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ON NORMALIZATION OF A CLASS OF POLYNOMIAL HAMILTONIANS: FROM ORDINARY AND INVERSE POINTS OF VIEW

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1 Introduction

It is widely known that the Birkhoff-Gustavson normal form (BGNF) expansion works effectively to study a behavior of nonlinear dynamical systems; the Hénon-Heiles system and Toda linear chain (TLC) are often taken as typical examples [1] to describe such an efficiency.

Since a core part of the BGNF expansion is made on the polynomial algebra [2], it fits very well the symbolic computing on computers: For example, the symbolic computing program named GITA realizes the algebraic procedure of converting power-series Hamiltonians into their BGNF [3] with REDUCE 3.3 or later versions of REDUCE.[1].

One of the aims of this paper is to demonstrate how the normalization into BGNF works around integrable systems. Although one might not think it necessary to normalize the integrable Hamiltonian systems, the normalization of integrable systems is worth discussing especially in the case where they admit the trajectories tending to singularities; the truncated three-particle Toda linear chain (3-TLC) is taken as an example to demonstrate how the normalization works in the integrable-system case.

The other aim of this paper is to present a symbolic computational approach to an 'inverse' problem of normalization recently posed by one of the authors (YU) with the aim of an application of quantum studies to certain BGNF systems [4, 5, 6]. The inverse problem reads as follows: '*Identify a class of dynamical systems which are reduced to the same BGNF up to a certain order*'. To solve it, the symbolic computing program named GITA^{-1} has been proposed by the authors [7], which will be reviewed in this paper together with an application to the regularized system of planar hydrogen atom with the linear Stark effect (HLSE) [8]. It is shown that a class of Liouville-type systems share the same BGNF with the regularized system of HLSE.

The aim of this talk is also to review another symbolic computing program named ANFER (*A*lgorithm of *N*ormal *F*orm *E*xpansion and *R*estoration) for the inverse problem proposed by the authors (YU and SV) [9]. ANFER is expected to work more effectively than GITA^{-1} does from various points of view; less steps of procedures, less memory expenses, and so on. The system of Hénon-Heiles type will be taken as a very simple but intuitive example to show how ANFER restores the Hénon-Heiles Hamiltonian from its BGNF expansion.

The contents of this paper are organized as follows. In Section 2, a brief reveiw of the ordinary normalization problem is given. In Section 3, the structures of GITA and GITA^{-1} for the general n-degree-of-freedom case is presented briefly. In Section 4, the direct problem of 3-TLC is discussed to show the normalization is effective not only for non-integrable systems but also for integrable ones. In Section 5, the run of the inverse problem of HLSE is demonstrated to show the way to identify a class of Hamiltonian systems which share certain BGNF Hamiltonian

[1]The authors are trying to implement the same procedure with Maple V

in common. In Section 6, the formulation of the inverse normalization problem and algorithm of ANFER are presented. In Section 7 a simple test example of the ANFER run is considered.

2 The Ordinary and Inverse Normalization Problems

In this Section, we review the ordinary problem of the BGNF expansion very briefly following [2]. Let $(\mathbf{R}^n \times \mathbf{R}^n, d\theta_n)$ be the phase space endowed with the canonical symplectic 2-form, $d\theta_n = \sum_{j=1}^{n} dp_j \wedge dq_j$, where (q,p) are the Cartesian coordinates of $\mathbf{R}^n \times \mathbf{R}^n$. Let us consider the Hamiltonian system on $(\mathbf{R}^n \times \mathbf{R}^n, d\theta_n)$ which admits a stable equilibrium point in a resonance of equal frequencies. Without loss of generality, such an equilibrium point can be put at the origin of $\mathbf{R}^n \times \mathbf{R}^n$, so that around it the Hamiltonian $H(q,p)$ of such a system is assumed to be expanded into a power series,

$$H(q,p) = \frac{1}{2}\sum_{j=1}^{n}\left(p_j^2 + q_j^2\right) + \sum_{k=3}^{\infty} H_k(q,p), \tag{1}$$

where $H_k(q,p)$ $(k = 3,4,\cdots)$ are homogeneous polynomials of degree k in (q,p).

The conversion of H into a BGNF power series is made as follows. Let us consider a local canonical transformation, $(q,p) \to (\xi,\eta)$ around the origin of $\mathbf{R}^n \times \mathbf{R}^n$ which is associated with a *type-2* generating function [1],

$$W(q,\eta) = \sum_{j=1}^{n} q_j\eta_j + \sum_{k=3}^{\infty} W_k(q,\eta), \tag{2a}$$

where $W_k(q,\eta)$ $(k = 3,4,\cdots)$ are homogeneous polynomials of degree k in (q,η). On choosing $W(q,\eta)$ suitably, the $H(q,p)$ is converted to the power series, say $G(\xi,\eta)$, through

$$H(q,\frac{\partial W}{\partial q}) = G(\frac{\partial W}{\partial \eta},\eta), \tag{3a}$$

which is written in the form

$$G(\xi,\eta) = \frac{1}{2}\sum_{j=1}^{n}\left(\eta_j^2 + \xi_j^2\right) + \sum_{k=3}^{\infty} G_k(\xi,\eta), \tag{4a}$$

where $G_k(\xi,\eta)$ $(k = 3,4,\cdots)$ are homogeneous polynomials of degree k in (ξ,η) subject to the Poisson-commuting relation,

$$\left\{\frac{1}{2}\sum_{j=1}^{n}\left(\eta_j^2 + \xi_j^2\right),\ G_k(\xi,\eta)\right\} = 0. \tag{4b}$$

Definition 2.1 *Let $D_{q,\eta}$ be the differential operator*

$$D_{q,\eta} = \sum_{j=1}^{n} \left(q_j \frac{\partial}{\partial \eta_j} - \eta_j \frac{\partial}{\partial q_j} \right) \tag{5}$$

associated with the variables (q, η) and let $f(q, \eta)$ be a power series or a polynomial in (q, η). Then f is said to be normal (resp. non-normal) to $D_{q,\eta}$ if $f \in Ker(D_{q,\eta})$ (resp. $f \in Im(D_{q,\eta})$.

We have the following fact [2] known well:

Theorem 2.2 *For any Hamiltonian $H(q, p)$ in the form of (1), there exists uniquely the pair of the BGNF, $G(\xi, \eta)$, in the form of (4) and the non-normal type-2 generating function $W(q, \eta)$ in the form of (2a), which satisfies (3a).*

Theorem 2.2 provides the ordinary problem of the BGNF expansion in the following form:

Definition 2.3 (The ordinary problem) *Convert a power-series or a polynomial Hamiltonian $H(q, p)$ of the form (1) into a BGNF power series $G(\xi, \eta)$ of the form (4) through a canonical transformation associated with a non-normal type-2 generating function $W(q, \eta)$ of the form (2a).*

In view of Definition 2.3 defining the ordinary problem, the inverse problem is defined as follows:

Definition 2.4 (The inverse problem) *For a given BGNF in the form (4), identify all possible power series $H(q, p)$ in the form (1), which are normalized to the given BGNF through the canonical transformations associated with the type-2 generating functions.*

It should be remarked here that we will present an alternative expression for the inverse problem posed below in Section 6, which will be a key to organize the algorithm ANFER .

3 A Review of GITA and GITA^{-1}

Let $H_{IN}(q, p)$ and $H_{OUT}(q, p)$ be the *input* and the *output* Hamiltonians, which are expressed as

$$H_\lambda(q, p) = \sum_{h=2}^{\infty} H_\lambda^{(h)}(q, p) \quad \text{with} \quad H_\lambda^{(2)}(q, p) = \frac{1}{2} \sum_{k=1}^{n} (p_k^2 + q_k^2), \tag{6}$$

where $H_\lambda^{(h)}$ ($\lambda = IN, OUT,\ h = 3, 4, \cdots$) is a degree-$h$ homogeneous polynomial in (q, p) expressed as

$$H_\lambda^{(h)}(q, p) = \sum_{|\alpha|+|\beta|=h} c_\lambda^{(h)}(\alpha, \beta) q^\alpha p^\beta \quad \text{with} \quad \begin{cases} q^\alpha p^\beta = q_1^{\alpha_1} \cdots q_n^{\alpha_n} p_1^{\beta_1} \cdots p_n^{\beta_n}, \\ |\alpha| = \sum_{k=1}^{n} \alpha_k,\ |\beta| = \sum_{k=1}^{n} \beta_k. \end{cases} \tag{7}$$

Let $G_{IN}(\xi,\eta)$ and $G_{OUT}(\xi,\eta)$ be the *input* and the *output* BGNF Hamiltonian,

$$G_\lambda(\xi,\eta)=\sum_{j=1}^{\infty}G_\lambda^{(2j)}(\xi,\eta) \quad \text{with} \quad G_\lambda^{(2)}(\xi,\eta)=\frac{1}{2}\sum_{k=1}^{n}(\eta_k^2+\xi_k^2), \tag{8}$$

where $G_\lambda^{(2j)}$ $(\lambda=IN,OUT,\ j=2,3,\cdots)$ is a degree-$2j$ homogeneous polynomial in (ξ,η) expressed as

$$G_\lambda^{(2j)}(\xi,\eta)=\sum_{|\alpha|+|\beta|=2j}\gamma_\lambda^{(2j)}(\alpha,\beta)\xi^\alpha\eta^\beta \quad \text{with} \quad \{G_\lambda^{(2)},G_\lambda^{(2j)}\}=0. \tag{9}$$

In equation (9), α and β are multi-indices used in the same way as in equation (7), and $\{\cdot,\cdot\}$ is the canonical Poisson bracket associated with the position variables ξ and the momentum ones η. The coefficients $\gamma_\lambda^{(2j)}(\alpha,\beta)$, $(j=2,3,\cdots)$ are found by solving the key BGNF equation

$$H_\lambda(q,\frac{\partial W}{\partial q})=G_\lambda(\frac{\partial W}{\partial \eta},\eta). \tag{10}$$

Here $W_\lambda(q,\eta)$ is the generating function of the form (2), which should be identified together with $G_\lambda(\xi,\eta)$ as the solutions of (10). We will not get the identification of $W_\lambda(q,\eta)$ in detail here (see [2, 3]).

