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ON EFFICIENCY OF CRITICAL-COMPONENT METHOD FOR SOLVING DEGENERATE AND ILL-POSED SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

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1. Introduction

In this paper, we present results of studies of efficiency of the critical- component direct method proposed in [1 ÷ 3] for solving *degenerate* and *ill-posed* systems of linear algebraic equations

$$AZ = F, \tag{1.1}$$

where A is a square matrix of the general form with real elements $a_{ij}, A = \{a_{ij}\}, Z$ is an unknown vector with coordinates $z_j, Z = \{z_j\}$, and F is a known vector with coordinates $f_i, F = \{f_i\}, i,j = 1,2,...,m$. It is shown that for systems like (1.1) the critical-component method makes it possible to numerically determine the only normal pseudosolution $(Z^+ = A^+F)$: $||AZ^+ - F|| = \inf_{Z\in Z_A} ||AZ - F||, ||Z^+|| = \inf_{Z\in Z_A} ||Z||$, where Z_A is a set of all pseudosolutions to system (1.1), and to obtain the unique matrix A^+, pseudoinverse of A: $||A^+A - E|| = \inf_{\mathring{A}^{-1}\in\Omega_A} || \mathring{A}^{-1}A - E||, ||A^+|| = \inf_{\mathring{A}^{-1}\in\Omega_A} || \mathring{A}^{-1}||, A^+A = AA^+$, where E is a unit matrix and Ω_A is a set of all $\mathring{A}^{-1}$, pseudoinverse of A. In this case, even if the problem (1.1) is substantially ill- posed, the quantities Z^+ and A^+ are stable to small changes of input data (A, F). Comparative analysis of results of the numerical solution performed for a large number of problems like (1.1) both by the new method and by those known earlier shows that the critical-component method is on the average more effective than any method compared to it. When $\det A \neq 0$ and a system is well-posed, the normal pseudosolution Z^+ of system (1.1) derived by the critical-component method coincides with its usual solution Z, and $A^+ = A^{-1}$ is a matrix inverse of A. One of the main problems in numerical solution of ill-posed systems of algebraic equations is well-known [4,5,6]: there can be large changes in the solution, beyond the scope of admissible values, corresponding to small changes in the matrix of a system or/and its right-hand side. The above breakdown of continuity of the inverse mapping $Z = A^{-1}F$, if A^{-1} exists, is caused by a great norm $||A^{-1}||$ and, as a result, by large $\mu = \text{cond } A$, the condition number of the system matrix ($\mu = ||A|| \cdot ||A^{-1}||$, if $\det A \neq 0$ and $\mu = \infty$, if $\det A = 0$, where $||\cdot||$ are the corresponding norms), i.e. even for an exactly given vector F a negligible relative error in calculating A^{-1} can produce a large distortion of the searched vector Z. This effect is to be taken into account since realistic calculations are carried out with a certain finite accuracy and , besides, sometimes one knows not the exact system $AZ = F$, but only a system $\tilde{A}\tilde{Z} = \tilde{F}$, approximate of it, which obeys the inequalities $||\tilde{A} - A|| \leq h^*$ and $||\tilde{F} - F|| \leq \delta^*$ (the meaning of norms is defined by the character of a problem). The numbers $h^* > 0$ and $\delta^* > 0$, specifying the norms of deviations of approximate data $(\tilde{A}, \tilde{F})$ from the exact ones (A, F) of problem (1.1) $(h^* \leq h_0 + h_1, \delta^* \leq \delta_0 + \delta_1, h_0 \geq 0, h_1 > 0, \delta_0 \geq 0, \delta_1 > 0)$, are sums of (h_0, δ_0),proper model (complete) errors of problem (1.1) and of (h_1, δ_1), round-off errors [7,8] when writing the data into the computer memory. Since there are, thus, infinitely many systems (1.1) with the input data (A, F), indistinguishable within the accuracy (h^*, δ^*), we can speak only about deriving an approximate solution to system (1.1). As a result, difficulties may arise in numerical computations for some systems of equations (1.1) with square matrices when answering the following questions:

– is the system degenerate "within accuracy (h^*, δ^*)" ill-posed?*) and

– is a given system ill-posed by virtue of its being degenerate or is it nondegenerate but ill-posed?

Indeed, if the system $AZ = F$ with a square matrix is degenerate, then $\det A = 0$, i.e., the matrix A has some of its eigenvalues equal to zero. But if $\det A \neq 0$, and the system is ill-posed, then the normal matrix $A^T A$ has some eigenvalues only close to zero $\mu_1^2, ..., \mu_m^2$ ($|\mu_i|$ are singular values of the matrix A). Consequently, systems of linear algebraic equations with square matrices, which are ill-posed and degenerate "within a given accuracy (h^*, δ^*)" may turn out to be indistinguishable in the process of computations. Besides, the problems (1.1) and $\tilde{A}\tilde{Z} = \tilde{F}$ can be inconsistent if one defines the criterion of consistency [9] determined by accuracies (h^*, δ^*). It may also happen that $\det A = 0$ (or $\det \tilde{A} = 0$), i.e. system (1.1) (or $\tilde{A}\tilde{Z} = \tilde{F}$) has an infinite number of solutions. Then, there arises the question: what is to be understood by the numerical solution to the initial system $AZ = F$. There are various conceptual approaches to solve this problem (see, for instance, reviews given in [4,6,10], etc.).

If one takes advantage of the regularization [4], the solution Z^+ to the system $AZ = F$ (1.1) will be the regularized normal pseudosolution Z^α that minimizes the discrepancy $||\tilde{A}\tilde{Z} - \tilde{F}||$ on the set of all its pseudosolutions Z_A if $||Z^\alpha|| = \inf_{\bar{Z} \in Z_A} ||\bar{Z}||$ and Z^α is stable to small variations in (h^*, δ^*) of input data (A, F). The parametric vector Z^α is directly computed by solving the sequence of normal systems of equations $(\tilde{A}^T\tilde{A} + \alpha E)Z^\alpha = \tilde{A}^T\tilde{F}$ with the aim of a more accurate iterative determination of the minimum of quadratic functional $M^\alpha[\bar{Z}, \tilde{F}, \tilde{A}] = ||\tilde{A}\bar{Z} - \tilde{F}||^2 + \alpha||\bar{Z}||^2$ with the regularization parameter $\alpha (\alpha > 0)$, determined from the discrepancy, i.e., from the condition $||\tilde{A}Z^\alpha - \tilde{F}|| = \delta_*$, where $\delta_* (\delta_* > 0)$ is a numerical function of (h^*, δ^*) and Z^α [4,5,6].

The other group of numerical methods of solving the problem (1.1) rely on searching for the generalized matrix A^+, which is (pseudo)inverse of $\tilde{A}$, either by the method of singular decomposition ($\tilde{A} = U\Sigma V$, where U and V are orthogonal matrices, Σ is a diagonal matrix, whose elements are singular numbers $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_m| \geq 0$ of the matrix A, and $A^+ = V^T\Sigma^+U^T$), or by some other method [7,9,10,11]. Common to both of the approaches is that in their program realization they solve (each by its own means and with its own efficiency) the problems of minimization of norms $||\tilde{A}\bar{Z} - \tilde{F}||$ and $||\bar{Z}||$ and of the continuous dependence of the solution Z^+ on small changes in (h^*, δ^*) of input data (A, F). Here it is set that $\mu = \text{cond } A = ||\tilde{A}|| \cdot ||A^+||$, and the main problem now is a stable calculation of the rank of $\tilde{A}$ [7,9].

*) Systems degenerate "within accuracy (h^*, δ^*)" are not always ill-posed (as example is system (1.1) with $A = A^T$, singular (eigen)values $\mu_1 = \mu_2 = ... = \mu_m = 10^{-6}$, determinant $\det A = 10^{-6m}$ and the condition number $\text{cond } A = \mu = ||A|| \cdot ||A^{-1}|| = 1$).

Conceptually, the critical-component method can be attributed to the second of indicated groups of methods. It is based on the idea of constructive search (under the condition that matrix and vector norms are consistent: $||Z^+|| \leq ||A^+|| \cdot ||\tilde{F}||$; and the matrix norm is induced by the vector norm: $||A^+|| = \sup_{||\tilde{F}|| \neq 0} (||A^+\tilde{F}||/||\tilde{F}||)$ [5,9]) of an optimal representation for the matrix A^+, pseudoinverse of the matrix $\tilde{A}$, in the process of decomposition of system (1.1) into subsystems, whose solution is stable to errors *) $(\varepsilon_1, \varepsilon_0)$ and small (h^*, δ^*) changes of input data (A, F). High efficiency of the critical-component method is provided by its basic constituents:

– the reduction, stable to errors $(h^*, \delta^*; \varepsilon_1, \varepsilon_0)$, of system (1.1) to two-(tri)diagonal systems;
– generalized processes $\{\Lambda, G\}$, stable to errors $(\varepsilon_1, \varepsilon_0)$ [14], for calculating ratios of upper (lower) corner minors of triangular matrices which allow one, accurate within constants (ε_1 and ε_0) of the computer arithmetic, to determine the structure and diagonal elements of matrices that are inverse of them (introduced in [12,13]);
– the algorithm of optimal (with $(\varepsilon_1, \varepsilon_0)$) decomposition of the system $\tilde{A}\tilde{Z} = \tilde{F}$ into well-posed subsystems;
– the algorithm of optimal sewing of the solution Z^+ to the system $\tilde{A}\tilde{Z} = \tilde{F}$ from well-posed subspace solutions.

In what follows, along with problem (1.1) of the general form, we will consider the problems of numerical solution of degenerate and ill-posed systems of linear algebraic equations

$$C_3 X = Y, \tag{1.2}$$
$$C_2 \hat{X} = \hat{Y} \tag{1.3}$$

with square real matrices C_3 and C_2 of order m, of the tridiagonal and two-diagonal form respectively:

$$C_3 = \begin{bmatrix} q_1 & r_2 & & & \\ p_2 & q_2 & r_3 & & \\ & \ddots & \ddots & \ddots & \\ & & p_{m-1} & q_{m-1} & r_m \\ & & & p_m & q_m \end{bmatrix}, \quad C_2 = \begin{bmatrix} q_1 & r_2 & & \\ & q_2 & r_3 & \\ & & \ddots & \ddots \\ & & q_{m-1} & r_m \\ & & & q_m \end{bmatrix}, \tag{1.4}$$

where $X = (x_1, x_2, ..., x_m)^T$ and $\hat{X} = (\hat{x}_1, \hat{x}_2, ..., \hat{x}_m)^T$ are unknown vectors, and $Y = (y_1, y_2, ..., y_m)^T$ and $\hat{Y} = (\hat{y}_1, \hat{y}_2, ..., \hat{y}_m)^T$ are given m-dimensional vectors, $\{q_i\}_{i=1}^m$ are diagonal elements and $\{p_i, r_i\}_{i=2}^m$ are sub(off)diagonal elements of matrices C_3 and C_2.

*) Throughout we use the notation: $\varepsilon_1 (\varepsilon_1 > 0)$ is the modulus of relative error of the arithmetic of computer operations with real numbers with a floating point; $\varepsilon_0 (\varepsilon_0 > 0)$ is the modulus of absolute error of the computer zero θ, i.e. of any small real number (except for 0) from the interval $\theta \in (0 - \varepsilon_0, \varepsilon_0 + 0)$, where 0 is the usual zeroth element of the real axis. If $\theta \in (0 - \varepsilon_0, \varepsilon_0 + 0)$ and $\theta \neq 0$, it is accepted that $\theta = 0$ [8,9]. Using constants ε_1 and ε_0, one can estimate [7] errors of arrangement (writing) of the real $[m, m]$ matrix A and m-dimensional vector F in the computer memory in the form $||A_{comp} - A||_E \leq (\varepsilon_1 ||A||_E + \varepsilon_0 m \equiv h_1)$, $||F_{comp} - F||_E \leq (\varepsilon_1 ||F||_E + \varepsilon_0 \sqrt{m} \equiv \delta_1)$, where $||\cdot||_E$ are the Euclidean norms of matrices and vectors, and $h_1 > 0, \delta_1 > 0$.

respectively. Without loss of generality, we consider systems (1.3) only with the right two-diagonal matrix.

Since these problems are a particular case of problem (1.1), all said above applies also to problems (1.2) and (1.3), whose solutions X^+ and $\hat{X}^+$ are constructed more easily than Z^+. Therefore, in the course of program realization of the above conceptions of solution of problem (1.1), the initial stage [4,5,7,9,13,15] consists in its reduction to problems (1.2) and (1.3), i.e.

$$(1.5)\qquad \begin{cases} C_2(Q^TZ) = PF, \text{ where } C_2 = PAQ \text{ is a two-diagonal matrix, if } A \neq A^T, \\ C_3(Q^TZ) = Q^TF, \text{ where } C_3 = Q^TAQ \text{ is a tridiagonal matrix, if } A = A^T. \end{cases}$$

Here $U^TU = E = UU^T, U: Q, P$ are matrices of reflections or rotations. The orthogonal transformations (1.5) stable*) to errors $(h^*, \delta^*; \varepsilon_1, \varepsilon_0)$, do not often improve the nature of the problem being ill-/well- posed. Ill-posed systems of type (1.1) sometimes can numerically be reduced to ill-posed systems of type (1.2) and (1.3), with the notation in (1.5): $X = Q^TZ, Y = Q^TF$ and $\hat{X} = Q^TZ, \hat{Y} = PF$. Therefore the basic problem is numerical solution of such degenerate and ill-posed systems. Once the vectors X and $\hat{X}$ are obtained, we determine the solution to system (1.1), vector Z, in the form

$$(1.6)\qquad Z = QX \text{ and } Z = Q\hat{X}.$$

Numerical solution of ill-posed systems (1.2),(1.3) with tridiagonal and upper two-diagonal matrices can be best realized by the following methods [4,7,9] : the inverse substitution with normalization, regularization, a singular decomposition with exhaustion. In sect.3, we present (in particular) the results of comparison between computations performed by these methodes and by the new one.

