\right)^{1/2}\left(\frac{mv^2-3\theta}{2\theta}(\alpha T)^{-1}(\gamma-1)\mp\frac{m}{\theta}c_0\vec{v}\cdot\vec{e}\right),$$

$$\Psi_{\binom{4}{5}}^{(R)}(\vec{e},\vec{v})=\left(\frac{1}{2}\right)^{1/2}\left(\frac{mv^2-3\theta}{2}\frac{\alpha}{C_p}\mp\frac{m}{Mc_0}\vec{v}\cdot\vec{e}\right).$$

So we finally obtain the explicit expression

$$Q_\ell(t)\chi(v_0)=\frac{na^4}{(2\pi)^3}\int_{|k|<k_{max}} d\vec{k}\sum_{(1\leq j\leq 5)} e^{-(\omega_j(k)+\omega_0(\vec{k}+\vec{\ell}))t} L_S\Psi_j^{(L)}(\vec{e},\vec{v}_0)\times$$

$$\times\int d\vec{v}_0'\Phi_0(v_0')\Psi_j^{(R)}(\vec{e},\vec{v}_0')L_S\chi(\vec{v}_0'), \text{ when } t\gg t_0,\ t>t_{rel} \tag{4.110}$$

which can be substituted in the equations (4.54), (4.56). We first consider the case when $\ell=0$.

* It is easy to see that we may add to the right-hand side of (4.109) any terms which do not depend upon $\vec{v}$ since their contribution will be zero.

Then (4.56) yields:

$$\frac{\partial\chi(t,\vec{v}_0)}{\partial t}=na^2w(a)L_S\chi(t,\vec{v}_0)+w^2(a)\int_0^t Q_0(t-\tau)\chi(\tau,\vec{v})d\vec{v}$$

$$Q_0(t-\tau)\chi(\vec{v}_0)= \tag{4.111}$$

$$=\frac{na^4}{(2\pi)^3}\int_0^{k_{max}}k^2dk\sum_{(1\leq j\leq 5)}e^{-(\omega_j(k)+\omega_0(k))(t-\tau)}\int d\vec{e}L_S\Psi_j^{(L)}(\vec{e},\vec{v}_0)\times$$

$$\times\int d\vec{v}_0'\Phi_0(v_0')\Psi_j^{(R)}(\vec{e},\vec{v}')L_S\chi(\vec{v}_0').$$

It is clear that if

$$\chi(0,\vec{v}_0)=\text{Const}$$

then also

$$\chi(t,\vec{v}_0)=\chi(0,\vec{v}_0)=\text{Const}$$

because

$$L_S\text{Const}=0.$$

From physical point of view this trivial solution corresponds to a change in the normalization of $\mathfrak{D}_{eq}(S,\Sigma)$.

By subtracting from $\chi(0,\vec{v}_0)$ a suitable constant, we can obtain

$$\int\Phi_0(v_0)\chi(0,\vec{v}_0)d\vec{v}_0=0. \tag{4.112}$$

Note that this property is conserved too

$$\int\Phi_0(v_0)\chi(t,\vec{v}_0)d\vec{v}_0=0 \tag{4.113}$$

because

$$\int\Phi_0(v_0)L_S g(\vec{v}_0)d\vec{v}_0=0.$$

Therefore, we shall restrict our attention to functions (4.112), that is to functions orthogonal to unity:

$$(1,\chi)=0. \qquad (4.114)$$

To obtain the first approximation for $\chi(t,\vec{v})$, we neglect the correction term with Q_0 in the equation (4.111) thus getting:

$$\chi(t,\vec{v}_0)=e^{tna^2w(a)L_S}\chi(0,\vec{v}_0). \qquad (4.115)$$

Since the spectrum of the operator

$$na^2w(a)L_S$$

is negative, in the space of functions (4.114) and is separated from zero by a gap of the order of t_0^{-1}, the function (4.115) decreases exponentially when $t\gg t_0$.