Let us denote by P_ℓ the space of degree-ℓ homogeneous polynomials in $2n$ variables with real coefficients, which can be identified with the vector space $\mathbf{R}^{N(n,\ell)}$, where $N(n,\ell)$ indicates the number of degree-ℓ monomials in $2n$ variables allowed to exist. Then, denoting such a correspondence by $\iota_\ell : P_\ell \to \mathbf{R}^{N(n,\ell)}$, we associate the vectors, $\vec{c}_\lambda^{(h)}$ and $\vec{\gamma}_\lambda^{(2j)}$, with H_λ and G_λ by

$$\vec{c}_\lambda^{(h)}=\iota_h(H_\lambda^{(h)})\in\mathbf{R}^{N(n,h)} \quad \text{and} \quad \vec{\gamma}_\lambda^{(2j)}=\iota_{2j}(G_\lambda^{(2j)})\in\mathbf{R}^{N(n,2j)}, \tag{11}$$

respectively. Further, using ι_ℓ, we express the differential operator D (5), restricted on P_ℓ by the matrix $M^{(\ell)}$ acting on $\mathbf{R}^{N(n,\ell)}$; $\iota_\ell\circ D=M^{(\ell)}\circ\iota_\ell$.

After the preparatory work done above, the hth order part of Eq. (10) is put into the series of algebraic equations,

$$\vec{\gamma}_\lambda^{(h)}=M^{(h)}\{\vec{c}_\lambda^{(h)}+\Phi^{(h)}(\vec{c}_\lambda^{(h-1)},\cdot,\cdot,\vec{c}_\lambda^{(2)})\} \quad (j=3,4,\cdots), \tag{12}$$

for $\vec{\gamma}_\lambda^{(h)}$ $(\lambda=IN,OUT)$ [2, 8], which are just the equations solved by GITA. Note that $\vec{\gamma}_\lambda^{(2j+1)}$ $(\lambda=IN,OUT)$ turn out to vanish [8].

We are now in position to present what GITA^{-1} computes: Let us recall the inverse problem posed in Section 1, which is put in the following: '*For a given H_{IN}, identify all the possible (or a part of) H_{OUT}subject to $G_{IN}=G_{OUT}$ up to a certain order*'. Since $G_{IN}=G_{OUT}$ can read $\vec{\gamma}_{IN}^{(h)}=\vec{\gamma}_{OUT}^{(h)}$ $(h=2,3,\cdots)$, GITA^{-1} solves the series of equations,

$$M^{(h)}\vec{c}_{OUT}^{(h)}=\vec{\gamma}_{IN}^{(h)}-M^{(h)}\Phi^{(h)}(\vec{c}_{OUT}^{(h-1)},\cdot,\cdot,\vec{c}_{OUT}^{(2)})\} \quad (j=3,4,\cdots), \tag{13}$$

for $\vec{c}^{(h)}_{OUT}$, where $\vec{\gamma}^{(h)}_{IN}$ are determined beforehand from H_{IN} through GITA (*i.e.* (6)-(12)). In the subsequent Sections, we demonstrate how GITA and GITA^{-1} (*i.e.* (6)-(13)) work in REDUCE 3.3 or later versions of REDUCE in the direct problem of 3-TLC and in the inverse problem of HLSE.

4 Truncated Three-Particle Toda Linear Chain

Let us consider the example of an integrable system: three identical particles on the line governed by the Toda Hamiltonian [10]. The original Toda Hamiltonian can be reduced to the two-dimensional one:

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{4}\{\exp\ (\xi) + \exp\ (\eta) + \exp\ (\zeta)\}, \qquad (14a)$$

$$\xi = \sqrt{2}q_1 + \sqrt{\frac{2}{3}}q_2, \quad \eta = -\sqrt{2}q_1 + \sqrt{\frac{2}{3}}q_2, \quad \zeta = -2\sqrt{\frac{2}{3}}q_2, \quad \xi + \eta + \zeta = 0. \qquad (14b)$$

It is easy to verify that the Hamiltonian system (14) possesses additional integral of motion (see Fig.1) in the form

$$I = \frac{4}{3}p_1(p_1^2 - 3p_2^2) + (p_1 - \sqrt{3}p_2)\exp\ (\xi) + (p_1 + \sqrt{3}p_2)\exp\ (\eta) - 2p_1\exp\ (\zeta). \qquad (15)$$

Note that the ansatz $q_1 = \sqrt{6}x,\ q_2 = \sqrt{6}y,\ p_1 = \sqrt{6}p_x,\ p_2 = \sqrt{6}p_y$ and $H \to 6H$ brings the expressions (14) and (15) to the same ones as in the book [11]. As the Toda Hamiltonian has the C_{3v} symmetry its power expansion is determined fully through the two invariant functions $f = q_1^2 + q_2^2$ and $g = q_1^2q_2 - \frac{1}{3}q_2^3$. Below the first power terms of the Taylor series for Hamiltonian (14) are written:

$$H - \frac{3}{4} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + \frac{1}{\sqrt{6}}(q_1^2q_2 - \frac{1}{3}q_2^3) + \frac{1}{12}(q_1^2 + q_2^2)^2$$

$$+\frac{\sqrt{6}}{36}(q_1^2 + q_2^2)(q_1^2q_2 - \frac{1}{3}q_2^3) + \frac{1}{180}[(q_1^2 + q_2^2)^3 + 2(q_1^2q_2 - \frac{1}{3}q_2^3)^2] + \dots \qquad (16a)$$

or

$$H - \frac{3}{4} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}f + \frac{1}{\sqrt{6}}g + \frac{1}{12}f^2 + \frac{\sqrt{6}}{36}fg + \frac{1}{180}(f^3 + 2g^2)$$

$$+\frac{1}{90\sqrt{6}}f^2g + \frac{1}{15120}f(3f^3 + 16g^2) + \dots \qquad (16b)$$

Thus, it is seen that any truncated Toda's polynomial series generates a generalized Henon-Heiles Hamiltonian. Here a surprising situation arises: while the full Toda Hamiltonian (14) is integrated, its power expansion truncated in any finite degree presents a nonintegrable system 3-TCL.

In some manner this phenomenon may be explained by the behavior of the negative Gaussian curvature(NGC) domain on the respective potential energy surface

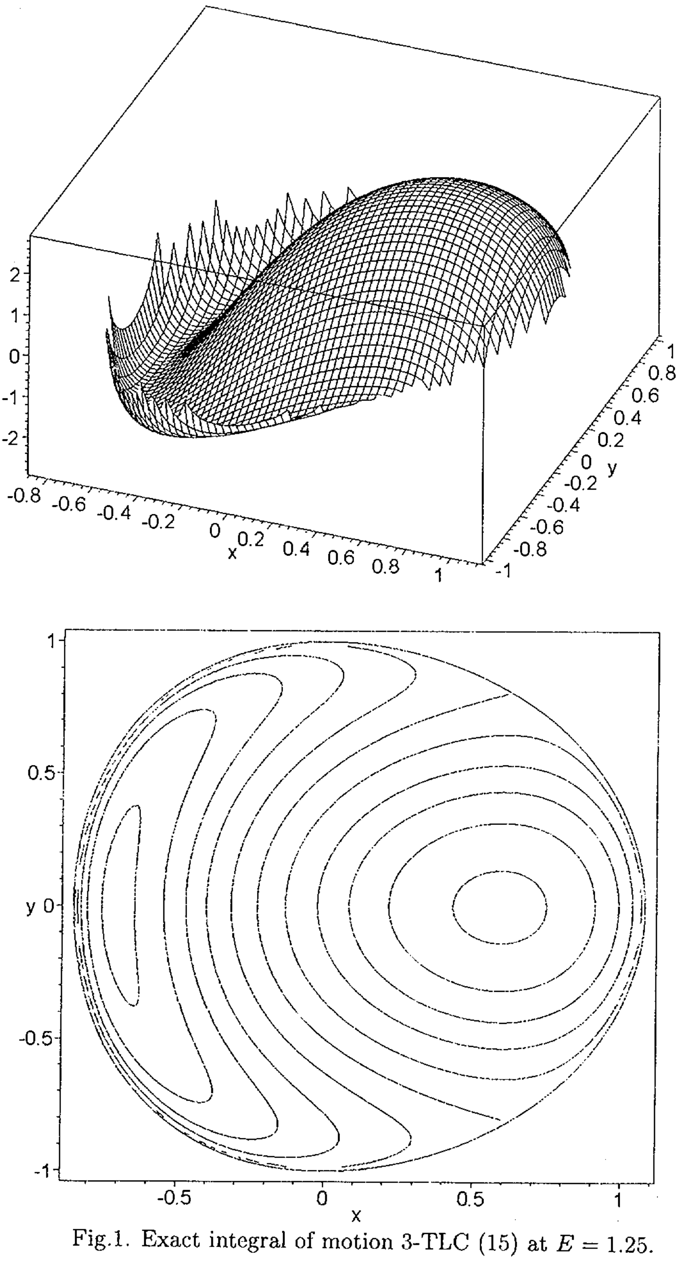


Fig.1. Exact integral of motion 3-TLC (15) at $E = 1.25$.