*)It is known [5,7,9] that the Euclidean and spectral norms of matrices are invariant (theoretically) under the orthogonal transformations (1.5), i.e. there hold the equalities: $||C_2||_E = ||PAQ||_E = ||A||_E, ||C_3||_E = ||Q^TAQ||_E = ||A||_E$; $||C_2||_2 = ||PAQ||_2 = ||A||_2, ||C_3||_2 = ||Q^TAQ||_2 = ||A||_2$ and $||P||_2 = ||Q||_2 = 1, ||P||_E = ||Q||_E = \sqrt{m}$. As a result, $\mu = \text{cond } A = \text{cond } C_3$ (or cond C_2). Here $||PF||_E = ||\hat{Y}||_E, ||Q^TF||_E = ||Y||_E$; $||A||_2$ is a norm induced by the Euclidean vector norms $||Z||_E$ and $||F||_E$; $M(A) = m \cdot \max_{1\le i,j\le m} |a_{ij}|$ and $||A||_E$ are norms consistent with norms $||Z||_E$ and $||F||_E$. However, in real computations in process (1.5) of the reduction of system (1.1) to form (1.2) or (1.3), using the Houscholder U transformations (reflections), we obtain the estimates [7]

$$||(C_2)_{comp} - C_2||_E = ||(PAQ)_{comp} - PAQ||_E \le \left\{\left[\left(\tfrac{(2m-3)\varepsilon_r}{1-(m-2)\varepsilon_r}||A||_E + \tfrac{(2m-3)\sqrt{m}\cdot 0_r}{1-(m-2)\varepsilon_r}\right) \equiv f_2(m)||A||_E\right] \equiv h_2\right\},$$
$$||(PF)_{comp} - PF||_E \le (\varepsilon_r ||F||_E + 0_r) \equiv \delta_2 \text{ and}$$
$$||(C_3)_{comp} - C_3||_E = ||(Q^TAQ)_{comp} - Q^TAQ|| \le \left\{\left[\left(\tfrac{(2m-4)\varepsilon_r}{1-(m-2,5)\varepsilon_r}||A||_E + \tfrac{(2m-4)\sqrt{m}\cdot 0_r}{1-(m-2,5)\varepsilon_r}\right) \equiv f_3(m)||A||_E\right] \equiv h_2\right\},$$
$$||(Q^TF)_{comp} - Q^TF||_E \le (\varepsilon_r ||F||_E + 0_r) \equiv \delta_2, \text{ where } \varepsilon_r \sim 29\varepsilon_1 \text{ and } 0_r \sim (2m + 2\sqrt{m})\varepsilon_0, h_2 > 0, \delta_2 > 0.$$

Similar inequalities could also be written for $\tilde{A}, \tilde{F}$, where matrix $\tilde{A}$ and vector $\tilde{F}$ differ from A and F by simultaneous inclusion of inherited errors and errors of writing into the computer memory. From the above inequalities it follows that problems $AZ = F$ and $\tilde{A}\tilde{Z} = \tilde{F}$ are continuous with respect to the orthogonal transformations (1.5). Though the inherited errors (h_0, δ_0), if known, are, as a rule, much larger than the total $(h_1 + h_2, \delta_1 + \delta_2)$ effect of the errors of writing and transformations (1.5), the latter can influence the character (degree) of problem (1.2) or (1.3) being well-/ill-posed. The cited monographs contain also simplified estimates for errors h_2 and δ_2.

2. Critical-component method for numerical solution of degenerate and ill-posed systems of linear algebraic equations with tri- and two-diagonal matrices

Below we formulate the theorem according to which one can numerically obtain the only stable non-iterated normal pseudosolution X^+ of the system of linear algebraic equations of the general form (1.2), stable to errors $(\varepsilon_1, \varepsilon_0)$ and (h, δ), by the critical-component method.

The vector X^+ and the representation for the matrix consistent with it $(C_3^+ \equiv B)$, pseudoinverse to C_3, are determined as functions of stably computed vector $\overset{\circ}{X}$ (a regular component of X^+) and matrix $\overset{\circ}{B}$ (a regular component of C_3^+). In contradistinction to the problem of computation of singular numbers of matrices C_3 being unstable in nature, the critical-component method is stable owing to the stable processes of computation of the ratios of upper (lower) corner minors $\{\Lambda, \mathrm{G}\}$ of this matrix. Thus, the method of solution based on the search for a non-parametric stable component of the pseudoinverse matrix [7,9] found one more argument for its being efficient (contrary to conclusions of perturbation theory according to which X^+ and C_3^+ are not valid for computer calculations).

Theorem. Let $C_3X = Y$ be either a degenerate or an ill-posed system of linear algebraic equations with a square, of order m, real tridiagonal matrix of the general form C_3 (1.4). Also, let the system $\tilde{C}_3\bar{X} = \tilde{Y}$, where $||\tilde{C}_3 - C_3|| \le h$ and $||\tilde{Y} - Y|| \le \delta$, being an image of the system $C_3X = Y$ in the computer memory, be ill-posed but nondegenerate. Then the only pseudosolution X of the system $C_3X = Y$ that is minimal in norm $(||X^+|| = \min)$, obeys the condition of the norm of discrepancy being minimal $(||\tilde{C}_3X^+ - \tilde{Y}|| = \min)$, and is stable to computation errors $(\varepsilon_1, \varepsilon_0)$ and to small changes (h, δ) of the input data (C_3, Y), can numerically be obtained by the following direct critical-component method*):

Start of computations:
$k = 1, i = m;$

(2.1) $l_k = i;$

(2.2) $\overset{[k]}{x}_i = \sum_{\xi=1}^{l_k} \overset{[k]}{B}_{i\xi}\, y_\xi, \quad \overset{[k]}{\phi}_i = \begin{cases} 0, \text{ if } k = 1, \\ -\overset{[k]}{B}_{il_k}\, r_{l_k+1} x^+_{l_k+1}, \text{ if } k > 1; \end{cases}$

if $i = l_k$, then (2.5), otherwise (2.3);

(2.3) if $|\overset{[k]}{\phi}_i| < 1/\varepsilon_1$, then (2.4), otherwise $k = k + 1$ and (2.1);

(2.4) $j = i + 1, \overset{[k]}{x}_{l_k+1} = 0;$

$$\Phi_j = \begin{cases} |y_j| - |p_j \overset{[k]}{x}_{j-1} + q_j \overset{[k]}{x}_j + r_{j+1} \overset{[k]}{x}_{j+1}|, \text{ at } |y_j| \le 1, \\ 1 - |p_j \overset{[k]}{x}_{j-1} + q_j \overset{[k]}{x}_j + r_{j+1} \overset{[k]}{x}_{j+1}|/|y_j|, \text{ at } |y_j| > 1; \end{cases}$$

*)Here $h \le h_0 + h_1 + h_2$ and $\delta \le \delta_0 + \delta_1 + \delta_2$ if the system $\tilde{C}_3\bar{X} = \tilde{Y}$ is a reduced image of the system $AZ = F$; and $h \le \bar{h}_0 + \bar{h}_1, \delta \le \bar{\delta}_0 + \bar{\delta}_1$, where $(\bar{h}_0 \ge 0, \bar{\delta}_0 \ge 0)$ are hereditary errors and $(\bar{h}_1 > 0, \bar{\delta}_1 > 0)$ are errors of writing the system $C_3X = Y$ into the computer memory if system (1.1) is initially of form (1.2).

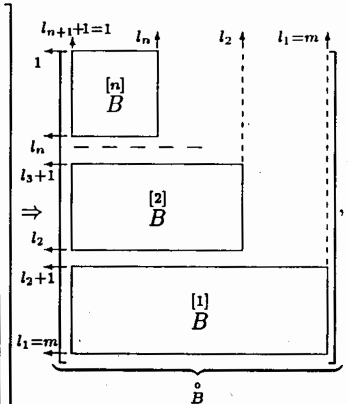
Since numerical solution is derived for the system $\tilde{C}_3\bar{X} = \tilde{Y}$ that is, within accuracy (h, δ), indistinguishable from the system $C_3X = Y$, for simplicity of the notation, the very algorithm of numerical method and its proof are given in the notation of the system $C_3X = Y$, i.e., without "~", if this does not cause misunderstanding. The requirement det $\tilde{C}_3 \neq 0$ of the theorem will be removed later. $X^+ = (x_1^+, x_2^+, ..., x_m^+)^T$.

if $|\Phi_j| \le 2\varepsilon_1$, then (2.5), otherwise $k = k+1$ and (2.1);

(2.5) $x_i^+ = \overset{[k]}{x}_i + \overset{[k]}{\phi}_i$;
if $i = 1$, computations are over, otherwise $i = i-1$ and (2.2);
End of computations.

Here:
$\overset{[k]}{B}_{ij}$ ($l_{k+1} \le i \le l_k$, $1 \le j \le l_k$ and $k = 1, 2, ..., n$) are elements of submatrices $\overset{[k]}{B}$ of the matrix $\overset{\circ}{B} = \overset{\circ}{C}{}_3^{-1}$ that is inverse of a well-posed matrix $\overset{\circ}{C}_3$ of the form:

$$(2.6)\qquad \overset{\circ}{C}_3 = \left[\begin{array}{ccc} \begin{bmatrix} q_1 & r_2 & & \\ p_2 & q_2 & r_3 & \\ & \cdots & & \\ & & p_{l_n} & q_{l_n} \end{bmatrix} = \overset{[n]}{C}{}_{l_n}^{(l_{n+1}+1=1)} & 0 & \\ - - - - - & & \\ p_{l_3+1} & \begin{bmatrix} \tilde q_{l_3+1} & r_{l_3+2} & \\ p_{l_3+2} & q_{l_3+2} & r_{l_3+3} \\ & \cdots\cdots & \\ & p_{l_2} & q_{l_2} \end{bmatrix} = \overset{[2]}{C}{}_{l_2}^{l_3+1} & 0 \\ & p_{l_2+1} & \overset{[1]}{C}{}_{(l_1=m)}^{l_2+1} = \begin{bmatrix} \tilde q_{l_2+1} & r_{l_2+2} & \\ p_{l_2+2} & q_{l_2+2} & r_{l_2+3} \\ & \cdots\cdots & \\ & p_m & q_m \end{bmatrix} \end{array}\right] \Rightarrow$$



,

where

$$(2.7)\qquad \tilde q_{l_{k+1}+1} = q_{l_{k+1}+1} - p_{l_{k+1}+1} \overset{[k+1]}{B}_{l_{k+1} l_{k+1}} r_{l_{k+1}+1}, \; k = 1, 2, .., n-1$$

and $\overset{[k+1]}{B}_{l_{k+1} l_{k+1}}$ are the last diagonal elements of submatrices $\overset{[k+1]}{B}$ which coincide with the last diagonal elements of submatrices, inverse of well-posed submatrices $\overset{[k+1]}{C}{}_{k+1}^{l_{k+2}+1}$ separated by the method; and n is the number of separated subspaces.

Elements $\overset{[k]}{B}_{ij}$ are calculated [1] by the formulae:

(2.8)

$$\overset{[k]}{B}_{ij} = \begin{cases} \omega_i \prod\limits_{\xi=j+1}^{i} \beta_\xi, \text{ if } 1 \le j < i, l_{k+1}+1 \le i \le l_k, \\ 0 \text{ for all } i \text{ from } j < i \le l_k, \text{ if } \Lambda_j = 0, \text{ for any } j \text{ from } l_{k+1}+2 \le j \le l_k, \\ 0 \text{ for all } j \text{ from } 1 \le j < i, \text{ if } \overset{[k]}{G}_i = 0, \text{ for any } i \text{ from } l_{k+1}+1 \le i \le l_k - 1, \\ \omega_i \prod\limits_{\xi=i+1}^{j} \overset{[k]}{\hat\beta}_\xi, \text{ if } l_{k+1}+1 \le i < j \le l_k, \\ 0 \text{ for all } i \text{ from } l_{k+1}+1 \le i < j, \text{ if } \overset{[k]}{G}_j = 0, \\ 0 \text{ for all } j \text{ from } i < j \le l_k, \text{ if } \Lambda_i = 0. \end{cases}$$

Diagonal elements $\overset{[k]}{B}_{ii}$ of submatrices $\overset{[k]}{B}$ and quantities ω_i in (2.8) are calculated [1] by

the formulae:

$$(2.9)\qquad \begin{cases} \overset{[k]}{B}_{ii} = (\Lambda_{i+1} + \overset{[k]}{G}_{i-1} - q_i)^{-1} \text{and } \omega_i = \overset{[k]}{B}_{ii}, \text{ if } \Lambda_i \ne 0 \ne \overset{[k]}{G}_i. \\ \overset{[k]}{B}_{ii} = 0, \overset{[k]}{B}_{i-1i-1} = \overset{[k]}{G}_{i-1}\,\omega_i, \overset{[k]}{B}_{i+1i+1} = \overset{[k]}{G}{}_i^{-1} \text{and } \omega_i = (-p_i r_i)^{-1}, \text{ if } \Lambda_i = 0. \\ \overset{[k]}{B}_{ii} = 0, \overset{[k]}{B}_{i-1i-1} = \Lambda_i^{-1}, \overset{[k]}{B}_{i+1i+1} = \Lambda_{i+1}\omega_i \text{ and } \omega_i = (-r_{i+1}p_{i+1})^{-1}, \text{ if } \overset{[k]}{G}_i = 0. \end{cases}$$

sequences $\{\Lambda\}$ and $\{\overset{[k]}{G}\}$ are computed by the formulae:

$$(2.10)\qquad \begin{cases} \Lambda_{i+1} = q_i - p_i\Lambda_i^{-1}r_i, \; \Lambda_2 = q_1, \; i = 2, ..., m, \text{ if } \Lambda_i \ne 0 \text{ for all } 2 \le i \le m; \\ \text{if } \Lambda_i = 0 \text{ for any } i \text{ from } (2 \le i \le m), \text{ then } \Lambda_{i+1} - \text{ is undefined, but } \Lambda_{i+2} = q_{i+1}; \end{cases}$$

$$(2.11)\qquad \begin{cases} \overset{[k]}{G}_{i-1} = q_i - r_{i+1}\overset{[k]}{G}{}_i^{-1}p_{i+1}, \overset{[k]}{G}_{l_k-1} = q_{l_k}, i = l_k-1, l_k-2, ..., l_{k+1}+1, \text{ if } \overset{[k]}{G}_i \ne 0; \\ \text{if } \overset{[k]}{G}_i = 0 \text{ for any } i \text{ from } (l_{k+1}+1 \le i \le l_k - 1), \text{ then } \overset{[k]}{G}_{i-1} - \text{is undefined.} \\ \text{but } \overset{[k]}{G}_{i-2} = q_{i-1}. \end{cases}$$

The structure elements β_ξ and $\overset{[k]}{\hat\beta}_\xi$ which determine the elements of submatrices $\overset{[k]}{B}$ and their products $\prod\beta_\xi$ and $\prod\overset{[k]}{\hat\beta}_\xi$ are computed [1] by the formulae:

$$(2.12)\qquad \beta_i = \begin{cases} -p_i\Lambda_i^{-1}, \text{ if } \Lambda_i \ne 0, \\ -p_i, \text{ and } \beta_{i+1} = -p_{i+1}\omega_i, \\ \text{if } \Lambda_i = 0; \end{cases} \quad \overset{[k]}{\hat\beta}_{i+1} = \begin{cases} -r_{i+1}\overset{[k]}{G}{}_i^{-1}, \text{ if } \overset{[k]}{G}_i \ne 0, \\ -r_{i+1}, \text{ and } \overset{[k]}{\hat\beta}_i = -r_i\omega_i, \\ \text{if } \overset{[k]}{G}_i = 0; \end{cases}$$

$$(2.13)\qquad \prod_{\xi=j+1}^{i}\beta_\xi = \begin{cases} \beta_i \cdot \beta_{i-1} \cdots \beta_{j+1}, \text{ if } j < i, \\ 1, \text{ if } j \ge i; \end{cases} \quad \prod_{\xi=i+1}^{j}\overset{[k]}{\hat\beta}_\xi = \begin{cases} \overset{[k]}{\hat\beta}_{i+1} \cdots \overset{[k]}{\hat\beta}_{j-1} \cdot \overset{[k]}{\hat\beta}_j, \text{ if } i < j, \\ 1, \text{ if } i \ge j; \end{cases}$$