We thus may write this approximation in the form:

$$\chi(t,\vec{v}_0)=\delta(t)\int_0^\infty e^{tna^2w(a)L_S}dt\,\chi(0,\vec{v})=$$

$$=-\delta(t)(na^2w(a))^{-1}L_S^{-1}\chi(0,\vec{v}_0).$$

By inserting it into the correction term in the right-hand side of (4.111), we obtain:

$$\frac{\partial\chi(t,\vec{v}_0)}{\partial t}=na^2w(a)L_S\chi(t,\vec{v}_0)-$$
$$-w(a)(na^2)^{-1}Q_0(t)L_S^{-1}\chi(0,\vec{v}_0)$$

from which it follows:

$$\chi(t,\vec{v}_0)=e^{tna^2w(a)L_S}\chi(0,\vec{v}_0)-$$

$$-w(a)(na^2)^{-1}\int_0^t e^{na^2w(a)L_S(t-\tau)}Q_0(\tau)d\tau\,L_S^{-1}\chi(0,\vec{v}_0)$$

and therefore the correction to a fast decaying term will be

$$\chi_c(t,\vec{v}_0)=(na^2)^{-2}L_S^{-1}Q_0(t)L_S^{-1}\chi(0,\vec{v}_0) \qquad (4.116)$$

$$\chi(t,\vec{v}_0)=\chi_c(t,\vec{v}_0), \quad \text{when} \quad t\gg t_0.$$

Now (4.111) yields:

$$\chi_c(t,\vec{v}_0)=\frac{1}{(2\pi)^3n}\int_0^{k_{max}}k^2dk\sum_{(1\leq j\leq 5)}e^{-(\omega_j(k)+\omega_0(k))t}\times \qquad (4.117)$$

$$\times\int d\vec{e}\,\Psi_j^{(L)}(\vec{e},\vec{v}_0)\int d\vec{v}_0'\,\Phi_0(v_0')\Psi_j^{(R)}(\vec{e},\vec{v}_0')\chi(0,\vec{v}_0').$$

Here, due to (4.67′) the asymptotic values of the integrals
$$\int_0^{k_{max}}e^{-(\nu+D_0)k^2t}k^2dk, \qquad \int_0^{k_{max}}e^{-(D_T+D_0)k^2t}k^2dk$$
for large $t\gg t_0$, are given, respectively, by:

$$\frac{\sqrt{\pi}}{4[(\nu+D_0)t]^{3/2}}, \qquad \frac{\sqrt{\pi}}{4[(D_T+D_0)t]^{3/2}}$$

Furthermore, note that (4.109) gives:

$$\int d\vec{e}\,\Psi_4^{(L)}(\vec{e},\vec{v}_0)\Psi_4^{(R)}(\vec{e},\vec{v}_0')=\int d\vec{e}\,\Psi_5^{(L)}(\vec{e},\vec{v}_0)\Psi_5^{(R)}(\vec{e},\vec{v}_0')$$

Hence, the time factors combine:

$$e^{-(\omega_4(k)+\omega_0(k))t}+e^{-(\omega_5(k)+\omega_0(k))t}=$$

$$=e^{-(\frac{1}{2}\Gamma_S+D_0)k^2t}(e^{-ickt}+e^{ickt})$$

and lead us to the integral:

$$\int_{-k_{max}}^{k_{max}} e^{-(\frac{1}{2}\Gamma_S+D_0)k^2t}e^{ickt}k^2dk$$

whose asymptotic value for large t will be

$$\frac{\sqrt{\pi}}{2(\xi t)^{3/2}}e^{-\frac{c^2t}{4\xi}},$$

where

$$\xi=\frac{1}{2}\Gamma_S+D_0.$$

Since this integral is exponentially decaying, we see that the sound modes do not contribute to the considered "hydrodynamical tail" and thus must be droped out from (4.117). That leaves us with two viscosity modes and one heat mode.