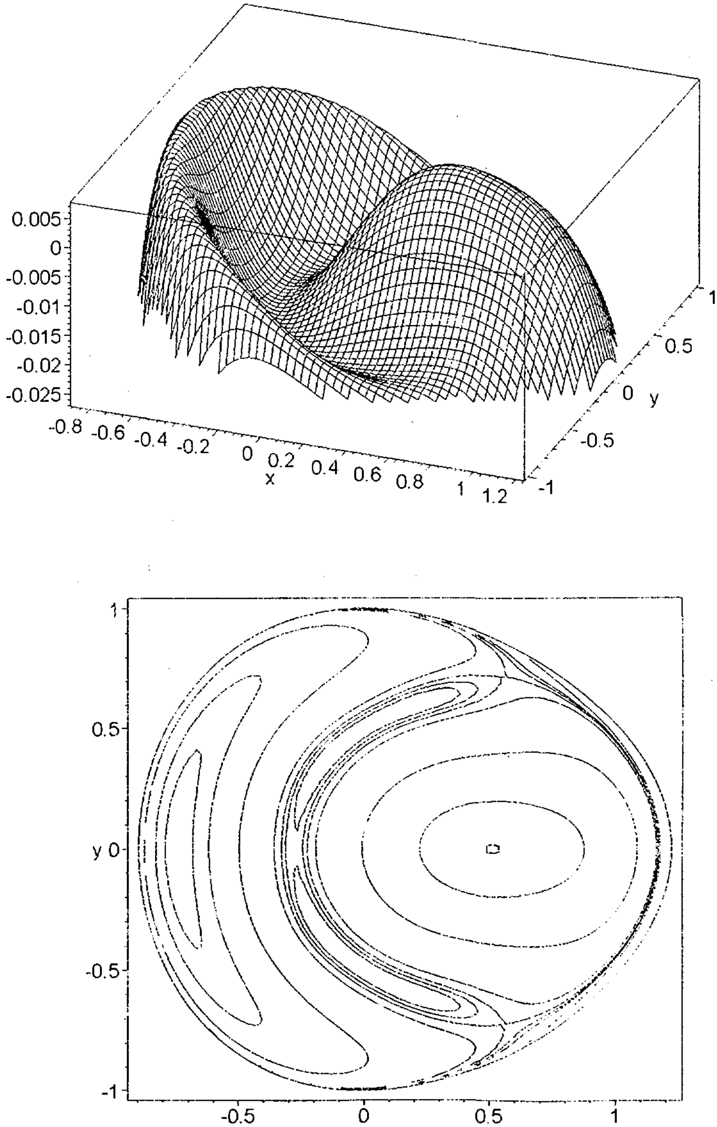


Fig.2. Approximate integral of motion 3-TLC(17b) $n_{max} = 3$, $s_{max} = 6$.

PES of the Hamiltonian (16). The NGC domain on the PES indeed emerges if the highest degree (n_{max}) in the truncated series (16) is an odd number and, on the contrary, if the highest degree is an even number then such NGC domain does not appear at all. The emergence of the NGC domain is linked with the saddle points on the PES. Moreover, if we take into account more and more terms in the expansion (16) then the NGC domain moves upwards on the PES and in the case of the infinite series the NGC region vanishes.

Below we have constructed the Birkhoff-Gustavson normal form and the approximate integrals of motion for some truncated Toda's Hamiltonians in order to understand a dependence of the structure of phase space on the inclusion of highest degree polynomial Hamiltonian (16). As an example, we present now the normal forms in the sixth $s_{max} = 6$ approach all but which are obtained for the different highest degree of Toda's series from the value $n_{max} = 3$ (Henon-Heiles's Hamiltonian) to $n_{max} = 5$. These Birkhoff-Gustavson normal forms are expressed below in the action-angle variables and are obtained with the aid of the GITA procedure:

$$\xi_\nu = \sqrt{2I_\nu}\cos(\phi_\nu), \quad \eta_\nu = \sqrt{2I_\nu}\sin(\phi_\nu), \quad (\nu = 1, 2).$$

$n_{max} = 3, \quad s_{max} = 6.$

$$G_3^{(6)} = (\frac{7}{1296}I_1^3 - \frac{7}{108}I_1^2 I_2 - \frac{7}{36}I_1^2 - \frac{7}{5184}I_1 I_2^2 + \frac{7}{432}I_2^3 - \frac{7}{144}I_2^2)cos(2\phi_2)$$
$$-\frac{155}{3888}I_1^3 - \frac{7}{108}I_1^2 I_2 + \frac{7}{5184}I_1 I_2^2 + \frac{35}{1296}I_2^3 - \frac{7}{144}I_2^2 - \frac{1}{12}I_1^2 + 2I_1 \qquad (17a)$$

$n_{max} = 4, \quad s_{max} = 6.$

$$G_4^{(6)} = (\frac{22}{81}I_1^3 + \frac{1}{18}I_1^2 I_2 - \frac{1}{9}I_1^2 - \frac{11}{162}I_1 I_2^2 - \frac{1}{72}I_2^3 + \frac{1}{36}I_2^2)cos(2\phi_2)$$
$$-\frac{2}{243}I_1^3 + \frac{1}{18}I_1^2 I_2 + \frac{11}{162}I_1 I_2^2 - \frac{5}{216}I_2^3 - \frac{1}{36}I_2^2 + \frac{1}{3}I_1^2 + 2I_1 \qquad (18a)$$

$n_{max} = 5, \quad s_{max} = 6.$

$$G_5^{(6)} = (\frac{4}{81}I_1^3 - \frac{1}{108}I_1^2 I_2 - \frac{1}{9}I_1^2 - \frac{1}{81}I_1 I_2^2 + \frac{1}{432}I_2^3 + \frac{1}{36}I_2^2)cos(2\phi_2)$$
$$-\frac{32}{243}I_1^3 - \frac{1}{108}I_1^2 I_2 + \frac{1}{81}I_1 I_2^2 + \frac{5}{1296}I_2^3 - \frac{1}{36}I_2^2 + \frac{1}{3}I_1^2 + 2I_1 \qquad (19a)$$

The second integrals in the corresponding approximation are also obtained by GITA procedure up to terms of degree s_{max} as quadratic form [3]

$$I^{(2)}(\xi[s_{max}], \eta[s_{max}]) = G(\xi[s_{max}], \eta[s_{max}]) - \sum_{\nu=1,2} \frac{1}{2}(\xi_\nu^2[s_{max}] + \eta_\nu^2[s_{max}]).$$

To obtain the integral in the original coordinates, GITA expresses the final variables $(\xi_\nu, \eta_\nu) = (\xi_\nu[s_{max}], \eta_\nu[s_{max}])$ in terms of variables $(q_\nu = \xi_\nu[2], p_\nu = \eta_\nu[2])$ making

$(s_{max}-2)$ coordinate transformations $\nu = 1,2;\quad s = 3,4,5,\ldots, s_{max}$ in accordance with eqs.(10)and definition of the generation function via coefficiens $W^{(s)}(\xi[s-1],\eta[s])$

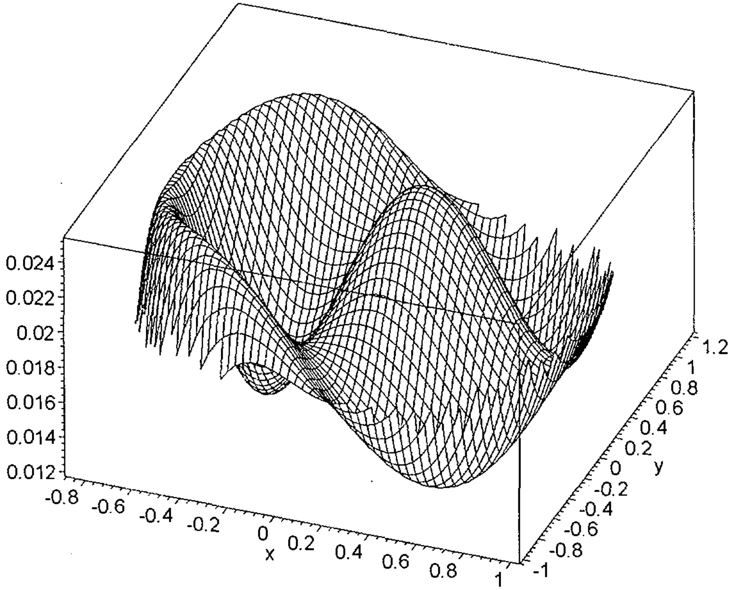
$$\xi_\nu[s] = \frac{\partial W^{(s)}}{\partial \eta_\nu[s]}(\xi[s-1],\eta[s]), \quad \eta_\nu[s-1] = \frac{\partial W^{(s)}}{\partial \xi[s-1]}(\xi[s-1],\eta[s]).$$