Proof. Let the system $\tilde C_3 \tilde X = \tilde Y$, according to the theorem condition, be ill-posed but non-degenerate. Then its solution X^+ with the properties given in the theorem does *theoretically* exist and it is unique. Let us show that it can numerically be obtained by the method (2.1) ÷ (2.13) called in [1] the critical-component method. To this end, we verify first that to the solution X^+ there corresponds the following generalized LDR [1]

decomposition of the matrix C_3 (1.4):

$$
(2.14)\qquad C_3 = LDR = \begin{bmatrix} \begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix} & & \\ (p_{l_3+1}\overset{[3]}{B}_{l_3 l_4+1})\cdots(p_{l_3+1}\overset{[3]}{B}_{l_3 l_3}) & \begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix} & \\ & (p_{l_2+1}\overset{[2]}{B}_{l_2 l_3+1})\cdots(p_{l_2+1}\overset{[2]}{B}_{l_2 l_2}) & \begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} \begin{bmatrix} q_1 & r_2 & & \\ p_2 & q_2 & r_3 & \\ & \cdots\cdots & & \\ & & p_{l_n} & q_{l_n} \end{bmatrix} = \overset{[n]}{C}{}_{l_n}^{(l_{n+1}+1=1)} & & \\ & \begin{bmatrix} \tilde{q}_{l_3+1} & r_{l_3+2} & & \\ p_{l_3+2} & q_{l_3+2} & r_{l_3+3} & \\ & \cdots\cdots\cdots & & \\ & & p_{l_2} & q_{l_2} \end{bmatrix} = \overset{[2]}{C}{}_{l_2}^{l_3+1} & \\ & & \overset{[1]}{C}{}_{(l_1=m)}^{l_2+1} = \begin{bmatrix} \tilde{q}_{l_2+1} & r_{l_2+2} & & \\ p_{l_2+2} & q_{l_2+2} & r_{l_2+3} & \\ & \cdots\cdots\cdots & & \\ & & p_m & q_m \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix} \begin{matrix} (\overset{[n]}{B}_{1 l_n} r_{l_n+1}) \\ \vdots \\ (\overset{[n]}{B}_{l_n l_n} r_{l_n+1}) \end{matrix} & & \\ & \begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix} \begin{matrix} (\overset{[2]}{B}_{l_3+1 l_2} r_{l_2+1}) \\ \vdots \\ (\overset{[2]}{B}_{l_2 l_2} r_{l_2+1}) \end{matrix} & \\ & & \begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix} \end{bmatrix},
$$

where it is assumed that tridiagonal matrices $\overset{[k]}{C}{}_{l_k}^{l_{k+1}+1} (k = 1, 2, ..., n; l_1 = m, l_{n+1} + 1 = 1)$ are well-posed and their first diagonal elements are denoted by

$$
(2.14)' \qquad \tilde{q}_{l_{k+1}+1} = q_{l_{k+1}+1} - p_{l_{k+1}+1} \overset{[k+1]}{B}_{l_{k+1} l_{k+1}} r_{l_{k+1}+1}, k = 1, 2, .., n-1,
$$

where $\{p_i, q_i, r_i\}_{i=l_{k+2}+1}^{l_{k+1}}$ are elements of the initial matrix C_3 (1.4), $\overset{[k+1]}{B}_{l_{k+1} j}$ $(j = l_{k+1}, l_{k+1}-1, ..., l_{k+2}+1)$ are the last rows and $\overset{[k+1]}{B}_{i l_{k+1}}$ $(i = l_{k+1}, l_{k+1}-1, ..., l_{k+2}+1)$ are the last columns of matrices, inverse of the matrices $\overset{[k+1]}{C}{}_{l_{k+1}}^{l_{k+2}+1}$, computed in accordance with $\overset{[k]}{B}_{ij}$ (2.8) since they are elements of rectangular submatrices $\overset{[k]}{B}$ (2.8). From the assumptions for C_3 being nonsingular and for square matrices $\overset{[k]}{C}{}_{l_k}^{l_{k+1}+1}$ being well-posed it follows that the LDR decomposition (2.14) is unique and stable to errors $(h, \varepsilon_1, \varepsilon_0)$.

And the matrix $B = C_3^+$ can uniquely be represented in the form $(B = (E + \Omega)\, \mathring{B})$:

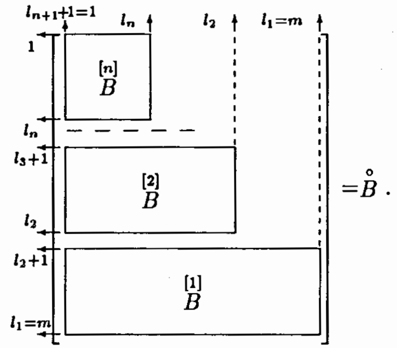
$$
(2.15)\qquad B = (\mathring{B} = \begin{bmatrix} \begin{bmatrix} q_1 & r_2 & & \\ p_2 & q_2 & r_3 & \\ & \cdots\cdots & & \\ & & p_{l_n} & q_{l_n} \end{bmatrix}^{-1} = (\overset{[n]}{C}{}_{l_n}^{(l_{n+1}+1=1)})^{-1} & & \\ & \begin{bmatrix} \tilde{q}_{l_3+1} & r_{l_3+2} & & \\ p_{l_3+2} & q_{l_3+2} & r_{l_3+3} & \\ & \cdots\cdots\cdots & & \\ & & p_{l_2} & q_{l_2} \end{bmatrix}^{-1} = (\overset{[2]}{C}{}_{l_2}^{l_3+1})^{-1} & \\ & & (\overset{[1]}{C}{}_{(l_1=m)}^{l_2+1})^{-1} = \begin{bmatrix} \tilde{q}_{l_2+1} & r_{l_2+2} & & \\ p_{l_2+2} & q_{l_2+2} & r_{l_2+3} & \\ & \cdots\cdots\cdots & & \\ & & p_m & q_m \end{bmatrix}^{-1} \end{bmatrix} \times
$$

$$
\times \begin{bmatrix} \begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix} & & \\ (-p_{l_3+1}\overset{[3]}{B}_{l_3 l_4+1}) \cdots (-p_{l_3+1}\overset{[3]}{B}_{l_3 l_3}) & \begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix} & \\ (p_{l_2+1}\overset{[2]}{B}_{l_2 l_3+1} p_{l_3+1}\overset{[3]}{B}_{l_3 l_4+1}) \cdots (p_{l_2+1}\overset{[2]}{B}_{l_2 l_3+1} p_{l_3+1}\overset{[3]}{B}_{l_3 l_3}) & (-p_{l_2+1}\overset{[2]}{B}_{l_2 l_3+1}) \cdots (-p_{l_2+1}\overset{[2]}{B}_{l_2 l_2}) & \begin{bmatrix}1 & & \\ & \ddots & \\ & & 1\end{bmatrix} \end{bmatrix}) +
$$

$$
+ (\Omega = \begin{bmatrix} \begin{bmatrix}0 & & \\ & \ddots & \\ & & 0\end{bmatrix} \begin{matrix} (-\overset{[n]}{B}_{1 l_n} r_{l_n+1})\, 0 \ldots 0\, (\overset{[n]}{B}_{1 l_n} r_{l_n+1} \overset{[2]}{B}_{l_3+1 l_2} r_{l_2+1}) \\ \vdots \qquad \cdots \qquad \vdots \\ (-\overset{[n]}{B}_{l_n l_n} r_{l_n+1})\, 0 \ldots 0\, (\overset{[n]}{B}_{l_n l_n} r_{l_n+1} \overset{[2]}{B}_{l_3+1 l_2} r_{l_2+1}) \end{matrix} & \\ \begin{bmatrix}0 & & \\ & \ddots & \\ & & 0\end{bmatrix} \begin{matrix} (-\overset{[2]}{B}_{l_3+1 l_2} r_{l_2+1}) \\ \vdots \\ (-\overset{[2]}{B}_{l_2 l_2} r_{l_2+1}) \end{matrix} & \\ & \begin{bmatrix}0 & & \\ & \ddots & \\ & & 0\end{bmatrix} \end{bmatrix}) \cdot \mathring{B}\,.
$$

Schematically, the matrix $\overset{\circ}{B}$ can be represented as follows:

(2.15)′



Representation (2.15) is easily established by a direct verification of the matrix equalities $C_3B = E = BC_3$, with representations (2.8) ÷ (2.13) taken into account for elements $\overset{[k]}{B}_{ij}$ of the matrix $\overset{\circ}{B}$ and decompositions of C_3 and B given by (2.14) and (2.15).

Now using representation (2.15) for B, we obtain components of the vector X^+ in form (2.2) and (2.5). From (2.15) it follows that X^+ can be written in the form

(2.16) $$X^+ = (E+\Omega)\overset{\circ}{X} = \overset{\circ}{X} + \Omega\overset{\circ}{X},$$

where the vector $\overset{\circ}{X}$ looks as follows

(2.17) $$\overset{\circ}{X} = (\overset{\circ}{B} \equiv \overset{\circ}{C}_3{}^{-1})Y$$

and is a unique, stable to errors (h,δ) and $(\varepsilon_1,\varepsilon_0)$, solution of the well-posed system of linear algebraic equations

(2.18) $$\left(\overset{\circ}{C}_3 = \begin{bmatrix} \begin{bmatrix} q_1 & r_2 & & \\ p_2 & q_2 & r_3 & \\ & \cdots\cdots & & \\ & & p_{l_n} & q_{l_n} \end{bmatrix} = \overset{[n]}{C}{}_{l_n}^{(l_{n+1}+1=1)} \quad 0 & & \\ - - - - - & & \\ p_{l_3+1} \begin{bmatrix} \tilde{q}_{l_3+1} & r_{l_3+2} & \\ p_{l_3+2} & q_{l_3+2} & r_{l_3+3} \\ & \cdots\cdots\cdots & \\ & p_{l_2} & q_{l_2} \end{bmatrix} = \overset{[2]}{C}{}_{l_2}^{l_3+1} \quad 0 & \\ & p_{l_2+1} \quad \overset{[1]}{C}{}_{(l_1=m)}^{l_2+1} = \begin{bmatrix} \tilde{q}_{l_2+1} & r_{l_2+2} & \\ p_{l_2+2} & q_{l_2+2} & r_{l_2+3} \\ & \cdots\cdots\cdots & \\ & p_m & q_m \end{bmatrix} \end{bmatrix}\right) \cdot \underbrace{\begin{bmatrix} \begin{bmatrix} \overset{[n]}{x}_1 \\ \vdots \\ \overset{[n]}{x}_{l_n} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \overset{[2]}{x}_{l_3+1} \\ \vdots \\ \overset{[2]}{x}_{l_2} \end{bmatrix} \\ \begin{bmatrix} \overset{[1]}{x}_{l_2+1} \\ \vdots \\ \overset{[1]}{x}_m \end{bmatrix} \end{bmatrix}}_{\overset{\circ}{X}} = \underbrace{\begin{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_{l_n} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} y_{l_3+1} \\ \vdots \\ y_{l_2} \end{bmatrix} \\ \begin{bmatrix} y_{l_2+1} \\ \vdots \\ y_m \end{bmatrix} \end{bmatrix}}_{Y},$$

which differs from the initial system $C_3X = Y$ by the change of the corresponding off-diagonal elements to zeros and of diagonal elements q to elements $\tilde{q}$ calculated by formulae

(2.14)′. Here the vector $\overset{\circ}{X}$ includes components given by sums (2.2), which follows from the representation $\overset{\circ}{B}$ (2.15) and (2.8).

For the matrix Ω (2.15) we can write the following decomposition

(2.19) $$\Omega = \begin{bmatrix} \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} \begin{matrix} (-\overset{[n]}{B}_{1l_n}r_{l_n+1}) \\ \vdots \\ (-\overset{[n]}{B}_{l_nl_n}r_{l_n+1}) \end{matrix} & \\ - - - - - & \\ & \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} \begin{matrix} (-\overset{[2]}{B}_{l_3+1l_2}r_{l_2+1}) \\ \vdots \\ (-\overset{[2]}{B}_{l_2l_2}r_{l_2+1}) \end{matrix} \\ & \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} \end{bmatrix} \cdots \begin{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} & \\ - - - & \\ & \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{matrix} (-\overset{[2]}{B}_{l_3+1l_2}r_{l_2+1}) \\ \vdots \\ (-\overset{[2]}{B}_{l_2l_2}r_{l_2+1}) \end{matrix} \\ & \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \end{bmatrix}.$$

Then for components of the vector $\Omega\cdot\overset{\circ}{X}$, denoted as vector ϕ, we obtain the explicit form

(2.20) $$\Omega\cdot\overset{\circ}{X} = [(-\overset{[n]}{B}_{1l_n}r_{l_n+1})x^+_{l_n+1}, \ldots, (-\overset{[n]}{B}_{l_nl_n}r_{l_n+1})x^+_{l_n+1}; \ldots; (-\overset{[2]}{B}_{l_3+1l_2}r_{l_2+1})x^+_{l_2+1}, \ldots$$
$$\ldots, (-\overset{[2]}{B}_{l_2l_2}r_{l_2+1})x^+_{l_2+1}; 0, \ldots, 0]^T = \phi.$$

Consequently, components of the vector ϕ are also calculated by formulae (2.2).

As a result, we have established that if X^+ is a normal (pseudo)solution of the system $C_3X = Y$ with the properties given by the theorem, then to it there correspond consistent with it decompositions (2.14) and (2.15) for matrix C_3 and its (pseudo)inverse matrix $B \equiv C_3^+$. In this case, representations for $X^+ = \overset{\circ}{X} + \Omega\cdot\overset{\circ}{X}$ (2.16) and $C_3^+ = \overset{\circ}{B} + \Omega\cdot\overset{\circ}{B}$ (2.15) being consistent with each other (since they are calculated with the same matrix Ω (2.15)) are unique and stable to small errors (h,δ) and $(\varepsilon_1,\varepsilon_0)$ in view of decompositions of C_3 (2.14) and C_3^+ (2.15) being unique. Stability is a consequence of the matrix $\overset{\circ}{C}_3$ (2.18) being well-posed.

Now let us show that if the numerical solution of the system $\tilde{C}_3\bar{X} = \check{Y}$ is obtained by the method*) (2.1) ÷ (2.13), then it is minimal in norm and provides a minimum of the discrepancy norm. Indeed, let X^+ is determined in the form (2.1) ÷ (2.13). Then from (2.5) it follows that the vector X^+ can be represented as a sum of two vectors,

(2.21) $$X^+ = \overset{\circ}{X} + \phi,$$

*)As we can see, this method includes the algorithm and criterion (2.3) ÷ (2.4) of separation of well-posed subspaces and, respectively, the procedure of numerical finding of X^+. It is to be kept in mind that the quantities Φ_j (2.4) obey the inequalities $|\Phi_j| \le |\Delta_j|$, where Δ_j is a discrepancy. As a matter of fact, $|\Phi_j| = ||y_j| - |y_j - [y_j - (p_j \overset{[k]}{x}_{j-1} + q_j \overset{[k]}{x}_j + r_{j+1} \overset{[k]}{x}_{j+1})]|| = ||y_j| - |y_j - \Delta_j|| \le |\Delta_j|$,since $||y_j| - |y_j - \Delta_j|| \le (|y_j - (y_j + \Delta_j)| = |\Delta_j|)$.

whose components are determined according to (2.2). The vectors $\overset{\circ}{X}$ and ϕ consist of n subvectors of proper dimensions, i.e.,

$$\overset{\circ}{X}=[(\overset{[n]}{x}_1,\dots,\overset{[n]}{x}_{l_n}),\dots,(\overset{[k]}{x}_{l_{k+1}+1},\dots,\overset{[k]}{x}_{l_k}),\dots,(\overset{[2]}{x}_{l_3+1},\dots,\overset{[2]}{x}_{l_2}),\dots,(\overset{[1]}{x}_{l_2+1}=x^+_{l_2+1},\dots,\overset{[1]}{x}_m=x^+_m)]^T,$$
$$\phi=[(\overset{[n]}{\phi}_1,\dots,\overset{[n]}{\phi}_{l_n}),\dots,(\overset{[k]}{\phi}_{l_{k+1}+1},\dots,\overset{[k]}{\phi}_{l_k}),\dots,(\overset{[2]}{\phi}_{l_3+1},\dots,\overset{[2]}{\phi}_{l_2}),\dots,(\overset{[1]}{\phi}_{l_2+1}=0,\dots,\overset{[1]}{\phi}_m=0)]^T,$$

corresponding to well-posed n-subspaces that are separated in accordance with the criterion (2.3), (2.4).