By noticing that

$$\int e_{j,\alpha}e_{j,\beta}d\vec{e}=\frac{4\pi}{3}\delta_{\alpha,\beta},\ j=1,2;\ \alpha,\beta=x,y,z$$

we easily perform the integration over $\vec{e}$ and get:

$$\chi_c(t,\vec{v})=(\frac{t_0}{t})^{3/2}\{\frac{1}{12n}\{\pi(\nu+D_0)t_0\}^{-3/2}\frac{m^2}{M\theta}\int(\vec{v}\cdot\vec{v}')\chi(0,\vec{v}')\Phi_0(v')d\vec{v}'+$$

$$+\frac{1}{8n}\frac{k_B}{C_p}\{\pi(D_T+D_0)t_0\}^{-3/2}\frac{mv^2-3\theta}{2\theta}\int\frac{mv'^2-3\theta}{2\theta}\chi(0,\vec{v}')\Phi_0(v')d\vec{v}'\}; \qquad (4.118)$$

$t\gg t_0$.

This asymptotic formula can be used to obtain the slow decaying part of the equilibrium time correlation functions.

Put, for example:

$$\chi(0,\vec{v})=v_x$$

Then (4.118) yields:

$$\langle v_x(t)v_x(0)\rangle_{eq}=\int v_x\chi_c(t,\vec{v})\Phi_0(v)d\vec{v}=$$

$$=(\frac{t_0}{t})^{3/2}\frac{m2}{12nM\theta}\{\pi(\nu+D_0)t_0\}^{-3/2}(\int v_x^2\Phi_0(v)d\vec{v})^2= \qquad (4.119)$$

$$=(\frac{t_0}{t})^{3/2}\frac{m}{12nM}\{\pi(\nu+D_0)t_0\}^{-3/2}\langle v_x^2\rangle_{eq}.$$

Consider the situation when S is a tagged particle of Σ and the hydrodynamic part of $U_k(t;1)$ is calculated by making use of the Enskog equation for moderately dense hard sphere gas.

Then in (4.119) ν is to be replaced by ν_E. Since D_0 itself is the Enskog diffusion coefficient we here obtain the formula found in the paper /13/ by J.R.Dorfman and E.G.D.Cohen.

On the other hand, if we replace D_0 by the "total" diffusion coefficient, (4.119) yields the well-known result of the mode-mode coupling theory.

Let us now make some comments concerning equation (4.54) for $\ell\neq 0$ where the expression (4.110) had been inserted.

By applying the Laplace transformation method, we can write it in the form:

$$(z - na^2 w(a) L_S)\tilde{\chi}_\ell(z, \vec{v}_0) = -i\,\vec{\ell}\,\vec{v}_0\,\tilde{\chi}_\ell(z, \vec{v}_0) + $$
$$+ w^2(a)\,\tilde{Q}_\ell(z)\,\tilde{\chi}_\ell(z, \vec{v}_0) + \chi_\ell(0, \vec{v}_0), \tag{4.120}$$

where

$$\tilde{\chi}_\ell(z, \vec{v}_0) = \int_0^\infty e^{-zt}\,\chi_\ell(t, \vec{v}_0)\,dt$$
$$\tilde{Q}_\ell(z)\,g(\vec{v}_0) = \frac{na^4}{(2\pi)^3}\int_{|k|<k_{max}} d\vec{k}\ \times \tag{4.121}$$
$$\times \sum_{(1\leqq j\leqq 5)} \frac{1}{\omega_j(k)+\omega_0(\vec{k}+\vec{\ell})+z}\, L_S \Psi_j^{(L)}(\vec{e},\vec{v}_0) \int d\vec{v}_0{}'\,\Phi_0(v_0')\,\Psi_j^{(R)}(\vec{e},\vec{v}_0{}')\,L_S\, g(v_0').$$