As an example we present the integrals (see Fig.2-4) in the explicit form : $n_{max}=3,\quad s_{max}=6.$

$$\begin{aligned}
I^{(2)} = &-\frac{385}{124416}p_2^6 + \frac{3311}{41472}p_2^4p_1^2 - \frac{385}{41472}p_2^4q_2^2 - \frac{1589}{41472}p_2^4q_1^2 - \frac{5}{288}p_2^4 \\
&+\frac{1225}{5184}p_2^3p_1q_2q_1 + \frac{7}{18\sqrt{6}}p_2^3p_1q_1 - \frac{2849}{41472}p_2^2p_1^4 + \frac{287}{6912}p_2^2p_1^2q_2^2 - \frac{7}{18\sqrt{6}}p_2^2p_1^2q_2 \\
&-\frac{35}{256}p_2^2p_1^2q_1^2 - \frac{5}{144}p_2^2p_1^2 - \frac{385}{41472}p_2^2q_2^4 + \frac{5}{216\sqrt{6}}p_2^2q_2^3 \\
&+\frac{245}{2304}p_2^2q_2^2q_1^2 - \frac{5}{144}p_2^2q_2^2 + \frac{23}{72\sqrt{6}}p_2^2q_2q_1^2 + \frac{763}{41472}p_2^2q_1^4 \\
&+\frac{1}{16}p_2^2q_1^2 - \frac{7}{5184}p_2p_1^3q_2q_1 - \frac{7}{54\sqrt{6}}p_2p_1^3q_1 + \frac{217}{5184}p_2p_1q_2^3q_1 \\
&-\frac{7}{36\sqrt{6}}p_2p_1q_2^2q_1 - \frac{791}{5184}p_2p_1q_2q_1^3 - \frac{7}{36}p_2p_1q_2q_1 \\
&-\frac{7}{36\sqrt{6}}p_2p_1q_1^3 + \frac{847}{124416}p_1^6 - \frac{2821}{41472}p_1^4q_2^2 + \frac{7}{54\sqrt{6}}p_1^4q_2 \\
&+\frac{847}{41472}p_1^4q_1^2 - \frac{5}{288}p_1^4 + \frac{1771}{41472}p_1^2q_2^4 - \frac{37}{216\sqrt{6}}p_1^2q_2^3 - \frac{91}{2304}p_1^2q_2^2q_1^2 \\
&+\frac{1}{16}p_1^2q_2^2 + \frac{1}{8\sqrt{6}}p_1^2q_2q_1^2 + \frac{623}{41472}p_1^2q_1^4 - \frac{5}{144}p_1^2q_1^2 - \frac{545}{124416}q_2^6 \\
&+\frac{5}{216\sqrt{6}}q_2^5 + \frac{943}{41472}q_2^4q_1^2 - \frac{5}{288}q_2^4 - \frac{5}{108\sqrt{6}}q_2^3q_1^2 - \frac{1537}{41472}q_2^2q_1^4 - \frac{5}{288}q_1^4 \\
&-\frac{5}{144}q_2^2q_1^2 - \frac{5}{72\sqrt{6}}q_2q_1^4 - \frac{49}{124416}q_1^6
\end{aligned} \qquad (17b)$$

$n_{max}=4,\quad s_{max}=6.$

$$\begin{aligned}
I^{(2)} = &\frac{35}{7776}p_2^6 + \frac{77}{2592}p_2^4p_1^2 + \frac{35}{2592}p_2^4q_2^2 - \frac{143}{2592}p_2^4q_1^2 + \frac{1}{72}p_2^4 \\
&+\frac{55}{324}p_2^3p_1q_2q_1 + \frac{2}{9\sqrt{6}}p_2^3p_1q_1 + \frac{7}{2592}p_2^2p_1^4 - \frac{11}{432}p_2^2p_1^2q_2^2 - \frac{2}{9\sqrt{6}}p_2^2p_1^2q_2 \\
&-\frac{25}{432}p_2^2p_1^2q_1^2 + \frac{1}{36}p_2^2p_1^2 + \frac{47}{2592}p_2^2q_2^4 - \frac{1}{54\sqrt{6}}p_2^2q_2^3
\end{aligned}$$



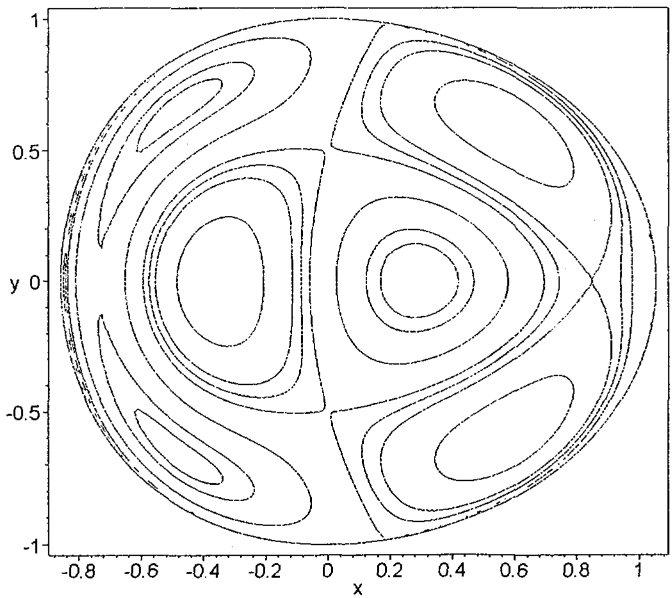


Fig 3. Approximate integral of motion 3-TLC(18b) $n_{max} = 4, s_{max} = 6$.

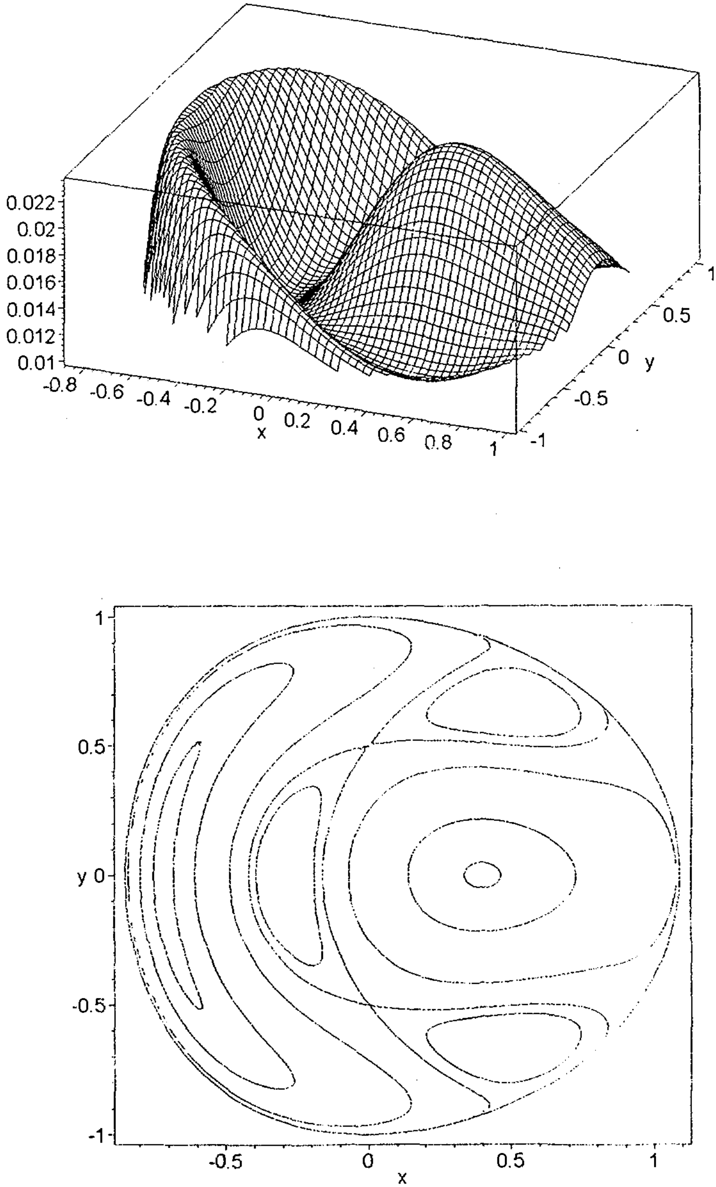


Fig 4. Approximate integral of motion 3-TLC(19b) $n_{max} = 5, s_{max} = 6$.

