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A PARTICULAR INVERSE PROBLEM FOR SCHRÖDINGER DISCRETE EQUATION IN TWO AND HIGHER DIMENSIONS UNDER APRIORI INFORMATION OF WAVE FUNCTIONS

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Частичная обратная задача
для дискретного уравнения Шредингера в многомерном случае
при некоторой априорной информации о волновых функциях

Определяются все значения потенциала и остальные значения волновых функций в области параллелепипеда, если весь спектральный набор задан вместе с априорной информацией о волновых функциях на одной боковой стенке параллелепипеда. Дана формулировка частичной обратной задачи для дискретного уравнения Шредингера в двумерном и многомерном случаях, а также выведены новые формулы для решения задачи. Приведены два примера для двумерной и один пример для трехмерной задачи.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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A Particular Inverse Problem
for Schrödinger Discrete Equation in Two and Higher Dimensions
under Apriori Information of Wave Functions

The entire potential and the rest of wave functions are determined in parallelepiped domain if the entire discrete spectrum and the apriori information about the wave functions on one side of parallelepiped are given. Formulation for solving the Shrödinger discrete equation in two and higher dimensions is proposed and new formulas are derived for their solution. Two examples for a 2D case and one example for a 3D case are demonstrated.

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1 INTRODUCTION

In general, the inverse problem for Schrödinger equation is to find all potential values and wave functions if spectral parameters are given. The uniqueness of the continuous inverse problem was studied in [1]. Then it was developed in [2] and in number of other works.

The problem was studied, developed and applied especially in its discrete form to 1D case. A good review of the works is done in [3]. We briefly remind the 1D problem given in [3]. Let us consider the Schrödinger discrete equation (SDE)

$$-\frac{\Psi_{i-1}(\lambda) - 2\Psi_i(\lambda) + \Psi_{i+1}(\lambda)}{\Delta^2} + V_i\Psi_i(\lambda) = E_\lambda \Psi_i(\lambda), \tag{1}$$

$$\Psi_0(\lambda) = 0, \qquad \Psi_{L+1}(\lambda) = 0, \tag{2}$$

where Δ is a positive equidistant step and $\lambda = 1, 2, \dots, L$. If eigenvalues E_λ and also first components of eigenvectors

$$\Psi_1(\lambda) \tag{3}$$

are given for all $\lambda = 1, 2, \dots, L$ then according to [3] the potential values

$$V_i = -\frac{2}{\Delta^2} + \Delta \sum_{\lambda=1}^{L} E_\lambda [\Psi_i(\lambda)]^2 \tag{4}$$

and eigenvectors

$$\Psi_{i+1}(\lambda) = [\Delta^2(V_i - E_\lambda) + 2]\Psi_i(\lambda) - \Psi_{i-1}(\lambda) \tag{5}$$

can be determined sequentially for $i = 1, 2, \dots, L$ and $\lambda = 1, 2, \dots, L$. We can see the problem (1) – (3) is not a full inverse problem for SDE because apriori information – the potential values (3) must be given. Hence, the existence of the solution of the problem (1) – (3) is tied together with the existence of the solution of the direct problem (1) – (2) because of the apriori information.

At present, perhaps the closest solution to the solution of the inverse problem for 1D SDE is the solution given in [4]. There the 3-diagonal symmetric matrix $\tilde{D}$ is determined

$$\tilde{D} = \begin{bmatrix} 2+x_1 & -1+u_1 & 0 & \dots & 0 & 0 \\ -1+u_1 & 2+x_2 & -1+u_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1+u_{L-1} & 2+x_L \end{bmatrix}$$

that fulfills the equation $\tilde{D}\vec{\Psi}(\lambda) = E_\lambda \vec{\Psi}(\lambda)$ for arbitrary $E_\lambda : E_1 < E_2 < \ldots < E_L$. Two cases are studied. The first case, the first components of eigenvectors (3) are given. The second case, the symmetrical wave functions are supposed. If solutions $u_1 = u_2 = \ldots = u_{L-1} = 0$, one can get a solution of the inverse problem for SDE in 1D case. However, conditions $u_1 = u_2 = \ldots = u_{L-1} = 0$ are not guaranteed.
The same method of "perturbation matrix" was used in [5] for the solution of the inverse problem of the discrete Dirac system.