From (2.1)$\div$(2.5) we have that to the solution X^+ there corresponds the decomposition of $B=C^+$ of the form (2.15), which results in the representation for C_3 (1.4) of form (2.14). Then, as mentioned above, the solution X^+ is written in form (2.16), where $\overset{\circ}{X}$ is in a unique way represented in form (2.17), (2.18). Owing to system (2.18) being well-posed, which results from criterion (2.3) and (2.4), the vector $\overset{\circ}{X}$ is unique, obeys the condition $\min||\overset{\circ}{X}||$, and is stable to small errors (h,δ) and $(\varepsilon_1,\varepsilon_0)$. From the uniqueness of matrix Ω (2.15) that contains the last columns of matrices, inverse of the well-posed matrices $\overset{[k]}{C}{}^{l_{k+1}+1}_{l_k}$, and from (2.16) and (2.15) it follows that X^+ and $(B\equiv C_3^+)$ are unique and minimal in norm.

Let us now show that the vector X^+ determined by formulae (2.1)$\div$(2.5) satisfies the condition of minimum of the discrepancy norm $(\min||\tilde{C}_3X^+-\tilde{Y}||)$. Taking advantage of the representation of X^+ (2.15), we get

$$||\tilde{C}_3X^+-\tilde{Y}||=||\tilde{C}_3(E+\Omega)\overset{\circ}{X}-\tilde{Y}||=||\overset{\circ}{C}_3\overset{\circ}{X}-\tilde{Y}||,$$

where $\overset{\circ}{X}=(\overset{\circ}{B}=\overset{\circ}{C}{}_3^{-1})\tilde{Y}$, $\overset{\circ}{C}{}_3^{-1}$, and $\overset{\circ}{B}$ are, respectively, defined by (2.18) and (2.15). Owing to the system (2.16) being well-posed, the minimum $\min||\overset{\circ}{C}_3\overset{\circ}{X}-\tilde{Y}||$ and, consequently, $\min||\tilde{C}_3X^+-\tilde{Y}||$ are attainable. So, the theorem is proved.

Corollary. The norm of discrepancy $||\tilde{C}_3X^+-\tilde{Y}||$ complies with the following estimate:

(2.22)
$$\begin{cases}||\tilde{C}_3X^+-\tilde{Y}||_\infty\le\varepsilon_1\tau\rho\gamma\max\limits_{1\le i\le m}|\tilde{y}_i|+\Delta, \text{ where } \tau=\max\limits_{2\le i\le m}(|\tilde{p}_i|,|\tilde{r}_i|), \rho=\max\limits_{i,j;k}(|\overset{[k]}{B}_{ij}|),\\ \gamma=\sqrt{\sum\limits_{k=1}^{n}l_k(l_k-l_{k+1})}, l_1=m, l_{n+1}=0, 0\le\Delta\le h||X^+||+\delta.\end{cases}$$

Proof. In view of all said above, we have

$$||\tilde{C}_3X^+-\tilde{Y}||=||\tilde{C}_3(E+\Omega)\overset{\circ}{X}-\tilde{Y}||=||\overset{\circ}{C}_3\overset{\circ}{X}-\tilde{Y}||.$$

Since the system $\overset{\circ}{C}_3\overset{\circ}{X}=\tilde{Y}$ is well-posed, the Euclidean norm of errors $||\overset{\circ}{C}_3\overset{\circ}{X}-\tilde{Y}||_E$ can be estimated by using the known results [9]:

$$(2.23)\qquad ||\tilde{C}_3X^+-\tilde{Y}||_E=||\overset{\circ}{C}_3\overset{\circ}{X}-\tilde{Y}||_E\le 4f(m)\varepsilon_1||\overset{\circ}{C}_3||_E||\overset{\circ}{X}||_E.$$

However, in the case of the method considered above, this estimate turns out to be excessive. Actually, performing obvious transformations and making use of the definition of matrix norms consistent with the corresponding vector norms,

we get

$$(2.24)\qquad ||\overset{\circ}{C}_3\overset{\circ}{X}-\tilde{Y}||=||\overset{\circ}{C}_3\overset{\circ}{B}\tilde{Y}-\tilde{Y}||=||(\overset{\circ}{C}_3\overset{\circ}{B}-E)\tilde{Y}||\le||\overset{\circ}{C}_3\overset{\circ}{B}-E||\cdot||\tilde{Y}||.$$

Next, we estimate the norm of matrix discrepancy $||\overset{\circ}{C}_3\overset{\circ}{B}-E||$, using the explicit form of $\overset{\circ}{C}_3$ (2.18) and $\overset{\circ}{B}$ (2.15), as well as the condition for the matrix $\overset{\circ}{C}_3$ being well-posed. Taking account of the explicit form of elements of the matrix $(\overset{\circ}{C}_3\overset{\circ}{B}-E)$ and introducing the notation $\nu_{ij}(\nu=(\nu_{ij})=\overset{\circ}{C}_3\overset{\circ}{B}-E)$ for them, we can write the system of scalar identities

$$(2.25)\qquad\begin{cases}\tilde{p}_i\overset{[k]}{B}_{i-1j}+\tilde{q}_i\overset{[k]}{B}_{ij}+\tilde{r}_{i+1}\overset{[k]}{B}_{i+1j}\equiv\nu_{ij}, l_{k+1}+1\le i<j\le l_k,\\ \tilde{p}_i\overset{[k]}{B}_{i-1i}+\tilde{q}_i\overset{[k]}{B}_{ii}+\tilde{r}_{i+1}\overset{[k]}{B}_{i+1i}\equiv 1+\nu_{ii}, l_{k+1}+1\le(i=j)\le l_k,\\ \tilde{p}_i\overset{[k]}{B}_{i-1j}+\tilde{q}_i\overset{[k]}{B}_{ij}+\tilde{r}_{i+1}\overset{[k]}{B}_{i+1j}\equiv\nu_{ij}, 1\le j<i\le l_k.\end{cases}$$

Hereafter, $k=1,2,\dots,n; l_1=m, l_{n+1}=0$. Utilizing the representations for $\overset{[k]}{B}_{ij}$ (2.8)$\div$(2.13), we write the system of identities (2.25) either in the form

$$(2.26)\qquad\begin{cases}[\Lambda_{i+1}(-\Lambda_{i+1}^{-1}\tilde{r}_{i+1})+\tilde{r}_{i+1}]\overset{[k]}{B}_{i+1i+1}\prod\limits_{\xi=i+2}^{j}\overset{[k]}{\hat{\beta}}_\xi\equiv\nu_{ij}, l_{k+1}+1\le i<j\le l_k,\\ (\Lambda_{i+1}+\overset{[k]}{G}_{i-1}-\tilde{q}_i)\overset{[k]}{B}_{ii}\equiv 1+\nu_{ii}, l_{k+1}+1\le(i=j)\le l_k,\\ [\overset{[k]}{G}_{i-1}(-\overset{[k]}{G}{}_{i-1}^{-1}\tilde{p}_i)+\tilde{p}_i]\overset{[k]}{B}_{i-1i-1}\prod\limits_{\xi=j+1}^{i}\overset{[k]}{\beta}_\xi\equiv\nu_{ij}, 1\le j<i\le l_k,\end{cases}$$

or in the form

$$(2.27)\qquad\begin{cases}(1-\Lambda_{i+1}\Lambda_{i+1}^{-1})\tilde{r}_{i+1}\overset{[k]}{B}_{i+1j}\equiv\nu_{ij}, l_{k+1}+1\le i<j\le l_k,\\ (\Lambda_{i+1}+\overset{[k]}{G}_{i-1}-\tilde{q}_i)\overset{[k]}{B}_{ii}\equiv 1+\nu_{ii}, l_{k+1}+1\le(i=j)\le l_k,\\ (1-\overset{[k]}{G}_{i-1}\overset{[k]}{G}{}_{i-1}^{-1})\tilde{p}_i\overset{[k]}{B}_{i-1j}\equiv\nu_{ij}, 1\le j<i\le l_k.\end{cases}$$

Let us now estimate $(2.27)_{1)}, (2.27)_{2)}$ and $(2.27)_{3)}$; we have

$$(2.28)\quad\begin{cases}|(1-\Lambda_{i+1}\Lambda_{i+1}^{-1})\tilde{r}_{i+1}\overset{[k]}{B}_{i+1j}\equiv\nu_{ij}|\le|1-(1\pm\varepsilon_1)|\max\limits_{1\le i\le m-1}|\tilde{r}_{i+1}|\max\limits_{1\le i<j\le m-1}|\overset{[k]}{B}_{i+1j}|,\\ |(\Lambda_{i+1}+\overset{[k]}{G}_{i-1}-\tilde{q}_i)\overset{[k]}{B}_{ii}-1\equiv\nu_{ii}|\le|(1\pm\varepsilon_1)-1|\max\limits_{1\le i\le m}|\overset{[k]}{B}_{ii}|,\\ |(1-\overset{[k]}{G}_{i-1}\overset{[k]}{G}{}_{i-1}^{-1})\tilde{p}_i\overset{[k]}{B}_{i-1j}\equiv\nu_{ij}|\le|1-(1\pm\varepsilon_1)|\max\limits_{2\le i\le l_k}|\tilde{p}_i|\max\limits_{2\le j<i\le l_k}|\overset{[k]}{B}_{i-1j}|.\end{cases}$$

With estimates (2.28), we obtain $||\tilde{C}_3X^+-\tilde{Y}||_\infty=||(\overset{\circ}{C}_3\overset{\circ}{B}-E)\tilde{Y}||_\infty\le||\overset{\circ}{C}_3\overset{\circ}{B}-E||_M||\tilde{Y}||_\infty\le\sqrt{\sum_{k=1}^{n}l_k(l_k-l_{k+1})}\max_{i,j}|\nu_{ij}|||\tilde{Y}||_\infty\le\varepsilon_1\tau\rho\gamma\max_{1\le i\le m}|\tilde{y}_i|$, where τ,ρ,γ are defined by (2.22). Here we took advantage of the condition of consistency of vector norms $||\tilde{C}_3X^+-\tilde{Y}||_\infty=\max_{1\le i\le m}|(\tilde{C}_3X^+-\tilde{Y})_i|$ and $||\tilde{Y}||_\infty=\max_{1\le i\le m}|\tilde{y}_i|$ with the M-norm of the matrix $(\overset{\circ}{C}_3\overset{\circ}{B}-E)$, i.e. $||\overset{\circ}{C}_3\overset{\circ}{B}-E||_M=\sqrt{m^2}\max_{1\le i,j\le m}|(\overset{\circ}{C}_3\overset{\circ}{B}-E)_{ij}|$. The validity of inequality (2.22) is established. Since the Euclidean norm of the matrix $|| \overset{\circ}{C}_3\overset{\circ}{B} - E||_E$ is consistent only with the vector Euclidean norm $||\tilde{Y}||_E$, instead of (2.22), one can, by analogous arguments, obtain the estimate $||\tilde{C}_3X^+ - \tilde{Y}||_E = ||(\overset{\circ}{C}_3\overset{\circ}{B} - E)\tilde{Y}||_E \leq \varepsilon_1\tau\rho\gamma||\tilde{Y}||_E$, where τ, ρ, γ are defined by (2.22).

Remark 1. To save the volume of publication, we do not present the method of solution of system (1.3) with the two-diagonal matrix C_2 (1.4). It is expounded in detail in ref.[3] and it is shown there that it results from the method (2.1) ÷ (2.13).

The estimate (2.22) for system (1.3) acquires the following form:

(2.29)

$$\begin{cases} ||\tilde{C}_2\hat{X}^+ - \hat{\tilde{Y}}|| \leq || \overset{\circ}{C}_2\overset{\circ}{B} - E|| \cdot ||\tilde{Y}|| \leq \varepsilon_1\hat{\tau}\hat{\rho}\hat{\gamma} \max\limits_{1\leq i\leq m} |\tilde{y}_i| + \Delta, \text{where } \hat{\tau} = \max\limits_{2\leq i\leq m}(|\tilde{r}_i|), \\ \hat{\rho} = \max\limits_{i,j;k}(| \overset{[k]}{B}_{ij} |), \hat{\gamma} = \sqrt{1/2 \sum\limits_{k=1}^{n}(l_k - l_{k+1})}, l_1 = m, l_{n+1} = 0, 0 \leq \Delta \leq h||\hat{X}^+|| + \delta. \end{cases}$$

Here $\overset{[k]}{B}_{ij}$ are elements of upper triangular matrices, inverse of well-posed two-diagonal matrices $[\overset{[k]}{C}_2]_{l_k}^{l_{k+1}+1}$.

Remark 2. Note that due to orthogonality of matrices P and Q in transformations (1.5), the following estimates take place for the norms of discrepancy $||\tilde{A}Z^+ - \tilde{F}||$:

$$\begin{cases} ||\tilde{A}Z^+ - \tilde{F}|| \leq \varepsilon_1\tilde{\tau}\tilde{\rho}\tilde{\gamma} \max\limits_{1\leq i\leq m} |\tilde{y}_i| + \Delta, \text{ if } A = A^T, \\ ||\tilde{A}Z^+ - \tilde{F}|| \leq \varepsilon_1\hat{\tilde{\tau}}\hat{\tilde{\rho}}\hat{\tilde{\gamma}} \max\limits_{1\leq i\leq m} |\tilde{y}_i| + \Delta, \text{ if } A \neq A^T, \end{cases} \quad (2.30)$$

where $\tilde{\tau}, \tilde{\rho}, \tilde{\gamma}$ and $\hat{\tilde{\tau}}, \hat{\tilde{\rho}}, \hat{\tilde{\gamma}}$ are defined in analogy with (2.22) and (2.29), $0 \leq \Delta \leq h||Z^+|| + \delta$.

Remark 3. The above estimates (2.22), (2.29) and (2.30) can also be used for problems of inversion i.e., $C_3C_3^+ = E, C_2C_2^+ = E, AA^+ = E$: the matrices C_3^+, C_2^+ and A^+ are to be obtained by solving the matrix system of equations

$$C_3C_3^+ = E, C_2C_2^+ = E \text{ and } AA^+ = E.$$

by the critical-component method. In the case when systems (1.1), (1.2) and (1.3) are ill-posed, one should not take, as C_3^+, C_2^+ and A^+, the corresponding matrices obtained by the critical-component method in solving these systems of equations with a given right-hand side. The reason is that the norms of matrices C_3^+, C_2^+ и A^+ are consistent with the norms of concrete vectors $X_3^+, X_2^+, Z^+, \tilde{Y}, \hat{\tilde{Y}}$ and $\tilde{F}$.

Remark 4. The theorem is formulated under the assumption $\det \tilde{C} \neq 0$. Let us remove this restriction. The critical-component method does not explicitly use the quantity $\det \tilde{C}$. Rather, it is based on the processes (2.10) and (2.11) for computing elements of m-dimensional vectors $\{\Lambda, \text{G}\}$. As established in ref. [14], if $\det \tilde{C} = 0$, then components of these vectors get into one of the following three situations: either $\Lambda_{m+1} = 0$ and $\text{G}_0 = 0$, or $\Lambda_i = 0$ and $\text{G}_i = 0$, or $[(\Lambda_i = 0$ and $\Lambda_i\Lambda_{i+1} = 0)$ or $(\text{G}_i = 0$ and $\text{G}_i\text{G}_{i-1} = 0)]$. In this case we replace some zero quantities by the quantity $o(\varepsilon_1)$. This does not essentially impair the quality of solution, since such perturbations can already be present in these quantities. Consequently, one may consider the critical-component method to be applicable for any value of $\det \tilde{C}$, including $\det \tilde{C} = 0$.