$$+\frac{1}{48}p_2^2q_2^2q_1^2+\frac{1}{36}p_2^2q_2^2+\frac{5}{18\sqrt{6}}p_2^2q_2q_1^2-\frac{17}{2592}p_2^2q_1^4$$
$$+\frac{1}{12}p_2^2q_1^2+\frac{41}{324}p_2p_1^3q_2q_1-\frac{2}{27\sqrt{6}}p_2p_1^3q_1+\frac{19}{324}p_2p_1q_2^3q_1$$
$$-\frac{1}{9\sqrt{6}}p_2p_1q_2^2q_1+\frac{13}{324}p_2p_1q_2q_1^3-\frac{1}{9}p_2p_1q_2q_1$$
$$-\frac{1}{9\sqrt{6}}p_2p_1q_1^3+\frac{49}{7776}p_1^6-\frac{157}{2592}p_1^4q_2^2+\frac{2}{27\sqrt{6}}p_1^4q_2$$
$$+\frac{49}{2592}p_1^4q_1^2+\frac{1}{72}p_1^4-\frac{11}{2592}p_1^2q_2^4-\frac{7}{54\sqrt{6}}p_1^2q_2^3+\frac{1}{144}p_1^2q_2^2q_1^2$$
$$+\frac{1}{12}p_1^2q_2^2+\frac{1}{6\sqrt{6}}p_1^2q_2q_1^2+\frac{53}{2592}p_1^2q_1^4+\frac{1}{36}p_1^2q_1^2+\frac{79}{7776}q_2^6$$
$$-\frac{1}{54\sqrt{6}}q_2^5+\frac{1}{2592}q_2^4q_1^2+\frac{1}{72}q_2^4+\frac{1}{27\sqrt{6}}q_2^3q_1^2+\frac{131}{2592}q_2^2q_1^4+\frac{1}{72}q_1^4$$
$$+\frac{1}{36}q_2^2q_1^2+\frac{1}{18\sqrt{6}}q_2q_1^4+\frac{53}{7776}q_1^6 \qquad (18b)$$

$n_{max}=5, \quad s_{max}=6.$

$$I^{(2)}=-\frac{35}{15552}p_2^6+\frac{175}{5184}p_2^4p_1^2-\frac{35}{5184}p_2^4q_2^2-\frac{205}{5184}p_2^4q_1^2+\frac{1}{72}p_2^4$$
$$+\frac{95}{648}p_2^3p_1q_2q_1+\frac{2}{9\sqrt{6}}p_2^3p_1q_1-\frac{175}{5184}p_2^2p_1^4-\frac{5}{846}p_2^2p_1^2q_2^2-\frac{2}{9\sqrt{6}}p_2^2p_1^2q_2$$
$$-\frac{25}{288}p_2^2p_1^2q_1^2+\frac{1}{36}p_2^2p_1^2-\frac{11}{5184}p_2^2q_2^4-\frac{1}{54\sqrt{6}}p_2^2q_2^3$$
$$+\frac{35}{864}p_2^2q_2^2q_1^2+\frac{1}{36}p_2^2q_2^2+\frac{5}{18\sqrt{6}}p_2^2q_2q_1^2+\frac{5}{5184}p_2^2q_1^4$$
$$+\frac{1}{12}p_2^2q_1^2+\frac{25}{648}p_2p_1^3q_2q_1-\frac{2}{27\sqrt{6}}p_2p_1^3q_1+\frac{23}{648}p_2p_1q_2^3q_1$$
$$-\frac{1}{9\sqrt{6}}p_2p_1q_2^2q_1-\frac{31}{648}p_2p_1q_2q_1^3-\frac{1}{9}p_2p_1q_2q_1$$
$$-\frac{1}{9\sqrt{6}}p_2p_1q_1^3+\frac{35}{15552}p_1^6-\frac{275}{5184}p_1^4q_2^2+\frac{2}{27\sqrt{6}}p_1^4q_2$$
$$+\frac{35}{5184}p_1^4q_1^2+\frac{1}{72}p_1^4+\frac{59}{5184}p_1^2q_2^4-\frac{7}{54\sqrt{6}}p_1^2q_2^3-\frac{19}{864}p_1^2q_2^2q_1^2$$
$$+\frac{1}{12}p_1^2q_2^2+\frac{1}{6\sqrt{6}}p_1^2q_2q_1^2+\frac{43}{5184}p_1^2q_1^4+\frac{1}{36}p_1^2q_1^2+\frac{53}{15552}q_2^6$$
$$-\frac{1}{54\sqrt{6}}q_2^5+\frac{23}{5184}q_2^4q_1^2+\frac{1}{72}q_2^4+\frac{1}{27\sqrt{6}}q_2^3q_1^2+\frac{73}{5184}q_2^2q_1^4+\frac{1}{72}q_1^4$$

$$+\frac{1}{36}q_2^2q_1^2+\frac{1}{18\sqrt{6}}q_2q_1^4+\frac{43}{15552}q_1^6 \qquad (19b)$$

The above integrals (17b)–(19b) and corresponding Poincare sections are presented in Figs. 2-4. One can observe how the sequence of these sections step by step tends to the limited section of the exact Toda integral (15) which is shown in Fig.1. Note that the approximate integrals of motion will describe well theoretically the regular phase trajectories similar to other generalization of the Henon-Heiles dynamical system [12]. In this way one may expect to find additional criteria of a true choice of BGNF structure related to exact integrals.

5 GITA^{-1} and the Inverse Problem of HLSE

GITA^{-1} consists of a core part and a subsidiary part. The core part is derived from GITA [3] and the subsidiary part contains the procedures characteristic of GITA^{-1}, both of which are put together in a single file. The procedure list and the input data (the input Hamiltonian) are loaded at the beginning of running GITA^{-1}.

As an example of program fulfillment, we take the inverse problem of HLSE. Let the input Hamiltonian H_{IN} be

$$H_{IN}(q,p)=\frac{1}{2}(p_1^2+p_2^2+q_1^2+q_2^2)+\frac{8}{3}\varepsilon(q_1^4-q_2^4), \qquad (20)$$

the Hamiltonian of regularized system of HLSE [8]. The input BGNF Hamiltonian, G_{IN}, for H_{IN} is calculated up to the fourth order to be

$$G_{IN}^{(4)}(\xi,\eta)=\varepsilon(\xi_1^2+\xi_2^2+\eta_1^2+\eta_2^2)(\xi_1^2+\eta_1^2-\xi_2^2-\eta_2^2), \qquad (21)$$

where ε is a parameter. The vector $\vec{\gamma}_{IN}^{(4)}$ in equation (13) is hence fixed by the coefficients, $\gamma_{IN}^{(4)}(\alpha_1,\alpha_2,\beta_1,\beta_2)$, of $G_{IN}^{(4)}$ through (10). It is worth noting that all these preliminary calculations can be made by GITA.

The GITA^{-1} starts with generating the field P_ℓ to describe the output Hamiltonian $H_{OUT}^{(4)}$, all of whose coefficients $c_{OUT}^{(4)}(\alpha,\beta)$ are unidentified. Next, eqs. (6)-(12) with $\lambda = OUT$ are proceeded in GITA^{-1} to calculate $G_{OUT}^{(4)}$, all of whose coefficients, $\gamma_{OUT}^{(4)}(\alpha,\beta)$, expressed in terms of $c_{OUT}^{(4)}(\alpha,\beta)$, are unidentified. On equating G_{IN} with G_{OUT} in GITA^{-1}, eq. (13) takes the following form:

$$\begin{aligned}3c_{OUT}^{(4)}(0,4,0,0)+c_{OUT}^{(4)}(0,2,0,2)+3c_{OUT}^{(4)}(0,0,0,4)&=-8\varepsilon,\\ 3c_{OUT}^{(4)}(4,0,0,0)+c_{OUT}^{(4)}(2,0,2,0)+3c_{OUT}^{(4)}(0,0,4,0)&=8\varepsilon,\end{aligned} \qquad (22)$$

where the coefficients, $c_{OUT}^{(4)}(k_1,k_2,\ell_1,\ell_2)$, not listed in (22) are set to zero.

Applying the subroutine SOLVE in REDUCE to equation (22), we have

$$\begin{aligned}&c_{OUT}^{(4)}(0,0,4,0)=u(4), && c_{OUT}^{(4)}(0,0,0,4)=u(2),\\ &c_{OUT}^{(4)}(0,4,0,0)=(-3u(2)-u(1)-8\varepsilon)/3, && c_{OUT}^{(4)}(0,2,0,2)=u(1),\\ &c_{OUT}^{(4)}(4,0,0,0)=(-3u(4)-u(3)+8\varepsilon)/3, && c_{OUT}^{(4)}(2,0,2,0)=u(3),\end{aligned}$$

as the solution of equation (22), where $u(i)$ $(i = 1,2,3,4)$ denote the unidentified constants 'arbcomplex(i)' introduced automatically in SOLVE.