The main goal of this work is to derive the formulas analogous to (4) – (5) that give solution (potential and eigenvectors) in case of a particular inverse problem of SDE in two and higher dimensions if apriori information, analogous to (3), about the wave functions is given. In Section 2 we formulate and study a 2D particular inverse problem for SDE where we derive analogous to formulas (4) – (5). Here we also give two examples that verify the new formulas derived. Section 3 contains a formulation of a particular inverse problem for SDE in higher dimensions and new formulas of the type (4) – (5) for these higher dimensions. We give an example for 3D case as well.

2 PARTICULAR INVERSE PROBLEM FOR SDE IN TWO DIMENSIONS

Let us formulate a particular inverse problem for SDE in two dimensions in the rectangular domain:

$$-\frac{\Psi_{i-1,j}(\lambda) - 2\Psi_{ij}(\lambda) + \Psi_{i+1,j}(\lambda)}{\Delta_x^2} - \frac{\Psi_{i,j-1}(\lambda) - 2\Psi_{ij}(\lambda) + \Psi_{i,j+1}(\lambda)}{\Delta_y^2} +$$

$$+ V_{ij}\Psi_{ij}(\lambda) \quad = \quad E_\lambda \Psi_{ij}(\lambda) \tag{6}$$

$$\Psi_{0j}(\lambda) = 0, \qquad \Psi_{L+1,j}(\lambda) = 0, \qquad \Psi_{i0}(\lambda) = 0, \qquad \Psi_{i,M+1}(\lambda) = 0 \tag{7}$$

where Δ_x, Δ_y are some positive equidistant steps and $\lambda = 1, 2, \ldots, LM$; $i = 1, 2, \ldots, L$; $j = 1, 2, \ldots, M$. If eigenvalues E_λ :

$$E_1 < E_2 < \ldots < E_{LM}$$

and the following components of eigenvectors

$$\Psi_{1j}(\lambda) \tag{8}$$

are given for each $\lambda = 1, 2, \ldots, LM$; $j = 1, 2, \ldots, M$ then we have the particular inverse problem (6) – (8) for SDE in two dimensions in the rectangular domain. Beside spectral data – different E_λ $\lambda = 1, 2, \ldots, LM$, we also use here apriori information – wave functions components (8) as input data of the particular inverse problem. The problem is to find the rest of the wave functions and all the values of the potential. The existence of the problem (6) – (8) is tied together with the existence of the direct problem (6) – (7) because of using of apriori information (8).

Lemma 1 *For eigenvectors of the particular inverse problem* (6) – (8) *it holds*

$$\Delta_x \Delta_y \sum_{i=1}^{L} \sum_{j=1}^{M} \Psi_{ij}(\lambda)\Psi_{ij}(\lambda') = \delta_{\lambda\lambda'} \tag{9}$$

where $\delta_{\lambda\lambda'}$ is Kronecker's symbol and $\lambda, \lambda' = 1, 2, \ldots, LM$.

Proof. Let us multiply the equation (6) by $\Delta_x\Delta_y\Psi_{ij}(\lambda')$ and consider the equation (6) with $\lambda = \lambda'$. We multiply the equation (6) with $\lambda = \lambda'$ by $\Delta_x\Delta_y\Psi_{ij}(\lambda)$. Subtracting these two equations, we receive

$$\begin{aligned}
&[\Psi_{i+1,j}(\lambda')\Psi_{ij}(\lambda) - \Psi_{i+1,j}(\lambda)\Psi_{ij}(\lambda')]\,\Delta_y/\Delta_x + \\
&+ [\Psi_{i-1,j}(\lambda')\Psi_{ij}(\lambda) - \Psi_{i-1,j}(\lambda)\Psi_{ij}(\lambda')]\,\Delta_y/\Delta_x + \\
&+ [\Psi_{i,j+1}(\lambda')\Psi_{ij}(\lambda) - \Psi_{i,j+1}(\lambda)\Psi_{ij}(\lambda')]\,\Delta_x/\Delta_y + \\
&+ [\Psi_{i,j-1}(\lambda')\Psi_{ij}(\lambda) - \Psi_{i,j-1}(\lambda)\Psi_{ij}(\lambda')]\,\Delta_x/\Delta_y = \\
&= (E_\lambda - E_{\lambda'})\Psi_{ij}(\lambda')\Psi_{ij}(\lambda)\Delta_x\Delta_y.
\end{aligned} \tag{10}$$