3. Results of numerical; experiments and their analysis

In this section, we discuss the results of numerical experiments performed in the computer arithmetic with double accuracy ($\varepsilon_1 = 2^{-52} \approx 2,2 \cdot 10^{-16}$) for computing basic numerical characteristics of the solutions X of systems $WX = Y$, W : $C_2; C_3; A \neq A^T; A = A^T$. Let us first explain the notation and abbreviations adopted in Tables 1 ÷ 12 : $\delta_M^{(m)}$ — the relative error of $\tilde{X}^{(m)}$ — the obtained numerical solution of system $W^{(m)}X^{(m)} = Y^{(m)}$ ($W^{(m)}$: $C_2^{(m)}; C_3^{(m)}; A^{(m)} \neq (A^{(m)})^T; A^{(m)} = (A^{(m)})^T$ — the above indicated types of matrices, $X^{(m)}$ — the exact solution, m — the order of the system under consideration); $\mu(W^{(m)}) = \text{cond}(W^{(m)})$ — the condition number of $W^{(m)}$; $t^{(m)}(sec.) = \text{com.time(sec.)}$ — the time of computing solutions $\tilde{X}^{(m)}$, $\delta_L^{(m)}$, $\delta_R^{(m)}$ — the lower and upper bounds $\delta_M^{(m)}$ i.e.*)

$$(3.1) \quad (\delta_L^{(m)} = \frac{|||\tilde{X}^{(m)}|| - ||X^{(m)}|||}{||X^{(m)}||}) \leq (\delta_M^{(m)} = \frac{||\tilde{X}^{(m)} - X^{(m)}||_E}{||X^{(m)}||}) \leq (\frac{||(W^{(m)})^{-1}|| \cdot ||W^{(m)}\tilde{X}^{(m)} - Y^{(m)}||_E}{||X^{(m)}||} = \delta_R^{(m)}),$$

where $||\tilde{X}^{(m)}||_E$, $||X^{(m)}||_E$ are norms of approximate and exact solutions; $\delta_L = \frac{1}{N_j}\sum\limits_{l=1}^{N_j}(\delta_L^{(m)})_l$. $\delta_M = \frac{1}{N_j}\sum\limits_{l=1}^{N_j}(\delta_M^{(m)})_l$, $\delta_R = \frac{1}{N_j}\sum\limits_{l=1}^{N_j}(\delta_R^{(m)})_l$, $\delta_{\tilde{X}}^* = \frac{1}{N_j}\sum\limits_{l=1}^{N_j}(||\tilde{X}^{(m)}||)_l$, $\delta_X = \frac{1}{N_j}\sum\limits_{l=1}^{N_j}(||X^{(m)}||)_l$. $\bar{\mu}(W) = \frac{1}{N_j}\sum\limits_{l=1}^{N_j}(\mu(W^{(m)}))_l$, $t(sec.) = \frac{1}{N_j}\sum\limits_{l=1}^{N_j}(t^{(m)})_l$ are arithmetic means of the characteristics listed above, $\bar{N}_i = \sum\limits_{j=1}^{s} N_j$, where N_j is the number of examples of a given type, s - the number of examples in a Table, i - the number of a Table; `MCS` and `MCC` — our programs `DCSOL` (access through www `http://cv.jinr.ru/lcta/sap/lib/f499.f`) from library `LIBJINR` [17] (algorithms of the critical-component method [1 ÷ 3]); `GS` — programs `DBEQN` and `DEQN` from library `CERNLIB` [21] (a modified algorithm of the Gauss exclusion method); `OSM` — program `DTSYS` from library `LIBJINR` [19] (algorithm of nonmonotone orthogonal run); `QR` — programs `F01AXF` from library `NAGLIB` [20] (algorithms of `QR` — method); `SVD` — subprogram-function `PSOL` from library `LINA` [7] (algorithm of the singular-expansion method with the use of exhaustion); `TRM` — subprogram `SLAY` from library `LIBJINR` [19] (algorithms of the Tikhonov regularization method). We have solved $N = 278$ ($N = \sum\limits_{i=1}^{10} \bar{N}_i, i \neq 4$ and $i \neq 8$) different systems of linear algebraic equations at different orders**) m_k ($k = 1, 2, 3, 4, 4', 5$), which is presented below. *A system of the type* $C_2X = Y$. Examples 1 ÷ 5 from [2,3,16] (see §5 Appendix). $\{m_1 : 10; 20.\ m_2 : 3.\ m_3 : 5; 10; 15.\ m_5 : 10; 20.$ Here $1 < \mu(C^{(m_k)}) \leq 1/\sqrt{\varepsilon_1}\}$, $\{m_1 : 30; 40; 50.\ m_2 : 4; 5; 6.\ m_3 : 20; 25; 30; 35.\ m_5 : 30; 40.$ Here $1/\sqrt{\varepsilon_1} < \mu(C^{(m_k)}) \leq 1/\varepsilon_1\}$, $\{m_1 : 60; 70; ...; 100; 150; 200.\ m_2 : 8; 9; 10.\ m_3 : 40; 45.\ m_4 : 5; 6; ...; 18.\ m_5 : 50; 60; ...; 100; 150; 200; 300; 400; 500.$ Here $1/\varepsilon_1 < \mu(C^{(m_k)})\}$. *A system of the type* $C_3X = Y$. Examples†*). 6 ÷ 10 from [2,3,16] (see §5 Appendix).

*)The left-hand side of inequality (3.1) is a property [10] of the norm $|| \cdot ||$, and the right-hand side is obtained by using the exact solution $X = W^{-1}Y$ and equality $W^{-1}W = E$. We have $(\delta_M = ||\tilde{X} - X||/||X||) = ||\tilde{X} - W^{-1}Y||/||X|| = ||W^{-1}(W\tilde{X} - Y)||/||X|| \leq (||W^{-1}|| \cdot ||W\tilde{X} - Y||/||X|| = \delta_R)$. Note that in practice, inequalities (3.1) can be broken (see, for instance, Table 7). This occurs when calculating $W\tilde{X} - Y$. In this case, the solution $\tilde{X}$ can be surely considered acceptable.

**)lower index k of order m_k indicates the number of an example from the set of given-type examples.

†*)Example 4′ is example 4 from [2] (system 9, see §5 Appendix), but with $\varepsilon_0^* = 0.00000001$.

$\{m_1 : 10; 20; \ldots; 100; 150; 200; 300; \ldots; 900.$ $m_2 : 10; 20; \ldots; 100; 150; 200; 300; \ldots; 900.$ $m_3 : 10; 21; 3\ 0; 40; 51; 60; 70; 81; 90; 100; 151; 201; 300; 400; 501; 600; 700; 801; 900.$ $m_4 : 10; 30; 40; 60; 70; 90; 100; 300; 400; 600; 700; 900.$ $m_{4'} : 10; 30; 40; 60; 70; 90; 100; 300; 400; 600; 700; 900.$ $m_5 : 10; 20; \ldots; 100.$ Here $1 < \mu(C^{(m_k)}) \leq 1/\sqrt{\varepsilon_1}\}$, $\{m_4 : 20; 50; 80; 150; 200; 500; 800.$ $m_5 : 150; 200.$ Here $1/\sqrt{\varepsilon_1} < \mu(C^{(m_k)}) \leq 1/\varepsilon_1\}$, $\{m_{4'} : 20; 50; 80; 150; 200; 500; 800.$ Here $1/\varepsilon_1 < \mu(C^{(m_k)})\}$. *A system of the type* $AX = Y$ $(A \neq A^T)$. Examples $11 \div 15$ from [2,3,16] (see §5 Appendix). $\{m_1 : 10; 20; \ldots; 100; 150; 200; 300; 400; 500.$ $m_2 : 5.$ $m_3 : 10; 20; \ldots; 100; 150; 200; 250; 300; 400; 500.$ $m_4 : 3; 4; \ldots; 10.$ $m_5 : 5; 6.$ Here $1 < \mu(C^{(m_k)}) \leq 1/\sqrt{\varepsilon_1}\}$, $\{m_2 : 6; 7; \ldots; 10.$ $m_4 : 11; 12; \ldots; 17.$ $m_5 : 7; 8; \ldots; 11.$ Here $1/\sqrt{\varepsilon_1} < \mu(C^{(m_k)}) \leq 1/\varepsilon_1\}$, $\{m_2 : 11; 12.$ $m_4 : 18.$ $m_5 : 12.$ Here $1/\varepsilon_1 < \mu(C^{(m_k)})\}$. *A system of the type* $AX = Y$ $(A = A^T)$. Examples $16 \div 20$ from [2,3,16] (see §5 Appendix). $\{m_1 : 10; 20; \ldots; 100; 150; 200; 300.$ $m_2 : 5; 6.$ $m_3 : 5; 6.$ $m_4 : 10; 20; \ldots; 100; 150; 200; 300; 400; 500.$ Here $1 < \mu(C^{(m_k)}) \leq 1/\sqrt{\varepsilon_1}\}$, $\{m_2 : 7; 8; \ldots; 11.$ $m_3 : 7; 8; \ldots; 11.$ Here $1/\sqrt{\varepsilon_1} < \mu(C^{(m_k)}) \leq 1/\varepsilon_1\}$, $\{m_2 : 12.$ $m_3 : 12.$ $m_5 : 5; 10; 15; 20; 25; 30; 35.$ Here $1/\varepsilon_1 < \mu(C^{(m_k)})\}$.

Below, in Tables $1 \div 3$, $5 \div 7$ and $9 \div 11$, we report the obtained numerical values of the indicated characteristics δ_L, δ_R, $\delta^*_{\tilde{X}}$, δ_X, of approximate $\tilde{X}$ and exact X solutions of systems $WX = Y$ $(W : C_2; C_3; A \neq A^T; A = A^T)$ at $1 < \mu(W) \leq 1/\sqrt{\varepsilon_1}$ — being well-posed, $1/\sqrt{\varepsilon_1} < \mu(W) \leq 1/\varepsilon_1$ — being ill-posed, $1/\varepsilon_1 < \mu(W)$ — being pathologically ill-posed of these systems, respectively. In Tables 4, 8 and 12, we present averaged results of Tables $1 \div 3$, $5 \div 7$ and $9 \div 11$. Note also that when $1/\varepsilon_1 < \mu(W)$, the subprogram `SVD` stops to work producing information `INF`$= -1$. The program `TRM` does not work [19], when $m > 100$. Tables 9, 12 do not contain $(*****)$ values of t(сек.) above 100 sec.

For an easier apprehension of the calculation results reported in Tables 4, 8 and 12, we plot their "graphic images-figures".

Remark 5. In Tables 4, 8 and 12 (as well as in Figures $1 \div 9$) , we also present the averaged results of computations by subprograms `MCS` when $W = (A = A^T)$ and `OSM` when $W = C_3$, which is to be kept in mind when analyzing the above Tables and Figures. *Explanations* to some Tables and Figures: on the horizontal axis of Figs. $1 \div 9$, in units 10^t where values of order t are indicated below the axis, approximate values of quantities $\delta_L(W), \delta_M(W)$ and $\delta_R(W)$ from Tables 4,8 and 12 are plotted. On the vertical axis of these Figures, we point out the names of programs through which those values have been obtained. The horizontal axis of Figs. $11 \div 13$ represents relative errors of the r.h.s. of a system with the Hilbert matrix (see §5 of Appendix, example 17), whereas on their vertical axes, we plot the corresponding relative errors found by various programs. For names of programs, see the notation. Table 13 contains the numerical results that are drawn in Figs. $11 \div 13$. Note that: $<\delta_Y>=<\frac{\|\Delta Y\|}{\|Y\|}>$ – the mean value of the relative error of perturbation of the r.h.s. of the system; $<\delta_X>=<\frac{\|\Delta X\|}{\|X\|}>$ – exact mean values corresponding to $<\delta_Y>$. Numbers in Table 13 written in line with a program are average values really obtained by this program for $<\delta_{\tilde{X}}>$. In Fig.10, the matrix, inverse of the Hilbert matrix, of order $m = 14$ is graphically shown. Along the axis Z, the values of elements of this inverse matrix are indicated. As a result, its complicated structure is easily visualized. Also, numbers of subspaces separated by the critical-component method are given in the Figure .

Table 1 $(0.173E02 < \mu(W) \leq 0.547E08, 3 \leq m \leq 100, \bar{\mu}(W) = 0.533E07, \bar{N}_1 = 117)$

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
	MCC	0.0004	0.238E-11	0.301E-11	0.470E-10	0.136E01	0.136E01
$C_2X = Y$	GS	0.0005	0.238E-11	0.301E-11	0.119E-08	0.136E01	0.136E01
$N_1 = 8$	SVD	0.1568	0.261E-11	0.335E-11	0.723E-08	0.136E01	0.136E01
$\bar{\mu}(C_2) =$	QR	0.0036	0.113E-10	0.175E-10	0.405E-08	0.136E01	0.136E01
0.576E07	TRM	0.0366	0.448E-05	0.546E-05	0.575E-05	0.136E01	0.136E01
	MCC	0.0051	0.839E-13	0.868E-13	0.137E-12	0.609E01	0.609E01
$C_3X = Y$	GS	0.0028	0.994E-13	0.153E-11	0.289E-10	0.609E01	0.609E01
$N_2 = 54$	QR	6.9793	0.187E-11	0.302E-10	0.294E-08	0.609E01	0.609E01
$\bar{\mu}(C_3) =$	SVD	0.3091	0.609E-12	0.505E-10	0.704E-09	0.609E01	0.609E01
$0.106E07$	TRM	5.3795	0.238E-07	0.212E-05	0.212E-05	0.609E01	0.609E01
	OSM	0.0039	0.561E-01	0.164E00	0.850E04	0.655E01	0.609E01
$AX = Y$	QR	0.1776	0.526E-11	0.101E-09	0.122E-07	0.254E01	0.254E01
$A \neq A^T$	MCC	1.0925	0.973E-11	0.117E-09	0.461E-08	0.254E01	0.254E01
$N_3 = 31$	SVD	2.8117	0.973E-11	0.117E-09	0.683E-08	0.254E01	0.254E01
$\bar{\mu}(A) =$	GS	0.1091	0.752E-11	0.144E-09	0.256E-08	0.254E01	0.254E01
0.549E07	TRM	2.9615	0.364E-04	0.618E-04	0.639E-04	0.254E01	0.254E01
	MCC	0.9636	0.163E-10	0.951E-10	0.178E-07	0.434E01	0.434E01
$AX = Y$	SVD	2.5378	0.163E-10	0.951E-10	0.307E-07	0.434E01	0.434E01
$A = A^T$	QR	0.1564	0.129E-10	0.160E-09	0.383E-07	0.434E01	0.434E01
$N_4 = 24$	GS	0.0977	0.373E-10	0.179E-09	0.230E-07	0.434E01	0.434E01
$\bar{\mu}(A) =$	MCS	0.9367	0.153E-09	0.429E-09	0.171E-07	0.434E01	0.434E01
0.900E07	TRM	2.5976	0.603E-08	0.328E-07	0.453E-07	0.434E01	0.434E01

Table 2 $(0.968E08 < \mu(W) \leq 0.399E16, 6 \leq m \leq 80, \bar{\mu}(W) = 0.365E15, \bar{N}_2 = 43)$