Finally, we identify the output Hamiltonian to be

$$\begin{aligned} H^{(4)}_{OUT}(q,p) &= u(4)p_1^4 + u(2)p_2^4 + u(3)q_1^2p_1^2 + u(1)q_2^2p_2^2 \\ &+ \frac{1}{3}\{-3u(4) - u(3) + 8\varepsilon\} q_1^4 + \frac{1}{3}\{-3u(2) - u(1) - 8\varepsilon\} q_2^4, \end{aligned} \tag{23}$$

which admits G_{IN} given by eq. (21) as the BGNF. Note that if all the $u(i)$ vanish in eq. (23), H_{OUT} becomes identical with H_{IN} given by (20).

On setting $u(2) = u(4) = 0$ in eq. (23), H_{OUT} becomes

$$\begin{aligned} H_{OUT}(q,p) &= \frac{1+2u(3)q_1^2}{2}p_1^2 + \frac{1+2u(1)q_2^2}{2}p_2^2 \\ &+ q_1^2 + \frac{1}{3}\{-u(3) + 8\varepsilon\} q_1^4 + q_2^2 - \frac{1}{3}\{u(1) + 8\varepsilon\} q_2^4. \end{aligned}$$

Surprisingly, H_{OUT} turns out to be a Hamiltonian admitting the separation of variables, which provides an integrable system accordingly. Although it seems to be incidental that we encounter such integrable systems after GITA^{-1} , one may expect to find a class of integrable systems whose Hamiltonians reduce to a given BGNF Hamiltonian.

6 The Inverse Problem and Algorithm of ANFER

To pose the inverse problem appropriately to the ordinary problem given by Def. 2.3, we look at equation (3a), the key equation of the ordinary problem, in more detail: Let us regard $-W(q,\eta)$ (see (2)) as the non-normal type-3 generating function [13] associated with the canonical transformation, $(\xi,\eta) \to (q,p)$ through the relation

$$\xi = -\frac{\partial(-W)}{\partial\eta}, \qquad p = -\frac{\partial(-W)}{\partial q}.$$

Then equation (4) can read that $H(q,p)$ is restored from its BGNF series $G(\xi,\eta)$ through the canonical transformation associated with the non-normal type-3 generating function $-W(q,\eta)$. Hence we pose the inverse problem as follows:

Definition 6.1 (The inverse problem) *For a given BGNF in the form (4) identify all possible pairs of the power series $H(q,p)$ in the form (1) and the non-normal type-3 generating function,*

$$S(q,\eta) = -\sum_{j=1}^{n} q_j\eta_j - \sum_{k=3}^{\infty} S_k(q,\eta), \tag{2b}$$

which satisfy

$$H(q, -\frac{\partial S}{\partial q}) = G(-\frac{\partial S}{\partial \eta}, \eta). \tag{3b}$$

Since we usually dealt with the BGNF polynomials of even order, it would be better to restrict the inverse problem to the following form:

Definition 6.2 (The restricted inverse problem) *For a given BGNF, $G(\xi,\eta)$, in 2ρ***th order polynomial,** *identify all the possible pairs of the 2ρ***th order** *Hamiltonian $H(q,p)$ in the form (1) and the 2ρ***th order** *non-normal type-3 generating function $S(q,\eta)$ in the form of (2b), which satisfy Eq. (3b) up to the 2ρth order.*

Remark: For any 2ρth order BGNF $G(\xi,\eta)$, 2ρ is the highest order up to which both $H(q,p)$ and $S(\eta,q)$ can be identified completely.

We present an algorithm of solving equation (3b), part of ANFER , up to 2ρth order, where $G(\xi,\eta)$ is a 2ρ-th order polynomial in BGNF to be inverted.

Like in the case of solving the ordinary problem [2], we have to deal with a highly combinatorial problem of picking up the homogeneous part of degree k ($3 \le k \le 2\rho$) from equation (3b). To settle this problem, it is of great use to consider the composition of canonical transformations as follows.

Let us define a series of canonical transformations,

$$\phi_r : (\xi^{(r-1)},\eta^{(r-1)}) \to (\xi^{(r)},\eta^{(r)}) \quad \text{with} \quad (\xi^{(2)},\eta^{(2)}) = (\xi,\eta) \quad (r = 3,4,\cdots),$$

associated with the type-3 generating functions,

$$S^{(r)}(\xi^{(r)},\eta^{(r-1)}) = -\sum_{j=1}^{n} \xi_j^{(r)}\eta_j^{(r-1)} - S_r(\xi^{(r)},\eta^{(r-1)}) \quad (r = 3,4,\cdots),$$

where S_r is the homogeneous part of degree r of the generating function $S(q,\eta)$ in the form (2b). Using the $\{\phi_r\}$'s we have the following Lemmas:

Lemma 6.1 *The composition, $\phi_r \circ \phi_{r-1} \circ \cdots \circ \phi_3$, ($3 \le r$) of the canonical transformations, $\phi_3,\cdots,\phi_r$, is generated by the type-3 generating function of the form*

$$\tilde{S}^{(r)}(\xi^{(r)},\eta^{(2)}) = \sum_{k=3}^{r-1}\left(\sum_{j=1}^{n} \xi_j^{(k+1)}\eta_j^{(k)}\right) + \sum_{s=3}^{r} S^{(s)}(\xi^{(s)},\eta^{(s-1)}), \tag{24}$$

where $(\xi^{(s)},\eta^{(s-1)})$ ($s = 3,\cdots,r$) on the r. h. s. of (24) are regarded as the functions of $(\eta^{(2)},\xi^{(r)})$ through $\phi_3,\cdots,\phi_r$.

Lemma 6.2 *Let $H^{(r)}(\xi^{(r)},\eta^{(r)})$, be the power series determined by*

$$H^{(r)}(\xi^{(r)}, -\frac{\partial \tilde{S}^{(r)}}{\partial \xi^{(r)}}) = G(-\frac{\partial \tilde{S}^{(r)}}{\partial \eta^{(2)}}, \eta^{(2)}) \qquad (r = 3,4,\cdots). \tag{25}$$

Then up to rth order, $H^{(r)}(q,p)$ is identical with $H(q,p)$ determined by (2b).

With account of Lemma 6.2, solving Eq. (3b) up to 2ρth order amounts to solving equation (25). Since equation (25) is put together with Lemma 6.1 to imply

$$H^{(r)} \circ \phi_r \circ \cdots \circ \phi_3 = G \qquad (r = 3,4,\cdots),$$

we see that the $H^{(r)}$'s satisfy the equations

$$H^{(r)}(\xi^{(r)}, -\frac{\partial S^{(r)}}{\partial \xi^{(r)}}) = H^{(r-1)}(-\frac{\partial S^{(r)}}{\partial \eta^{(r-1)}}, \eta^{(r-1)}) \quad (r = 3, 4, \cdots) \qquad (26a)$$

with

$$H^{(2)}(\xi^{(2)}, \eta^{(2)}) = G(\xi^{(2)}, \eta^{(2)}), \qquad (26b)$$

which are the basic equations of constructing ANFER .

We proceed to solve equations (26a) up to $r = 2\rho$ now. Equating the kth order homogeneous part ($k = 2, 3, \cdots, 2\rho$) in (26a), we have

$$H_k^{(r)}(\xi^{(r)}, \eta^{(r-1)}) = H_k^{(r-1)}(\xi^{(r)}, \eta^{(r-1)}) \qquad (27a)$$

for $k = 2, \cdots, r-1$,

$$H_r^{(r)}(\xi^{(r)}, \eta^{(r-1)}) = (D_{q,\eta} S_r)(\xi^{(r)}, \eta^{(r-1)}) + H_r^{(r-1)}(\xi^{(r)}, \eta^{(r-1)}) \qquad (27b)$$

for $k = r$, and

$$H_k^{(r)}(\xi^{(r)}, \eta^{(r-1)}) = H_k^{(r-1)}(\xi^{(r)}, \eta^{(r-1)}) + \Theta_k^{(r)}(\xi^{(r)}, \eta^{(r-1)}) \qquad (27c)$$

for $k = r+1, \cdots, 2\rho$. The $\Theta_k^{(r)}(\xi^{(r)}, \eta^{(r-1)})$ in (27c) is the homogeneous polynomial of degree k ($k = r+1, \cdots, 2\rho$) given by [3]

$$\begin{aligned} &\Theta_k^{(r)}(\xi^{(r)}, \eta^{(r-1)}) \\ &= \sum_{|\alpha|=1}^{[\frac{k-2}{r-2}]} \frac{1}{\alpha!} \left[\left(\frac{\partial S_r}{\partial \eta}\Big|_{(\xi^{(r)}, \eta^{(r-1)})} \right)^{\alpha} \left(\left(\frac{\partial}{\partial \xi^{(r-1)}} \right)^{\alpha} H_{k-(r-2)|\alpha|}^{(r-1)} \right)\Big|_{(\xi^{(r)}, \eta^{(r-1)})} \right. \\ &\left. - \left(\frac{\partial S_r}{\partial q}\Big|_{(\xi^{(r)}, \eta^{(r-1)})} \right)^{\alpha} \left(\left(\frac{\partial}{\partial \eta^{(r)}} \right)^{\alpha} H_{k-(r-2)|\alpha|}^{(r)} \right)\Big|_{(\xi^{(r)}, \eta^{(r-1)})} \right], \end{aligned} \qquad (28a)$$