Making the sum of the equation (10) for all $i = 1, 2, \ldots, L$; $j = 1, 2, \ldots, M$ we obtain

$$\begin{aligned}
&(E_\lambda - E_{\lambda'}) \sum_{i=1}^{L} \sum_{j=1}^{M} \Psi_{ij}(\lambda')\Psi_{ij}(\lambda)\Delta_x\Delta_y = \\
&= \sum_{i=1}^{L} \sum_{j=1}^{M} [\Psi_{i+1,j}(\lambda')\Psi_{ij}(\lambda) - \Psi_{i+1,j}(\lambda)\Psi_{ij}(\lambda')]\,\Delta_x\Delta_y + \\
&+ \sum_{i=1}^{L} \sum_{j=1}^{M} [\Psi_{i-1,j}(\lambda')\Psi_{ij}(\lambda) - \Psi_{i-1,j}(\lambda)\Psi_{ij}(\lambda')]\,\Delta_x\Delta_y + \\
&+ \sum_{i=1}^{L} \sum_{j=1}^{M} [\Psi_{i,j+1}(\lambda')\Psi_{ij}(\lambda) - \Psi_{i,j+1}(\lambda)\Psi_{ij}(\lambda')]\,\Delta_x\Delta_y + \\
&+ \sum_{i=1}^{L} \sum_{j=1}^{M} [\Psi_{i,j-1}(\lambda')\Psi_{ij}(\lambda) - \Psi_{i,j-1}(\lambda)\Psi_{ij}(\lambda')]\,\Delta_x\Delta_y.
\end{aligned} \tag{11}$$

We shift the index $i+1 \to i$ in the first double sum and the index $j+1 \to j$ in the third double sum in the right hand side of the equation (11). Subtracting the terms with opposite signs, we come to the following expression (12) for the right hand side of the equation (11)

$$\sum_{j=1}^{M} [\Psi_{L+1,j}(\lambda')\Psi_{Lj}(\lambda) - \Psi_{L+1,j}(\lambda)\Psi_{Lj}(\lambda')]\,\Delta_y/\Delta_x +$$
$$+\sum_{j=1}^{M} [\Psi_{0,j}(\lambda')\Psi_{1j}(\lambda) - \Psi_{0,j}(\lambda)\Psi_{1j}(\lambda')]\,\Delta_y/\Delta_x +$$
$$+\sum_{i=1}^{L} [\Psi_{i,M+1}(\lambda')\Psi_{Mj}(\lambda) - \Psi_{i,M+1}(\lambda)\Psi_{Mj}(\lambda')]\,\Delta_x/\Delta_y +$$
$$+\sum_{i=1}^{L} [\Psi_{i,0}(\lambda')\Psi_{i1}(\lambda) - \Psi_{i,0}(\lambda)\Psi_{i1}(\lambda')]\,\Delta_x/\Delta_y. \tag{12}$$

Due to the boundary conditions (7) the expression (12) is equal to zero. That means the left hand side of the equation (11) is equal to zero as well. So we have proved

$$(E_\lambda - E_{\lambda'})\sum_{i=1}^{L}\sum_{j=1}^{M}\Psi_{ij}(\lambda')\Psi_{ij}(\lambda)\Delta_x\Delta_y = 0.$$

The last equality means the eigenvectors are orthogonal if $\lambda \neq \lambda'$. After normalization of the eigenvectors we come to desired equation (9). The proof is complete.