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
	MCC	0.0008	0.191E-04	0.353E-04	0.289E-02	0.180E01	0.180E01
$C_2X = Y$	GS	0.0009	0.191E-04	0.353E-04	0.517E-02	0.180E01	0.180E01
$N_1 = 13$	SVD	0.7276	0.598E-04	0.110E-03	0.259E00	0.180E01	0.180E01
$\bar{\mu}(C_2) =$	QR	0.0200	0.954E-03	0.221E-02	0.276E00	0.180E01	0.180E01
0.351E15	TRM	0.3108	0.208E12	0.208E12	0.210E12	0.131E13	0.180E01
	MCC	0.0067	0.638E-08	0.240E-04	0.277E-02	0.754E01	0.754E01
$C_3X = Y$	GS	0.0035	0.672E-06	0.213E-03	0.313E-02	0.754E01	0.754E01
$N_2 = 3$	SVD	8.6764	0.335E-05	0.463E-03	0.453E-01	0.754E01	0.754E01
$\bar{\mu}(C_3) =$	QR	0.4031	0.102E-04	0.962E-03	0.960E-02	0.754E01	0.754E01
0.586E15	OSM	0.0048	0.468E-01	0.141E00	0.136E11	0.793E01	0.754E01
	TRM	6.6831	0.148E14	0.148E14	0.562E14	0.125E15	0.754E01
AX=Y	MCC	1.0966	0.200E-03	0.418E-03	0.239E00	0.254E01	0.254E01
$A \neq A^T$	SVD	2.8427	0.200E-03	0.418E-03	0.255E00	0.254E01	0.254E01
$N_3 = 17$	GS	0.1095	0.278E-03	0.514E-03	0.473E00	0.255E01	0.254E01
$\bar{\mu}(A) =$	QR	0.1784	0.338E-03	0.615E-03	0.316E00	0.255E01	0.254E01
0.271E15	TRM	2.9711	0.133E12	0.133E12	0.261E12	0.167E12	0.254E01
	MCC	0.9660	0.654E-06	0.161E-04	0.215E00	0.435E01	0.435E01
$AX = Y$	SVD	2.5544	0.654E-06	0.161E-04	0.467E00	0.435E01	0.435E01
$A = A^T$	MCS	0.9389	0.119E-05	0.548E-04	0.189E00	0.435E01	0.435E01
$N_4 = 10$	GS	0.0979	0.166E-05	0.828E-04	0.452E00	0.435E01	0.435E01
$\bar{\mu}(A) =$	QR	0.1568	0.109E-05	0.976E-04	0.734E00	0.435E01	0.435E01
0.253E15	TRM	2.6032	0.435E05	0.435E05	0.460E05	0.543E05	0.435E01

Table 3 ($\mu(W)$ >0.450E16, $5 \leq m \leq 80$, $\bar{\mu}(W)$ >0.450E16, $\bar{N}_3 = 46$)

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
	MCC	0.0010	0.133E00	0.179E00	0.952E02	0.259E01	0.253E01
$C_2X = Y$	GS	0.0012	0.482E00	0.103E01	0.924E03	0.378E01	0.253E01
$N_1 = 30$	QR	0.0879	0.274E02	0.276E02	0.211E07	0.182E03	0.253E01
$\bar{\mu}(C_2) >$	TRM	1.6955	0.602E55	0.602E55	0.145E75	0.602E56	0.253E01
0.450E16	SVD	INF = − 1					
	MCC	0.0069	0.298E00	0.828E00	0.126E02	0.887E01	0.683E01
$C_3X = Y$	QR	0.1659	0.724E00	0.127E01	0.398E02	0.129E02	0.683E01
$N_2 = 3$	GS	0.0022	0.174E02	0.184E02	0.940E03	0.142E03	0.683E01
$\bar{\mu}(C_3) >$	OSM	0.0034	0.221E02	0.231E02	0.140E18	0.178E03	0.683E01
0.450E16	TRM	3.1464	0.777E15	0.777E15	0.611E16	0.430E16	0.683E01
	SVD	INF = − 1					
$AX = Y$	MCC	0.0179	0.109E-02	0.333E-01	0.106E03	0.856E00	0.856E00
$A \neq A^T$	GS	0.0022	0.330E-02	0.681E-01	0.193E03	0.861E00	0.856E00
$N_3 = 4$	QR	0.0040	0.610E-02	0.895E-01	0.209E03	0.861E00	0.856E00
$\bar{\mu}(A) >$	TRM	0.0421	0.166E14	0.166E14	0.278E14	0.208E14	0.856E00
0.450E16	SVD	INF = − 1					
	MCC	0.0479	0.428E-01	0.365E00	0.139E03	0.994E00	0.989E00
$AX = Y$	MCS	0.0439	0.197E00	0.545E00	0.149E03	0.107E01	0.989E00
$A = A^T$	GS	0.0051	0.104E05	0.104E05	0.297E22	0.483E04	0.989E00
$N_4 = 9$	QR	0.0090	0.261E12	0.261E12	0.332E25	0.123E12	0.989E00
$\bar{\mu}(A) >$	TRM	0.1165	0.457E16	0.457E16	0.593E18	0.213E16	0.989E00
0.450E16	SVD	INF = − 1					

Table 4 (Mean values of characteristics of Tables $1 \div 3$, $\bar{N}_4 = 206$)

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
	MCC	0.5154	0.712E-11	0.538E-10	0.561E-08	0.358E01	0.358E01
$\bar{\mu}(W) =$	SVD	3.1214	0.763E-11	0.614E-10	0.119E-07	0.358E01	0.358E01
0.533E07	GS	0.0525	0.118E-10	0.819E-10	0.669E-08	0.358E01	0.358E01
$\bar{N}_1 = 117$	QR	0.1617	0.752E-11	0.822E-10	0.138E-07	0.358E01	0.358E01
	MCS	0.9367	0.153E-09	0.429E-09	0.171E-07	0.434E01	0.434E01
	TRM	2.7438	0.102E-04	0.174E-04	0.180E-04	0.358E01	0.358E01
	OSM	0.0039	0.561E-01	0.164E00	0.850E04	0.655E01	0.609E01
	MCS	0.9389	0.119E-05	0.548E-04	0.189E00	0.435E01	0.435E01
	MCC	0.5175	0.549E-04	0.123E-03	0.115E00	0.406E01	0.406E01
$\bar{\mu}(W) =$	GS	0.0529	0.749E-04	0.211E-03	0.233E00	0.406E01	0.406E01
0.365E15	SVD	3.7003	0.660E-04	0.252E-03	0.257E00	0.406E01	0.406E01
$\bar{N}_2 = 43$	QR	0.1896	0.326E-03	0.971E-03	0.334E00	0.406E01	0.406E01
	OSM	0.0048	0.468E-01	0.141E00	0.136E11	0.793E01	0.754E01
	TRM	3.1420	0.379E13	0.379E13	0.142E14	0.316E14	0.406E01
	MCC	0.0184	0.119E00	0.351E00	0.882E02	0.333E01	0.280E01
$\bar{\mu}(W) >$	MCS	0.0439	0.197E00	0.545E00	0.149E03	0.107E01	0.989E00
0.450E16	OSM	0.0034	0.221E02	0.231E02	0.140E18	0.178E03	0.683E01
$\bar{N}_3 = 46$	GS	0.0027	0.260E04	0.260E04	0.742E21	0.124E04	0.280E01
	QR	0.0667	0.653E11	0.653E11	0.830E24	0.308E11	0.280E01
	TRM	1.2501	0.151E55	0.151E55	0.362E74	0.151E56	0.280E01
	SVD	INF = − 1					

Table 5 (0.250E03< $\mu(W)$ ≤0.323E07, $150 \leq m \leq 200$, $\bar{\mu}(W) =$ 0.827E08, $\bar{N}_5 = 16$)

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
	MCC	0.0133	0.344E-12	0.362E-12	0.724E-12	0.788E01	0.788E01
$C_3X = Y$	GS	0.0065	0.345E-12	0.363E-12	0.727E-12	0.788E01	0.788E01
$N_2 = 8$	QR	6.0108	0.622E-12	0.777E-12	0.170E-10	0.788E01	0.788E01
$\bar{\mu}(C_3) =$	SVD	31.137	0.138E-10	0.104E-10	0.274E-10	0.788E01	0.788E01
0.202E05	OSM	0.0093	0.306E00	0.467E00	0.114E01	0.116E02	0.788E01
$AX = Y$	MCC	22.606	0.331E-13	0.311E-09	0.120E-07	0.719E01	0.719E01
$A \neq A^T$	GS	2.2529	0.134E-13	0.374E-09	0.275E-07	0.719E01	0.719E01
$N_3 = 4$	SVD	37.433	0.331E-13	0.436E-09	0.189E-07	0.719E01	0.719E01
$\bar{\mu}(A) =$	QR	3.6669	0.236E-13	0.617E-09	0.306E-07	0.719E01	0.719E01
0.123E07							
$AX = Y$	MCC	10.312	0.122E-11	0.155E-09	0.121E-06	0.134E02	0.134E02
$A = A^T$	SVD	18.469	0.122E-11	0.155E-09	0.364E-06	0.134E02	0.134E02
$N_4 = 4$	MCS	10.181	0.115E-11	0.215E-09	0.100E-06	0.134E02	0.134E02
$\bar{\mu}(A) =$	QR	1.6768	0.808E-12	0.263E-09	0.371E-06	0.134E02	0.134E02
0.123E07	GS	1.0537	0.218E-11	0.336E-09	0.230E-06	0.134E02	0.134E02

Table 6 (0.657E11< $\mu(C_3)$ ≤0.594E15, $150 \leq m \leq 200$, $\bar{\mu}(C_3) =$ 0.308E15, $\bar{N}_6 = 3$)

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
	MCC	0.0591	0.117E-05	0.133E-02	0.207E-01	0.136E02	0.136E02
$C_3X = Y$	SVD	43.881	0.488E-05	0.179E-02	0.171E01	0.136E02	0.136E02
$N_2 = 3$	GS	0.0075	0.766E-05	0.277E-02	0.324E-01	0.136E02	0.136E02
$\bar{\mu}(C_3) =$	QR	4.5427	0.114E-04	0.336E-02	0.229E00	0.136E02	0.136E02
0.308E15	OSM	0.0109	0.135E-02	0.414E-01	0.101E12	0.138E02	0.136E02

Table 7 ($\mu(W)$ >0.450E16, $150 \leq m \leq 200$, $\bar{\mu}(W)$ >0.450E16, $\bar{N}_7 = 5$)

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
$C_2X = Y$	MCC	0.0049	0.615E-04	0.113E-03	0.000E00	0.719E01	0.719E01
$N_1 = 4$	GS	0.0060	0.615E-04	0.113E-03	0.000E00	0.719E01	0.719E01
$\bar{\mu}(C_2) >$	QR	3.3505	0.401E02	0.408E02	0.128E06	0.653E02	0.719E01
0.450E16	SVD	INF = − 1					
	MCC	0.0278	0.292E+00	0.819E+00	0.258E02	0.183E02	0.141E02
$C_3X = Y$	QR	5.3062	0.551E+00	0.837E+00	0.446E02	0.189E02	0.141E02
$N_2 = 1$	GS	0.0079	0.719E+02	0.728E+02	0.649E04	0.103E04	0.141E02
$\bar{\mu}(C_3) >$	OSM	0.0133	0.882E+02	0.892E+02	0.571E18	0.126E04	0.141E02
0.450E16	SVD	INF = − 1					

Table 8 (Mean values of characteristics of Tables $5 \div 7$, $\bar{N}_8 = 24$)

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
	MCC	10.977	0.532E-12	0.155E-09	0.443E-07	0.949E01	0.949E01
$\bar{\mu}(W) =$	SVD	29.013	0.502E-11	0.200E-09	0.128E-06	0.949E01	0.949E01
0.827E06	MCS	10.181	0.115E-11	0.215E-09	0.100E-06	0.134E02	0.134E02
$\bar{N}_5 = 16$	GS	1.1044	0.846E-12	0.237E-09	0.848E-07	0.949E01	0.949E01
	QR	3.7848	0.498E-12	0.294E-09	0.134E-06	0.949E01	0.949E01
	OSM	0.0093	0.306E00	0.467E00	0.114E01	0.116E02	0.788E01
	MCC	0.0591	0.175E-05	0.133E-02	0.207E-01	0.136E02	0.136E02
$\bar{\mu}(C_3) =$	SVD	43.881	0.488E-05	0.179E-02	0.171E01	0.136E02	0.136E02
0.308E15	GS	0.0075	0.766E-05	0.277E-02	0.324E-01	0.136E02	0.136E02
$\bar{N}_6 = 3$	QR	4.5427	0.114E-04	0.336E-02	0.229E00	0.136E02	0.136E02
	OSM	0.0109	0.135E-02	0.414E-01	0.101E12	0.138E02	0.136E02

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
$\bar{\mu}(W) >$	MCC	0.0164	0.146E00	0.410E00	0.129E02	0.127E02	0.106E02
0.450E16	QR	4.3284	0.203E02	0.208E02	0.640E05	0.421E02	0.106E02
$\bar{N}_7 = 5$	GS	0.0069	0.360E02	0.364E02	0.325E04	0.519E03	0.106E02
	OSM	0.0133	0.882E02	0.892E02	0.571E18	0.126E04	0.141E02
	SVD	INF = − 1					

Table 9 (0.498E03≤ $\mu(W)$ ≤0.318E08, 250≤ m ≤900, $\bar{\mu}(W)$ = 0.675E07, $\bar{N}_9 = 39$)

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
$C_3X = Y$	MCC	0.0524	0.762E-14	0.403E-13	0.168E-11	0.999E01	0.999E01
$\bar{\mu}(C_3) =$	GS	0.0230	0.718E-14	0.444E-13	0.179E-11	0.999E01	0.999E01
0.255E06	OSM	0.0324	0.179E02	0.186E02	0.682E75	0.618E02	0.999E01
$N_2 = 31$							
$AX = Y$	MCC	12.156	0.703E-12	0.611E-08	0.115E-05	0.101E02	0.101E02
$A \neq A^T$	GS	1.2256	0.227E-11	0.971E-08	0.179E-05	0.101E02	0.101E02
$N_3 = 4$							
$N_4 = 4$							
$A = A^T$	MCC	*****	0.256E-11	0.522E-09	0.182E-05	0.188E02	0.188E02
$\bar{\mu}(A) =$	MCS	*****	0.216E-12	0.227E-08	0.174E-05	0.188E02	0.188E02
0.999E07	GS	*****	0.703E-12	0.240E-08	0.197E-05	0.188E02	0.188E02

Table 10 (0.540E15≤ $\mu(C_3)$ ≤0.637E15, m : 500,800, $\bar{\mu}(C_3)$ =0.589E15, $\bar{N}_{10} = 2$)

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
$C_3X = Y$	MCC	0.0901	0.161E-05	0.174E-02	0.415E-01	0.253E02	0.253E02
$\bar{\mu}(C_3) =$	GS	0.0290	0.294E-04	0.765E-02	0.666E-01	0.253E02	0.253E02
0.589E15	OSM	0.0394	0.150E-02	0.915E-01	0.290E75	0.253E02	0.253E02
$N_2 = 2$							

Table 11 ($\mu(W)$ >0.450E16, 250≤ m ≤ 800, $\bar{\mu}(W)$ > 0.450E16, $\bar{N}_{11} = 7$)

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
$C_2X = Y$	MCC	0.0088	0.615E-04	0.113E-03	0.000E00	0.106E02	0.106E02
$\bar{\mu}(C_2) >$	GS	0.0115	0.615E-04	0.113E-03	0.000E00	0.106E02	0.106E02
0.450E16							
$N_1 = 4$							
$C_3X = Y$	MCC	0.0583	0.145E00	0.409E00	0.114E03	0.250E02	0.213E02
$\bar{\mu}(C_3) >$	GS	0.0192	0.167E03	0.168E03	0.190E06	0.394E04	0.213E02
0.450E16	OSM	0.0300	0.215E03	0.216E03	0.209E75	0.500E04	0.213E02
$N_2 = 3$							