where $[\cdot]$ denotes the Gauss symbol, and $\alpha = (\alpha_1, \cdots, \alpha_n)$ is the multi-index with nonnegative-integer valued components associating the notations,

$$\begin{aligned} &|\alpha| = \sum_{j=1}^{n} \alpha_j, \qquad \left(\frac{\partial S_r}{\partial q} \right)^{\alpha} = \left(\frac{\partial S_r}{\partial q_1} \right)^{\alpha_1} \cdots \left(\frac{\partial S_r}{\partial q_n} \right)^{\alpha_n}, \\ &\left(\frac{\partial S_r}{\partial \eta} \right)^{\alpha} = \left(\frac{\partial S_r}{\partial \eta_1} \right)^{\alpha_1} \cdots \left(\frac{\partial S_r}{\partial \eta_n} \right)^{\alpha_n}, \\ &\left(\frac{\partial}{\partial \xi} \right)^{\alpha} = \frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial \xi_n^{\alpha_n}}, \left(\frac{\partial}{\partial \eta} \right)^{\alpha} = \frac{\partial^{\alpha_1}}{\partial \eta_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial \eta_n^{\alpha_n}}. \end{aligned} \qquad (28b)$$

In ANFER , equations (27) are solved recursively from $r = 3$ to $r = 2\rho$ as follows: Let us assume that $H^{(2)}, \cdots H^{(r-1)}$ and $S^{(3)}, \cdots, S^{(r-1)}$ (*i.e.*, $S_3, \cdots, S_{r-1}$)

have been identified already. Then it turns out that equations (27) are closed for $H_k^{(r)}$ $(k = 2, \cdots, 2\rho)$ and S_r. Since equation (27a) means merely an incrementation, we start with solving equation (27b). A key of solving Eq. (27b) is the direct-sum decomposition induced by the operator $D_{\xi^{(r)},\eta^{(r-1)}}$ (see (5)) of the vector spaces of kth order homogeneous polynomials of $(\xi^{(r)}, \eta^{(r-1)})$ denoted by $V_k(\xi^{(r)}, \eta^{(r-1)})$;

$$V_k(\xi^{(r)}, \eta^{(r-1)}) = \mathrm{Ker}\left(D^{(k)}_{\xi^{(r)},\eta^{(r-1)}}\right) \oplus \mathrm{Im}\left(D^{(k)}_{\xi^{(r)},\eta^{(r-1)}}\right), \tag{29a}$$

where

$$D^{(k)}_{\xi^{(r)},\eta^{(r-1)}} = D_{\xi^{(r)},\eta^{(r-1)}}\Big|_{V_k(\xi^{(r)},\eta^{(r-1)})}. \tag{29b}$$

Accordingly, decomposing $H_r^{(r)}$ and $H_r^{(r-1)}$ as

$$H_r^{(r-1)} = H_r^{(r-1),N} + H_r^{(r-1),I}, \qquad \left(\begin{array}{l} H_r^{(r-1),N} \in \mathrm{Ker}\left(D^{(r)}_{\xi^{(r-1)},\eta^{(r-1)}}\right), \\ H_r^{(r-1),I} \in \mathrm{Im}\left(D^{(r)}_{\xi^{(r-1)},\eta^{(r-1)}}\right) \end{array}\right),$$

$$H_r^{(r)} = H_r^{(r),N} + H_r^{(r),I}, \qquad \left(\begin{array}{l} H_r^{(r),N} \in \mathrm{Ker}\left(D^{(r)}_{\xi^{(r)},\eta^{(r)}}\right), \\ H_r^{(r),I} \in \mathrm{Im}\left(D^{(r)}_{\xi^{(r)},\eta^{(r)}}\right) \end{array}\right),$$

we can rewrite (27b) as a pair of equations:

$$H_r^{(r),N}(\xi^{(r)}, \eta^{(r-1)}) = H_r^{(r-1),N}(\xi^{(r)}, \eta^{(r-1)}), \tag{30a}$$

and

$$H_r^{(r),I}(\xi^{(r)}, \eta^{(r-1)}) = (D^{(r)}_{q,\eta} S_r)\Big|_{(\xi^{(r)},\eta^{(r-1)})} + H_r^{(r-1),I}(\xi^{(r)}, \eta^{(r-1)}). \tag{30b}$$

Equation (30a) obviously identifies $H_r^{(r),N}$ to be equal to $H_{r-1}^{(r),N}$. In contrast with (30a), a pair of unidentified functions, $H_r^{(r),I}$ and S_r, exists in Eq. (30b), so that we cannot get rid of an ambiguity in the identification; we identify

$$H_r^{(r),I} : \text{chosen arbitrarily as long as it is in } \mathrm{Im}\left(D^{(r)}_{\xi^{(r)},\eta^{(r-1)}}\right), \tag{31a}$$

and

$$S_r(\xi^{(r)}, \eta^{(r-1)}) = \left(D^{(r)}_{\xi^{(r)},\eta^{(r-1)}}\Big|_{\mathrm{Im}(D^{(r)})}\right)^{-1} (H_r^{(r),I} - H_r^{(r-1),I}). \tag{31b}$$

Once one has solved (27b), $H_k^{(r)}$ $(k = r+1, \cdots, 2\rho)$ are identified by (27c) with simple substitutions.

After repeating the process described above from $r = 3$ to $r = 2\rho$, $H^{(2\rho)}(q,p)$ thus obtained identifies the inverted Hamiltonian $H(q,p)$ (see (2b)) up to the 2ρth order. In ANFER , the above described process (24)–(31) has been implemented with Reduce 3.3 or its later version.

7 Example

7.1 Introduction

In this short note we solve the ordinary problem for simple test Hamiltonian,

$$K(q,p) = \frac{1}{2}(p^2 + q^2) + q^3. \tag{32}$$

The homogeneous 4th order term of its BGNF is obtained by the direct way. The inverse problem is solved also following the algorithm ANFER . Through out this note, numbers of the variables are reduced as far as such a reduction causes no confusions.

7.2 The Ordinary Problem

Since we would like to convert K into the BGNF up to the 4th order, it is convenient to denote by K_k $(k = 3, 4)$, the kth order part of K;

$$K_3(q,p) = q^3, \qquad K_4(q,p) = 0. \tag{33}$$

Let us further assume the type-2 generating function $W(q,\eta)$ and the BGNF $G(\xi,\eta)$ to be in the power series form,

$$W(q,\eta) = q\eta + \sum_{k=3}^{\infty} W_k(q,\eta),$$

and

$$G(\xi,\eta) = \frac{1}{2}(\eta^2 + \xi^2) + \sum_{\ell=2}^{\infty} G_{2\ell}(\xi,\eta),$$

where W_k (resp. $G_{2\ell}$) stand for the homogeneous k (resp. 2ℓ)th order parts of W (resp. G).

According to the algorithm of BG(Birkhoff-Gustavson)- normalization, what we have to solve is a system of equations[2],

$$G_3(q,\eta) + D_{q,\eta}W_3 = K_3(q,\eta), \tag{34a}$$

and

$$G_4(q,\eta) + D_{q,\eta}W_4 = K_4(q,\eta) + \frac{\partial W_3}{\partial q}\cdot\frac{\partial K_3}{\partial \eta} - \frac{\partial W_3}{\partial \eta}\cdot\frac{\partial G_3}{\partial q}$$
$$+\frac{1}{2}(\frac{\partial W_3}{\partial q})^2 - \frac{1}{2}(\frac{\partial W_3}{\partial \eta})^2 \equiv \tilde{K}_4(q,p) \tag{34b}$$

for both G and W, where

$$D_{q,\eta} = q\frac{\partial}{\partial \eta} - \eta\frac{\partial}{\partial q}.$$

[2] G_3 vanishes from the definition of BGNF

On introducing the complex variable z by

$$z = q + i\eta, \quad \overline{z} = q - i\eta$$

equation (34a) may be rewritten as

$$i(z\partial - \overline{z}\overline{\partial})W_3 = \frac{1}{8}(z^3 + 3z^2\overline{z} + 3z\overline{z}^2 + \overline{z}^3),$$

with

$$\partial = \frac{\partial}{\partial z}, \qquad \overline{\partial} = \frac{\partial}{\partial \overline{z}},$$

so that we have

$$G_3(\xi, \eta) = 0, \qquad W_3(q, \eta) = -\frac{i}{24}\left(z^3 + 9z^2\overline{z} - 9z\overline{z}^2 - \overline{z}^3\right). \tag{35}$$

We now proceed to the solution of (34b). Using the relations

$$\frac{\partial}{\partial q} = \partial + \overline{\partial}, \qquad \frac{\partial}{\partial \eta} = i(\partial - \overline{\partial}),$$

we have

$$\frac{\partial K_3}{\partial \eta} = 0, \quad \frac{\partial W_3}{\partial q} = -\frac{i}{2}(z^2 - \overline{z}^2), \quad \frac{\partial W_3}{\partial \eta} = -\frac{1}{4}(z^2 - 6z\overline{z} + \overline{z}^2). \tag{36}$$