In order to examine the diffusion process, we shall consider the case when

$$\chi_\ell(0, \vec{v}_0) = \rho_\ell(0) \tag{4.122}$$

is constant with respect to $\vec{v}_0$.
Put

$$\tilde{\chi}_\ell(z, \vec{v}_0) = \tilde{\rho}_\ell(z) + \phi_\ell(z, \vec{v}_0), \tag{4.123}$$

where

$$\tilde{\rho}_\ell(z) = \int \Phi_0(z)\,\tilde{\chi}_\ell(z, \vec{v}_0)\,d\vec{v}_0$$
$$\int \Phi_0(v_0)\,\phi_\ell(z, \vec{v}_0)\,d\vec{v}_0 = 0. \tag{4.124}$$

Then (4.120) yields:

$$z\tilde{\rho}_\ell(z) = -i\,\vec{\ell}\int \vec{v}_0\,\Phi_0(v_0)\,\phi(z, \vec{v}_0)\,d\vec{v}_0 + \rho_\ell(0) \tag{4.125}$$

and

$$(z - na^2 w(a) L_S)\phi_\ell(z, \vec{v}_0) = -i\,\vec{\ell}\,\vec{v}_0\,\tilde{\rho}_\ell(z) + $$
$$+ w^2(a)\tilde{Q}_\ell(z)\,\phi_\ell(z, \vec{v}_0) - i\vec{\ell}\left(\vec{v}_0\,\phi_\ell(z, \vec{v}_0) - \int \vec{v}_0\,\phi_\ell(z, \vec{v}_0)\,\Phi_0(v_0)\,d\vec{v}_0\right).$$

Since ℓ is supposed to be sufficiently small:

$$\ell \ll \ell_0$$

we can leave in $\phi_\ell(z, \vec{v}_0)$ only the terms proportional to ℓ. So we shall write

$$(z - na^2 w(a) L_S)\,\phi_\ell(z, \vec{v}_0) = -i\,\vec{\ell}\,\vec{v}_0\,\rho_\ell(z) + $$
$$+ w^2(a)\,\tilde{Q}_\ell(z)\,\phi_\ell(z, \vec{v}_0).$$

Furthermore, neglecting the correction term with $\tilde{Q}_\ell$ we obtain in the first approximation:

$$\phi_\ell(z, \vec{v}_0) = -i(z - na^2 w(a) L_S)^{-1}\,\vec{\ell}\,\vec{v}_0\,\rho_\ell(z)$$

The insertion of this formula into the correction term leads us to the following expression:

$$\phi_\ell(z, \vec{v}_0) = -i(z - na^2 w(a) L_S)^{-1}\,\vec{\ell}\,\vec{v}_0 + $$
$$+ w^2(a)(z - na^2 w(a)\,L_S)^{-1}\tilde{Q}_\ell(z)(z - na^2 w(a) L_S)^{-1}(-i\,\vec{\ell}\vec{v}_0)\,\rho_\ell(z).$$

Let us now point out that L_S, when acting on functions $g(\vec{v}_0)$ orthogonal to unity, has only a negative spectrum, and that for the study of longtime behaviour, we are interested only in $z \ll t_0^{-1}$. So, z can be neglected in the term $(z - na^2 w(a) L_S)^{-1}$ and our approximation becomes:

$$\phi_\ell(z,\vec{v}_0) = -i(na^2 w(a))^{-1} L_S^{-1} \vec{\ell}\vec{v}_0 \rho_\ell(z) - -i(na^2)^{-2} L_S^{-1} \tilde{Q}_\ell(z) L_S^{-1} \vec{\ell}\vec{v}_0 \rho_\ell(z).$$