Table 12 (Mean values of characteristics of Tables 9 ÷ 11, $\bar{N}_{12} = 48$)

	PR.	t(sec)	δ_L	δ_M	δ_R	$\delta^*_{\tilde{X}}$	δ_X
$\bar{\mu}(W) =$	MCC	*****	0.109E-11	0.221E-08	0.990E-06	0.130E02	0.130E02
0.675E07	MCS	*****	0.216E-12	0.227E-08	0.174E-05	0.188E02	0.188E02
$\bar{N}_9 = 39$	GS	*****	0.993E-12	0.404E-08	0.125E-05	0.130E02	0.130E02
	OSM	*****	0.179E02	0.186E02	0.682E75	0.618E02	0.999E01
$\bar{\mu}(C_3) =$	MCC	0.0901	0.161E-05	0.174E-02	0.415E-01	0.253E02	0.253E02
0.589E15	GS	0.0290	0.294E-04	0.765E-02	0.666E-01	0.253E02	0.253E02
$\bar{N}_{10} = 2$	OSM	0.0394	0.150E-02	0.915E-01	0.290E75	0.253E02	0.253E02
$\bar{\mu}(W) >$	MCC	0.0335	0.725E-01	0.205E00	0.570E02	0.178E02	0.159E02
0.450E16	GS	0.0153	0.835E02	0.840E02	0.950E05	0.198E04	0.159E02
$\bar{N}_{11} = 7$	OSM	0.0300	0.215E03	0.216E03	0.209E75	0.500E04	0.213E02

Table 13

$3 \leq m \leq 6$, 0.524E03≤ $\mu(A)$ ≤0.150E08		$7 \leq m \leq 11$, 0.475E09≤ $\mu(A)$ ≤0.518E15		$12 \leq m \leq 13$, $\mu(A)$ >0.450E16	
$<\delta_X>$= 0.00, $<\delta_Y>$= 0.00		$<\delta_X>$= 0.00, $<\delta_Y>$= 0.00		$<\delta_X>$= 0.00, $<\delta_Y>$= 0.00	
MCS	0.384886772512E-10	MCC	0.64196E-04	MCS	0.182E01
QR	0.394340547190E-10	SVD	0.64196E-04	MCC	0.214E01
GS	0.432343309209E-10	MCS	0.23960E-03	GS	0.270E01
MCC	0.502178164479E-10	GS	0.23988E-03	QR	0.403E01
SVD	0.502178164479E-10	QR	0.44822E-03	SVD	INF= −1
TRM	0.334842097124E-07	TRM	0.58231E05	TRM	0.397E08
$<\delta_X>$= 0.10, $<\delta_Y>$= 0.054		$<\delta_X>$= 0.10, $<\delta_Y>$= 0.047		$<\delta_X>$= 0.10, $<\delta_Y>$= 0.045	
MCS	0.999999999782E-01	MCC	0.99945E-01	MCS	0.125E01
MCC	0.999999999809E-01	SVD	0.99945E-01	MCC	0.277E01
SVD	0.999999999809E-01	QR	0.99982E-01	GS	0.333E01
GS	0.999999999948E-01	MCS	0.10005E00	QR	0.465E01
QR	0.999999999991E-01	GS	0.10005E00	SVD	INF= −1
TRM	0.100000344252E00	TRM	0.23802E06	TRM	0.442E10
$<\delta_X>$= 0.20, $<\delta_Y>$= 0.107		$<\delta_X>$= 0.20, $<\delta_Y>$= 0.094		$<\delta_X>$= 0.20, $<\delta_Y>$= 0.090	
TRM	0.199999724752E00	QR	0.20003E00	MCC	0.219E01
MCC	0.199999999988E00	GS	0.20004E00	MCS	0.323E01
SVD	0.199999999988E00	MCS	0.20009E00	QR	0.342E01
MCS	0.199999999994E00	MCC	0.20016E00	GS	0.352E01
GS	0.199999999997E00	SVD	0.20016E00	SVD	INF= −1
QR	0.199999999999E00	TRM	0.10208E06	TRM	0.139E11
$<\delta_X>$= 0.30, $<\delta_Y>$= 0.161		$<\delta_X>$= 0.30, $<\delta_Y>$= 0.142		$<\delta_X>$= 0.30, $<\delta_Y>$= 0.135	
MCC	0.299999999983E00	MCS	0.29990E00	MCS	0.172E01
SVD	0.299999999983E00	GS	0.29991E00	MCC	0.279E01
MCS	0.299999999985E00	QR	0.29995E00	GS	0.350E01
GS	0.299999999992E00	MCC	0.29996E00	QR	0.465E01
QR	0.299999999996E00	SVD	0.29996E00	SVD	INF= −1
TRM	0.300000398356E00	TRM	0.60699E04	TRM	0.487E10
$<\delta_X>$= 0.39, $<\delta_Y>$= 0.209		$<\delta_X>$= 0.39, $<\delta_Y>$= 0.184		$<\delta_X>$= 0.39, $<\delta_Y>$= 0.176	
MCC	0.389999999988E00	QR	0.38988E00	MCS	0.333E01
SVD	0.389999999988E00	MCS	0.39002E00	MCC	0.377E01
GS	0.389999999997E00	MCC	0.39003E00	GS	0.392E01
MCS	0.389999999998E00	SVD	0.39003E00	QR	0.399E01
QR	0.389999999999E00	GS	0.39004E00	SVD	INF= −1
TRM	0.390000429338E00	TRM	0.31130E06	TRM	0.137E12
$<\delta_X>$= 0.60, $<\delta_Y>$= 0.320		$<\delta_X>$= 0.60, $<\delta_Y>$= 0.282		$<\delta_X>$= 0.60, $<\delta_Y>$= 0.269	
MCC	0.597599999976E00	MCC	0.59754E00	MCS	0.197E01
SVD	0.597599999976E00	SVD	0.59754E00	MCC	0.334E01
MCS	0.597599999983E00	QR	0.59761E00	QR	0.430E01
QR	0.597599999985E00	GS	0.59762E00	GS	0.471E01
GS	0.597599999990E00	MCS	0.59763E00	SVD	INF= −1
TRM	0.597600286570E00	TRM	0.19820E06	TRM	0.377E11

The notation used in Figures 1-9:

□ – $\delta_L(W)$;   ■ – $\delta_M(W)$;   ⊠ – $\delta_R(W)$.

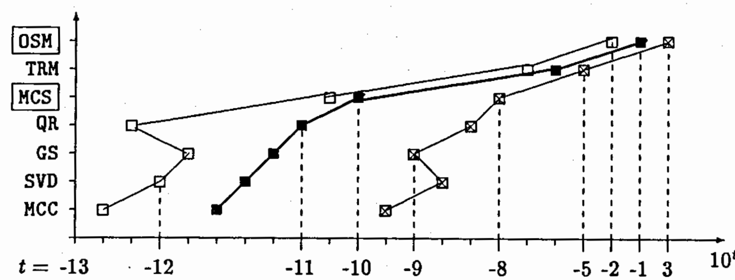


Fig.1. At $\bar{\mu}(W)$=0.533E07 — being well-posed, $\bar{N}_1 = 117$.

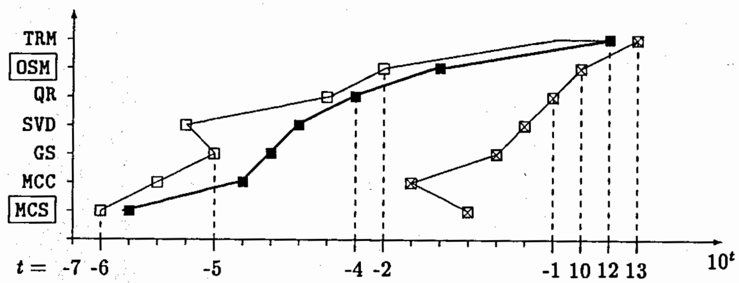


Fig. 2. At $\bar{\mu}(W)$=0.365E15 — being ill-posed, $\bar{N}_2 = 43$.

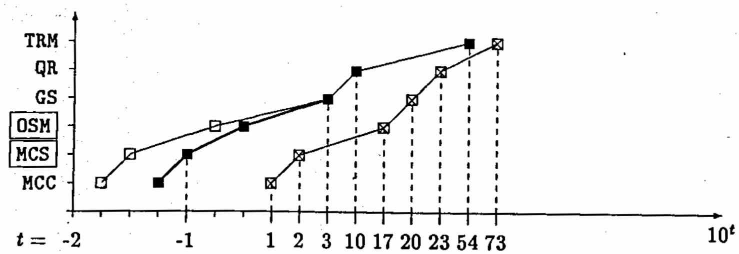


Fig. 3. At $\bar{\mu}(W)$>0.450E16 — beind pathologically ill-posed, $\bar{N}_3 = 46$.

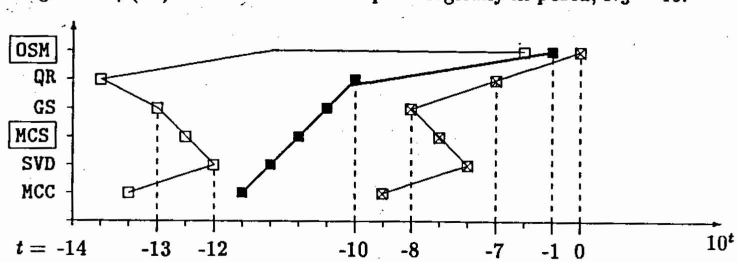


Fig. 4. At $\bar{\mu}(W)$=0.857E06 — being well-posed, $\bar{N}_5 = 16$.

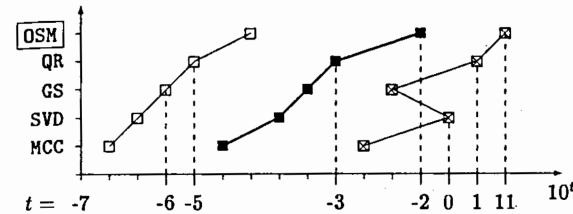


Fig. 5. At $\bar{\mu}(W)$=0.308E15 — being ill-posed, $\bar{N}_6 = 3$.

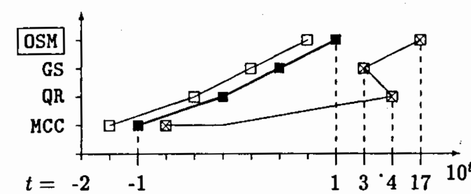


Fig. 6. At $\bar{\mu}(W)$ >0.450E16 — being pathologically ill-posed, $\bar{N}_7 = 5$.

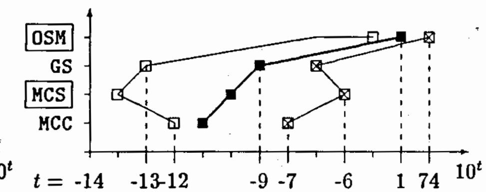


Fig. 7. At $\bar{\mu}(W)$ =0.675E07 — being well-posed, $\bar{N}_9 = 39$.

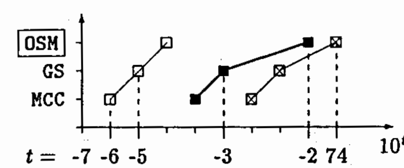


Fig. 8. At $\bar{\mu}(W)$ =0.589E15 — being ill-posed, $\bar{N}_{10} = 2$.

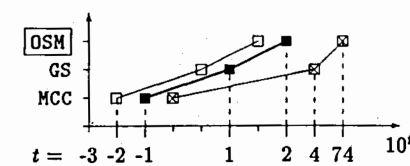


Fig. 9. At $\bar{\mu}(W)$ >0.450E16 — being pathologically ill-posed, $\bar{N}_{11} = 7$.

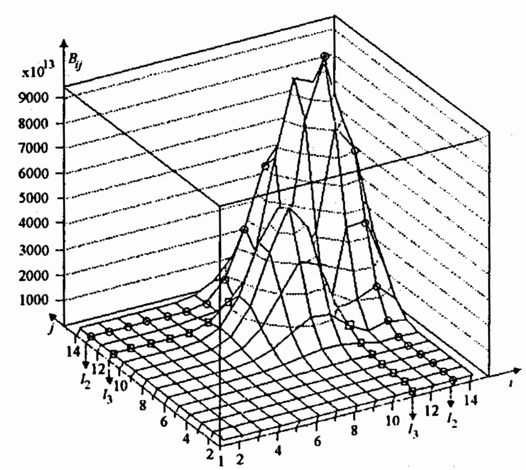


Fig. 10.

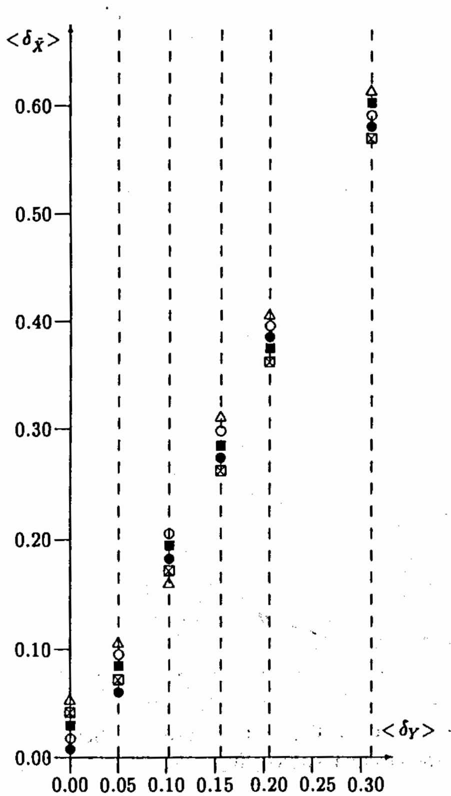


Fig.11 ($3 \leq m \leq 6$, $0.524E03 \leq \mu(A) \leq 0.150E08$)

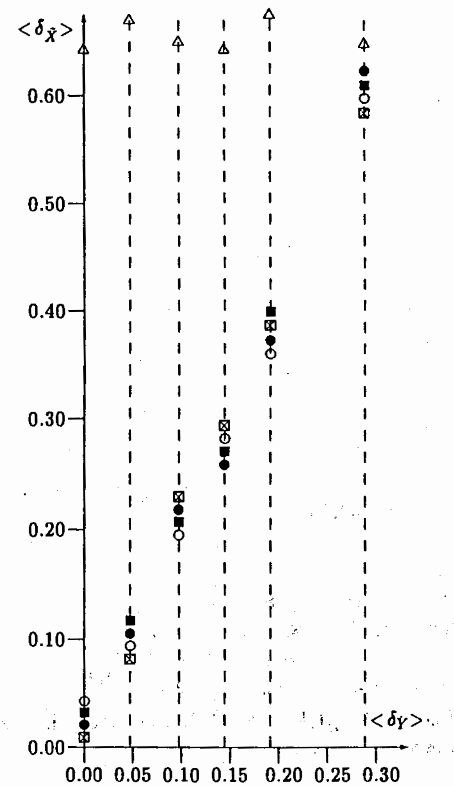


Fig.12 ($7 \leq m \leq 11$, $0.475E09 \leq \mu(A) \leq 0.518E15$)

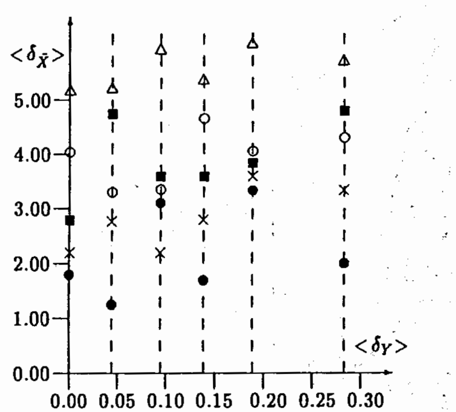
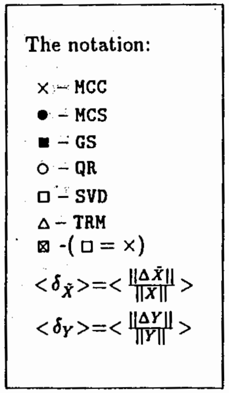


Fig.13 ($12 \leq m \leq 13$, $\mu(A) > 0.450E16$)



The analysis of numerical results reported in Tables 1 ÷ 13 and their graphical interpretation with the use of Figs. 1 ÷ 13 show that our programs MCC and MCS provide, on the average, better accuracy characteristics as compared to the most known analogous programs.