Hence equations (35) and (36) are combined with equation (33) to put equation (34b) into the form

$$G_4(q, \eta) + D_{q,\eta}W_4 = \tilde{K}_4,$$

where

$$\tilde{K}_4 = \frac{1}{32}(-5z^4 + 12z^3\overline{z} - 30z^2\overline{z}^2 + 12z\overline{z}^3 - 5\overline{z}^4).$$

Since $D_{q,\eta}W_4$ cannot be normal we have

$$G_4(\xi, \eta) = -\frac{15}{16}z^2\overline{z}^2.$$

7.3 The Inverse Problem

In this Section, we solve the inverse problem for the harmonic oscillator Hamiltonian, the BGNF for the Hamiltonian K given by (32). Let $H(q,p)$ and $S(q,\eta)$ be the 4th order polynomial Hamiltonian and the 4th order type-3 generating function of the form

$$H(q,p) = \frac{1}{2}(p^2 + q^2) + H_3(q,p) + H_4(q,p)$$

and

$$S(q, \eta) = -q\eta - S_3(q, \eta) - S_4(q, \eta),$$

where H_k (resp. S_k) ($k = 3, 4$) stand for the homogeneous kth order parts of H (resp. S). The inverse problem is the problem of identifying all the H whose BGNF is identical with the harmonic oscillator up to the 4th order.

We apply the algorithm of $ANFER$ (24)-(31) to the present case: Following the notation of Section 6, we define the starting Hamiltonian $H^{(2)}$ to be

$$H_2^{(2)}(q,\eta) = \frac{1}{2}(\eta^2 + q^2), \qquad H_3^{(2)}(q,\eta) = 0, \quad H_4^{(2)}(q,\eta) = -\frac{15}{16}z^2\overline{z}^2.$$

Then at the stage of $r = 3$, equation (27b) is solved as

$$H_3^{(3),N} = H_3^{(2),N} = H_3^{(2)} = 0, \quad H_3^{(2),I} = H_3^{(2)} = 0;$$

$H_3^{(3),I}$: chosen arbitrarily in $Im\left(D^{(3)}_{\xi^{(3)},\eta^{(2)}}\right)$, i.e.

$$H_3^{(3)}(q,\eta) = c_1 z^3 + c_2 z^2\overline{z} + \overline{c}_2 z\overline{z}^2 + \overline{c}_1\overline{z}^3; \tag{37a}$$

from (31b)

$$S_3(\xi^{(3)},\eta^{(2)}) = \left(D^{(3)}_{\xi^{(3)},\eta^{(2)}}\Big|_{\mathrm{Im}(D^{(3)})}\right)^{-1}(H_3^{(3),I} - H_3^{(2),I}),$$

we have

$$S_3(q,\eta) = -i\left(\frac{c_1}{3}z^3 + c_2 z^2\overline{z} - \overline{c}_2 z\overline{z}^2 - \frac{\overline{c}_1}{3}\overline{z}^3\right), \tag{37b}$$

where c_j ($j = 1, 2$) are complex numbers arbitrarily chosen, and $\overline{c}_j$ their complex conjugate. Accordingly, we have from(27c) at k=4:

$$\begin{aligned}
\frac{\partial S_3}{\partial q} &= -i(c_1 + c_2)z^2 - 2i(c_2 - \overline{c}_2)z\overline{z} + i(\overline{c}_1 + \overline{c}_2)\overline{z}^2,\\
\frac{\partial S_3}{\partial \eta} &= (c_1 - c_2)z^2 + 2(c_2 + \overline{c}_2)z\overline{z} + (\overline{c}_1 - \overline{c}_2)\overline{z}^2,\\
\frac{\partial H_3^{(3)}}{\partial \eta} &= i(3c_1 - c_2)z^2 + 2i(c_2 - \overline{c}_2)z\overline{z} - i(3\overline{c}_1 - \overline{c}_2)\overline{z}^2,
\end{aligned} \tag{38}$$

where $H_3^{(2)}(q,\eta) = 0$, and $H_4^{(3)}(q,\eta)$ turns out to be $H_4^{(2)} + \Theta_4^{(3)}$. Using the definition of $\Theta_4^{(3)}$:

$$\Theta_4^{(3)} = \frac{\partial S_3}{\partial \eta}\frac{\partial H_3^{(2)}}{\partial q} - \frac{\partial S_3}{\partial q}\frac{\partial H_3^{(3)}}{\partial \eta} + \frac{1}{2}\left(\frac{\partial S_3}{\partial \eta}\right)^2 - \frac{1}{2}\left(\frac{\partial S_3}{\partial q}\right)^2,$$

we have

$$\begin{aligned}
H_4^{(3)}(q,\eta) = {} & 2(-c_1^2 - c_1c_2 + c_2^2)z^4\\
& + 4(-c_1c_2 + 2c_1\overline{c}_2 - c_2\overline{c}_2)z^3\overline{z}\\
& + 4(2\overline{c}_1c_2 - \overline{c}_1\overline{c}_2 - c_2\overline{c}_2)z\overline{z}^3\\
& + 2(-\overline{c}_1^2 - \overline{c}_1\overline{c}_2 + \overline{c}_2^2)\overline{z}^4\\
& + (6c_1\overline{c}_1 + 6c_2\overline{c}_2 - \frac{15}{16})z^2\overline{z}^2.
\end{aligned} \tag{39}$$

Substituting equations (37)–(39) in equation (27b) with $r = 4$, we have the following: $H_4^{(4),N} = H_4^{(3),N}$; $H_4^{(4),I}$: chosen arbitrarily in $Im\left(D^{(4)}_{\xi^{(4)},\eta^{(3)}}\right)$.
The non-normal part of $H_4^{(4)}$ denoted by $H^{(4),I}$ can be chosen to be arbitrarily non-normal, *i.e.*,

$$H_4^{(4),I}(q,\eta) = f_1 z^4 + f_2 z^3\overline{z} + \overline{f}_2 z\overline{z}^3 + \overline{f}_1\overline{z}^4,$$

where f_1 and f_2 are arbitrary complex numbers. In contrast with this, the normal part, denoted by $H_4^{(4),N}$, is determined uniquely to be

$$H^{(4),N}(q,\eta) = (6c_1\overline{c}_1 + 6c_2\overline{c}_2 - \frac{15}{16})z^2\overline{z}^2.$$

To summarize,

$$H(q,\eta) = H_2^{(2)} + H_3^{(3),I} + H_4^{(4),I} + H_4^{(4),N},$$

we see that all the 4th order polynomial Hamiltonians of the form

$$\begin{aligned} H(q,\eta) = \frac{1}{2}z\overline{z} + \left(c_1 z^3 + c_2 z^2\overline{z} + \overline{c}_2 z\overline{z}^2 + \overline{c}_1\overline{z}^3\right) \\ +[f_1 z^4 + f_2 z^3\overline{z} + \overline{f}_2 z\overline{z}^3 + \overline{f}_1\overline{z}^4 \\ +(6c_1\overline{c}_1 + 6c_2\overline{c}_2 - \frac{15}{16})z^2\overline{z}^2] + \mathcal{O}_4, \end{aligned} \tag{40}$$

share $G = (1/2)(\eta^2 + \xi^2)$ as the BGNF up to the 4th order, where $c_1, c_2, f_1, f_2 \in \mathbf{C}$ can be chosen arbitrarily.

The type-3 generating function $S(q,\eta)$ is identified up to 3rd order[3],

$$S(q,\eta) = -q\eta - i\left(\frac{c_1}{3}z^3 + c_2 z^2\overline{z} - \overline{c}_2 z\overline{z}^2 - \frac{\overline{c}_1}{3}\overline{z}^3\right) + \mathcal{O}_3.$$

7.4 Restoring Test Example

We wish to show that the Hamiltonian K defined by equation (32) is in the form of equation (40). Indeed, setting

$$c_1 = 1/8, \quad c_2 = 3/8, \quad f_1 = f_2 = 0, \tag{41}$$

in equation (40), we immediately obtain K. Further, equation (41) is put together with equation (35) to show that $W(q,\eta)$ is equal, up to 3rd order[3], to $S(q,\eta)$ with (41). Thus the normalization of K into G, the BGNF, and the restoration of G to K are completed.

[3] To identify H up to 4th order, it is sufficient to identify S up to 3rd order.

8 Concluding Remarks

We would like to make a few remarks on the peculiarities of the realization of $GITA^{-1}$. (i) In generating the field P_ℓ and equation (13), we have used REDUCE to implement the combinatorial algorithms and the list processing. (ii) As is seen from the algorithm (6)–(13) presented in Section 3, $GITA^{-1}$ is proceeded by tracing back the procedures of GITA in principle. Since ANFER might have the performance features different from $GITA^{-1}$, we may expect that GITA and $GITA^{-1}$ can be put together with ANFER to provide a unified symbolic computing program for various calculation around the BGNF expansion and integrable models in future.

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