Then (4.125) yields:

$$z\tilde{\rho}_\ell(z) = -\ell^2 D(\ell,z)\tilde{\rho}_\ell(z) + \rho_\ell(0), \tag{4.126}$$

where

$$D(\ell,z) = D + \Delta D(\ell,z), \tag{4.127}$$

D is the "renormalized" diffusion coefficient:

$$D = D_0 + D_1$$

$$D_0 = -(na^2 w(a))^{-1} \int \Phi_0(v) v_x L_S^{-1} v_x d\vec{v} \tag{4.128}$$

$$D_1 = -\frac{1}{(2\pi)^3 n} \int_{k<k_{max}} d\vec{k} \sum_{(1\le j\le 5)} \frac{\int \Phi_0(v) v_x \Psi_j^{(L)}(\vec{e},\vec{v}) d\vec{v} \int \Phi_0(v) \Psi_j^{(R)}(\vec{e},\vec{v}) v_x d\vec{v}}{\omega_j(k) + \omega_0(k)}$$

and

$$\Delta D(\ell,z) = \frac{1}{(2\pi)^3 n} \int_{k<k_{max}} d\vec{k} \sum \frac{z+\omega_0(\vec{k}+\vec{\ell}) - \omega_0(\vec{k})}{(z+\omega_j(k)+\omega_0(\vec{k}+\vec{\ell}))(\omega_j(k)+\omega_0(k))} \times$$

$$\times \int \Phi_0(v)(\hat{\ell},\vec{v}) \Psi_j^{(L)}(\vec{e},\vec{v}) d\vec{v} \int \Phi_0(v) \Psi_j^{(R)}(\vec{e},\vec{v})(\hat{\ell},\vec{v}) d\vec{v}, \tag{4.129}$$

$\hat{\ell} = \vec{\ell}/\ell$ is the unit vector. The computation of the extra term D_1 due to the interaction with hydrodynamic modes shows that it is a small quantity of the second order in density.

Nevertheless, from the formal point of view, we must emphasize that D_1 contains the integral

$$\int_{k<k_{max}} \frac{k^2 dk}{(\nu + D_0)k^2} = \frac{k_{max}}{\nu + D_0}$$

which is proportional to k_{max}. Since k_{max} is determined only up to a numerical factor "of the order of 1" we see that the real value of D_1 must also depend upon nonhydrodynamical part of our operators.

Due to (4.109) we have from (4.129)

$$\int \Phi_0(v)(\hat{\ell}\vec{v}) \Psi_j^{(L)}(\vec{e},\vec{v}) d\vec{v} \int \Phi_0(v)(\hat{\ell},\vec{v}) \Psi_j^{(R)}(\vec{e},\vec{v}) d\vec{v} = \frac{m}{M}(\hat{\ell},\vec{e}_j)^2$$

$$j = 1,2 \tag{4.130}$$

$$\int \Phi_0(v) \Psi_{\binom{4}{5}}^{(L)}(\vec{e},\vec{v})(\hat{\ell},\vec{v}) d\vec{v} \int \Phi_0(v)(\hat{\ell},\vec{v}) \Psi_{\binom{4}{5}}^{(R)}(\vec{e},\vec{v}) d\vec{v} = \frac{1}{2}\frac{m}{M}(\hat{\ell},\vec{e})^2$$

Since

$$\omega_1 = \omega_2 = \nu k^2$$

two terms (4.130) combine giving

$$\frac{m}{M}\{(\hat{\ell},\vec{e}_1)^2 + (\hat{\ell},\vec{e}_2)^2\} = \frac{m}{M}\{1 - (\hat{\ell},\vec{e})^2\}.$$

Because of the symmetry of this expression with respect to the reflection $\vec{e} \to -\vec{e}$, we may write the terms in $\Delta D(\ell,z)$ corresponding to viscosity modes in the form:

$$\frac{m}{M}(\nu+D)^{-1}\int(1-(\hat{\ell}\vec{e})^2)\{\int_0^{k_{max}} dk \frac{z}{z+\nu k^2+D(k^2+\ell^2+2k\ell(\hat{\ell}\vec{e}))}+$$

$$+ k\ell(\vec{e}\hat{\ell})[\frac{1}{z+(\nu+D)k^2+2D k\ell(\vec{e}\hat{\ell})}-\frac{1}{z+(\nu+D)k^2-2Dk\ell(\vec{e}\hat{\ell})}]\}d\vec{e}=$$