The program MCC has also better time characteristics in the case $W = C_2$, no matter, whether a system of equations is well- or ill- or pathologocally ill-posed, but MCC and MCS are about twice as worse in time as the program GS (DBEQN) in the case $W = C_3$. This is owing to the time consumption on the analysis of zeros in computing B_{ij} — elements of matrices $B = C_3^+$ and on testing various inequalities in accordance with the algorithm (2.1) ÷ (2.5). The programs MCC and MCS work about 10 times as slow as the program GS (DEQN) in the general case $W : A = A^T, A \neq A^T$. This is due to considerable time consumption on reduction of the system $WX = Y$ of the general form to systems of the type (1.2) and (1.3).

From the analysis presented it follows that the critical-component method in its qualitative characteristics is the best one of the methods of solution of degenerate and ill-posed systems of linear algebraic equations.

4. Conclusion

In this paper, we have demonstrated the efficiency of the critical- component method for numerical solution of degenerate and ill-posed systems of linear algebraic equations.

We have proved the theorem according to which the only stable normal solution can surely be obtained for degenerate and ill-posed systems of linear algebraic equations by the critical-component method.

Results of numerical experiments (278 examples were computed) on the calculation of basic characteristics of solution of the system $WX = Y$ are presented, and a comparative analysis has been performed, which shows that the programs MCC and MCS have, on the average, better characteristics.

5. Appendix

I. Test examples of systems of equations $C_2X = Y$ with two-diagonal matrices of the general form:

System 1

$$C_2 = \begin{bmatrix} 1 & 2 & & \\ & \cdots & & \\ & & 1 & 2 \\ & & & 1 \end{bmatrix}, \quad \begin{array}{l} x_i = 1/i, \\ i = 1,2,...,M, \\ y_i = \frac{3i+1}{i(i+1)},\ y_M = 1/M, \\ i = 1,2,...,M-1; \end{array}$$

System 2

$$C_2 = \begin{bmatrix} \varepsilon_0^* & r & & \\ & \cdots & & \\ & & \varepsilon_0^* & r \\ & & & \varepsilon_0^* \end{bmatrix}, \quad \begin{array}{l} x_i = 1/(2i+\varepsilon_0^*),\ i = 1,2,...,M, \\ y_i = \frac{2i+3\varepsilon_0^*}{(2i+\varepsilon_0^*)(2i+\varepsilon_0^*+2)},\ y_M = \varepsilon_0^*/(2M+\varepsilon_0^*), \\ i = 1,2,...,M-1, \\ \text{where } r = 1-\varepsilon_0^*,\ \varepsilon_0^* = 0,01; \end{array}$$

System 3

$$C_2 = \begin{bmatrix} \frac{7}{5} & \frac{11}{3} & & \\ & \cdots & & \\ & & \frac{7}{5} & \frac{11}{3} \\ & & & \frac{7}{5} \end{bmatrix}, \quad \begin{array}{l} x_i = 1/(2i+1), \\ i = 1,2,...,M, \\ y_i = \frac{152i+118}{15(2i+1)(2i+3)},\ y_M = 7/5(2M+1), \\ i = 1,2,...,M-1; \end{array}$$

System 4

$$C_2 = \begin{bmatrix} \varepsilon_0^* & 2 & & & & & & & \\ -1 & 2 & & & & & & & \\ & \cdots & & & & & & & \\ & & -1 & 2 & & & & & \\ & & & \varepsilon_1^* & 2 & & & & \\ & & & & -1 & 2 & & & \\ & & & & & \cdots & & & \\ & & & & & & -1 & 2 & \\ & & & & & & & & \varepsilon_1^* \end{bmatrix}, \quad \begin{array}{l} x_i = (-1)^{i+1}a, \\ i = 1,2,...,M, \\ y_1 = (\varepsilon_0^*-2)a, \\ y_i = (-1)^i 3a, \\ i = 2,3,...,k-1,k+1,...,M-1, \\ y_k = (-1)^k(2-\varepsilon_1^*)a, \\ y_M = (-1)^{M+1}\varepsilon_1^* a, \\ \text{where } a = 1+\varepsilon_0^*,\ \varepsilon_0^* = 0,0000001, \\ \varepsilon_1^* = 0,0001; \end{array}$$

System 5

$$C_2 = \begin{bmatrix} 3 & 7 & & \\ & \cdots & & \\ & & 3 & 7 \\ & & & 3 \end{bmatrix}, \quad \begin{array}{l} x_i = 1, \\ i = 1,2,...,M, \\ y_i = 10,\ y_M = 3, \\ i = 1,2,...,M-1. \end{array}$$

II. Test examples of systems of linear equations $C_3X = Y$ with tridiagonal matrices of the general form:

System 6

$$C_3 = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \begin{array}{l} x_i = \frac{1}{i},\ i = 1,2,...,M,\ y_1 = 2,\ y_M = \frac{M-2}{M(M-1)}, \\ y_i = \frac{2}{(1-i)i(1+i)},\ i = 2,3,...,M-1; \end{array}$$

System 7

$$C_3 = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & a \end{bmatrix}, \quad \begin{array}{l} x_i = 1+(-1)^i\varepsilon_0^*,\ i = 1,2,...,M, \\ y_1 = 2\varepsilon_0^*,\ y_i = (-1)^{i-1}4\varepsilon_0^*,\ i = 2,3,...,M-1, \\ y_M = (-1)^{M-1}(a+\varepsilon_0^*)\varepsilon_0^*, \\ \text{where } a = \frac{1-M}{M},\ \varepsilon_0^* = 0,0000001; \end{array}$$

System 8

$$C_3 = \begin{bmatrix} 1 & -1 & & & \\ -1 & 1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1 & -1 \\ & & & -1 & 1 \end{bmatrix}, \quad \begin{array}{l} x_i = \frac{1}{2i},\ i = 1,2,...,M,\ y_1 = \frac{1}{4},\ y_M = \frac{1}{2M(1-M)}, \\ y_i = \frac{i^2+1}{2i(1-i)(1+i)},\ i = 2,3,...,M-1; \end{array}$$

System 9

$$C_3 = \begin{bmatrix} 1 & r & & & \\ p & 1 & r & & \\ & \ddots & \ddots & \ddots & \\ & & p & 1 & r \\ & & & p & 1 \end{bmatrix}, \quad \begin{array}{l} x_i = 1,\ i = 1,2,...,M,\ y_1 = 2-\varepsilon_0^*, \\ y_M = 2+\varepsilon_0^*,\ y_i = 3,\ i = 2,3,...,M-1, \\ \text{where } p = 1+\varepsilon_0^*,\ r = 1-\varepsilon_0^*,\ \varepsilon_0^* = 0,0000001; \end{array}$$

System 10

$$C_3 = \begin{bmatrix} 6 & 3 & & & \\ 4 & 6 & 3 & & \\ & \ddots & \ddots & \ddots & \\ & & 4 & 6 & 3 \\ & & & 4 & 6 \end{bmatrix}, \quad \begin{array}{l} x_i = 1,\ i = 1,2,...,M, \\ y_1 = 9, y_M = 10,\ y_i = 13,\ i = 2,3,...,M-1. \end{array}$$

III. Test examples of systems of equations $AX = Y$ with $A \neq A^T$ – filled matrices of the general form:

System 11

$$A = \begin{bmatrix} M & M-1 & M-2 & \cdots & 3 & 2 & 333 \\ M-1 & M-1 & M-2 & \cdots & 3 & 2 & 1 \\ M-2 & M-2 & M-2 & \cdots & 3 & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & 3 & 3 & \cdots & 3 & 2 & 1 \\ 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ \varepsilon_0^* & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix},$$

$x_i = 1/i,\ i = 1,2,...,M,$

$y_1 = \sum_{k=1}^{M-1} \frac{M-k+1}{k} + \frac{333}{M},\ y_M = \sum_{k=2}^{M} \frac{1}{k} + \varepsilon_0^*,$

$y_i = (M-i+1)\sum_{k=1}^{i} \frac{1}{k} + \sum_{k=i+1}^{M} \frac{M-k+1}{k},$

$i = 2,3,...,M-1,$

where $\varepsilon_0^* = 0,0000001;$

System 12

$A = (a_{ij}),\ a_{ij} = \frac{1}{i+j-1},$

$i = 1,2,...,M-1,\ j = 1,2,...,M,$

$a_{M1} = 333,$

$a_{M1} = \frac{1}{M+j-1},\ j = 2,3,...,M,$

$x_i = 1/(2i+1),\ i = 1,2,...,M,$

$y_i = \sum_{k=1}^{M} \frac{1}{(2k+1)(i+k-1)},$

$i = 1,2,...,M-1,$

$y_M = \sum_{k=2}^{M} \frac{1}{(2k+1)(i+k-1)} + 111;$

System 13

$A = (a_{ij}),\ a_{1j} = a_{j1} = \frac{1}{M-j+1},$

$j = 1,2,...,M-1,$

$a_{1M} = 1 + \varepsilon_0^*,\ a_{M1} = 1 - \varepsilon_0^*,$

$a_{ij} = a_{ji} = \frac{1}{M-i+1},$

$i = 2,3,...,M,\ j = 2,3,...,i,$

$x_i = 1 - \varepsilon_0^*,\ i = 1,2,...,M,$

$y_1 = (1-\varepsilon_0^*)(\sum_{k=1}^{M-1} \frac{1}{M-k+1} + 1 + \varepsilon_0^*),$

$y_i = (1-\varepsilon_0^*)(\frac{i}{M-i+1} + \sum_{k=1}^{M} \frac{1}{M-k+1}),$

$i = 2,3,...,M-1,$

$y_M = (1-\varepsilon_0^*)(1 - \varepsilon_0^* + \sum_{k=2}^{M} \frac{1}{M-k+1}),$

where $\varepsilon_0^* = 0,00001;$

System 14

$$A = \begin{bmatrix} M & M-1 & & & & & \\ M-1 & M-1 & M-2 & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 3 & 3 & 3 & \cdots & 3 & 2 & \\ 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix},$$

$x_i = (-1)^i/i,\ i = 1,2,...,M,$

$y_i = (1 + \frac{(-1)^i(i-M)}{i+1})\sum_{k=1}^{i} \frac{(-1)^k}{k},$

$i = 1,2,...,M-1,$

$y_M = \sum_{k=1}^{M} (-1)^k/k;$

System 15

$A = (a_{ij}),\ a_{ij} = \frac{1}{i-j+M},$

$i = 1,2,...,M,\ j = 1,2,...,M,$

$x_i = 1/i,\ i = 1,2,...,M,$

$y_i = \sum_{k=1}^{M} \frac{1}{k(i-k+M)},$

$i = 1,2,...,M.$

IV. Test examples of systems of equations $AX = Y$ with $A = A^T$ – filled matrices of the general form:

System 16

$A = (a_{ij}),\ a_{1j} = a_{j1} = \frac{1}{M-j+1},$

$j = 1,2,...,M-1,$

$a_{M1} = a_{1M} = M + \varepsilon_0^*,$

$a_{ij} = a_{ji} = \frac{1}{M-i+1},$

$i = 2,3,...,M,\ j = 2,3,...,i,$

$x_i = (i+1)/i,\ i = 1,2,...,M,$

$y_1 = \sum_{k=1}^{M-1} \frac{k+1}{k(M-k+1)} + \frac{(M+\varepsilon_0^*)(M+1)}{M},$

$y_i = \frac{1}{M-i+1}\sum_{k=1}^{i} \frac{k+1}{k} + \sum_{k=i+1}^{M} \frac{k+1}{k(M-k+1)},$

$y_M = \sum_{k=2}^{M} \frac{k+1}{k} + 2(M + \varepsilon_0^*),$

$i = 1,2,...,M-1,$ where $\varepsilon_0^* = 0,0000001;$

System 17(A – the Hilbert matrix)

$A = (a_{ij}),\ a_{ij} = \frac{1}{i+j-1},$

$i = 1,2,...,M,\ j = 1,2,...,M,$

$x_i = 1/i,\ i = 1,2,...,M,$

$y_i = \sum_{k=1}^{M} \frac{1}{k(i+k-1)},\ i = 1,2,...,M;$

System 18

$A = (a_{ij}),\ a_{1j} = a_{j1} = \frac{1}{i+j-1},$

$j = 1,2,...,M-1,$

$a_{M1} = a_{1M} = 333,\ a_{ij} = a_{ji} = \frac{1}{i+j-1},$

$i = 2,3,...,M,\ j = 2,3,...,i,$

$x_i = (-1)^i/i,\ i = 1,2,...,M,$

$y_1 = \sum_{k=1}^{M-1} \frac{(-1)^k}{k^2} + \frac{(-1)^M 333}{M},$

$y_{i-1} = \sum_{k=1}^{M} \frac{(-1)^k}{k(k+i-2)},\ i = 3,4,...,M,$

$y_M = \sum_{k=2}^{M} \frac{(-1)^k}{k(k+M-1)} + 333;$

System 19

$$A = \begin{bmatrix} \varepsilon_0^* & M-1 & M-2 & \cdots & 3 & 2 & M \\ M-1 & M-1 & M-2 & \cdots & 3 & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ M & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix},$$

$x_i = 1 - \varepsilon_0^*,\ i = 1,2,...,M,$

$y_1 = \frac{(1-\varepsilon_0^*)(M^2-M+2\varepsilon_0^*)}{2},$

$y_i = \frac{(1-\varepsilon_0^*)(M-i)(i+M-3)}{2},$

$y_M = (2M-1)(1-\varepsilon_0^*),$

$i = 2,3,...,M-1,$ where $\varepsilon_0^* = 0,0000001;$

System 20(det(A)=0)

$$A = \begin{bmatrix} a & b & b & \cdots & b & a \\ b & a & b & \cdots & b & b \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ b & b & b & \cdots & a & b \\ a & b & b & \cdots & b & a \end{bmatrix},$$

$x_i = \frac{(-1)^i}{2i+1},\ i = 1,2,...,M,$

$y_1 = y_M = b\sum_{k=2}^{M-1} \frac{(-1)^k}{2k+1} + a\left(\frac{(-1)^M}{2M+1} - \frac{1}{3}\right),$

$y_i = b\sum_{k=1}^{M} \frac{(-1)^k}{2k+1} - \frac{(-1)^i a}{2i+1},$

$i = 2,3,...,M-1,$ where $a = 1 - \varepsilon_0^*,$

$b = 1 + \varepsilon_0^*,\ \varepsilon_0^* = 1 \cdot 10^{-11}.$

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