$$=\frac{m}{M}(\nu+D)^{-1}\int d\vec{e}\,(1-(\hat{\ell}\vec{e})^2)\int_0^{k_{max}} dk\,\{\frac{z}{z+\nu k^2+D(k^2+\ell^2+2k\ell(\hat{\ell}\vec{e}))}-$$

$$-4Dk^2\ell^2(\hat{\ell}\vec{e})^2\frac{1}{\{z+\nu+D(k^2+\ell^2+2k\ell(\hat{\ell}\vec{e}))\}\{z+\nu+D(k^2+\ell^2-2k\ell(\hat{\ell}\vec{e}))\}}\}.$$

By using the variables

$$k=q\ell$$

$$\zeta=\frac{z}{D\ell^2}$$

then for finite ζ we see that the limit of the integration over q will be $k_{max}/\ell\to\infty$ when $\ell\to 0$, but as it is easy to see this integral will be convergent.

The same procedure can be performed for the sound modes but there the corresponding contribution on factor ℓ is smaller and, hence, is neglected in the adopted approximation.

Evidently the contribution of the heat mode will be zero. Note that equation (4.126) with (4.127), (4.129), (4.130) belongs to the type of equations considered in the monography [14] by I. de Schepper, and hence may be treated by the procedure elaborated in this monography.

We now wish to point out that all equations obtained in §4 by starting from the initial condition:

$$\mathfrak{D}_0(S,\Sigma)=V\chi_0(S)\mathfrak{D}_{eq}(S,\Sigma)$$

could also be found from the equations established in §2 on the basis of the initial condition [1]

$$\mathfrak{D}_0(S,\Sigma)=f_0(S)\,\mathfrak{D}_{eq}(\Sigma) \qquad (1.2)$$

$$f_0(S)=\chi_0(S)\Phi_0(v_0).$$

The difference between these two approaches could be described as follows:

First, (4.48) if derived starting from (1.2) would contain $\bar{T}_k$ instead of T_k on the right in the expression (4.48). But this difference disappears at the stage when we replace T_k, $\bar{T}_k$ by $T_0=\bar{T}_0$.

Second, the only remaining difference is that in case (1.2), we should replace $w(a)$ by its low density limit, i.e., by 1.

So all the results discussed in §4 could have been obtained starting from our old scheme developed in the paper [1]. The main new element in the technique of application of this method, which enabled us to include the new developments, was the introduction [4] of the collision operator.

It is also to be emphasized that the procedure, elaborated in this paper needs essential improvement.

In fact while the operator $U(t;1)$ referring to the system Σ could have been evaluated by using any sophisticated kinetic equation, the interaction term Π_{int} is here treated in a rather rough way. In fact we have supposed that it is small and only the terms formally belonging to "the second order of smallness" have been correctly included.

Suppose we consider the situation when:

$$Л_{int} = Л_{int}^{(\Phi)}$$

with $\Phi(r)$ corresponding to short range strong repulsive forces.

It is clear that such an interaction must lead to a kind of the collision operator but formally our scheme can work in this situation only if we replace $Л_{int}^{(\Phi)}$ by an ad hoc introduced collision interaction.

So we see that our scheme needs certain refinement. Such a refinement could have been achieved, for example, if instead of the considered zeroth approximation

$$\mathfrak{D}_t(S,\Sigma) = V\chi_t(S)\mathfrak{D}_{eq}(S,\Sigma)$$

we had used zeroth approximation in the form:

$$\mathfrak{D}_t(S,\Sigma) = V\{\chi_t(S) + \sum_{(1\leq j\leq N)} \eta_t(S,j)\}\mathfrak{D}_{eq}(S,\Sigma), \quad (4.131)$$

where $\eta_t(S,j)$ depends upon phases of S and of j-th particle from Σ.

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