Lemma 2 *If the orthonormal eigenvectors of the problem* (6) – (8) *hold the equation* (9), *they also hold*

$$\Delta_x\Delta_y \sum_{\lambda=1}^{LM} \Psi_{ij}(\lambda)\Psi_{i'j'}(\lambda) = \delta_{ij,i'j'}. \tag{13}$$

Proof. Let us multiply the equation (9) by $\Psi_{i'j'}(\lambda)$ and make the sum for all $\lambda = 1, 2, \ldots, LM$. Then we obtain

$$\sum_{i=1}^{L}\sum_{j=1}^{M} \Psi_{ij}(\lambda')\left[\Delta_x\Delta_y \sum_{\lambda=1}^{LM} \Psi_{ij}(\lambda)\Psi_{i'j'}(\lambda)\right] = \sum_{\lambda=1}^{LM} \delta_{\lambda,\lambda'}\Psi_{i'j'}(\lambda). \tag{14}$$

We have $\Psi_{i'j'}(\lambda')$ on the right hand side of the equation (14). Changing indices i', j' $i' = 1, 2, \ldots, L$; $j' = 1, 2, \ldots, M$ the equation (14) represents a system of linear algebraic equations $T\vec{\Psi}(\lambda') = I\vec{\Psi}(\lambda')$ where the elements of the matrix T are given by expression in the parenthesis [] of the equation (14) and I is a

identity matrix. Since the vectors $\vec{\Psi}(\lambda')$ for $\lambda' = 1, 2, \ldots, LM$ represent the system of linearly independent vectors, this means the equation $(T - I)\vec{\Psi}(\lambda') = 0$ results the equation $T = I$. However, $T = I$ is the equation (14). This brings the proof to an end.

Theorem 1 *The particular inverse problem* (6)–(8) *for SDE has the following solution*

$$\Psi_{i+1,j}(\lambda) = \Big[-E_\lambda + \Delta_x\Delta_y \sum_{\lambda=1}^{LM} E_\lambda[\Psi_{ij}(\lambda)]^2\Big]\Psi_{ij}(\lambda)\Delta_x^2 - \Psi_{i-1,j}(\lambda) -$$
$$-\frac{\Delta_x^2}{\Delta_y^2}[\Psi_{i,j+1}(\lambda) + \Psi_{i,j-1}(\lambda)], \tag{15}$$

$$V_{ij} = -\frac{2}{\Delta_x^2} - \frac{2}{\Delta_y^2} + \Delta_x\Delta_y \sum_{\lambda=1}^{LM} E_\lambda[\Psi_{ij}(\lambda)]^2. \tag{16}$$

Proof. Let us substitute from the equations (15) – (16) to the left hand side of the equation (6). We can find out the equation (6) is fulfilled. This ends the proof.

Now we explain how the equations (15) and (16) can be derived. Let us multiply the equation (6) by $\Psi_{ij}(\lambda)$ and make the sum for all $\lambda = 1, 2, \ldots, LM$. Due to the **Lemma 2** we receive the expression (16) for potential. Substituting from the equation (16) to the equation (6) and expressing the value $\Psi_{i+1,j}(\lambda)$, we obtain the equation (15).

Now we explain how to make the computations according to the formulas (15) – (16). Having boundary values (8) i.e. $\Psi_{1j}(\lambda)$, we can compute the values $\Psi_{2j}(\lambda)$ for all $j = 1, 2, \ldots, M$ and for all $\lambda = 1, 2, \ldots, LM$ according to the formula (15). Then we can do the same consequently for $\Psi_{3j}(\lambda), \ldots, \Psi_{Lj}(\lambda)$. Having all values of wave function $\Psi_{ij}(\lambda)$, we can calculate all values of potential V_{ij} according to the formula (15).

The uniqueness of the problem (6)–(8) follows from the formulas (15)–(16).

At the end of this section we give two examples.

Example 1.
Let us put $L = 2$, $M = 2$ and

$$V_{11} = V_{12} = V_{21} = V_{22} = 1/\Delta_x^2 + 1/\Delta_y^2, \tag{17}$$

$(\Delta_x < \Delta_y)$. Then the solution of the direct problem (6) – (7) is given in the following table

λ	1	2	3	4
E_λ	$\frac{2}{\Delta_x^2}+\frac{2}{\Delta_y^2}$	$\frac{2}{\Delta_x^2}+\frac{4}{\Delta_y^2}$	$\frac{4}{\Delta_x^2}+\frac{2}{\Delta_y^2}$	$\frac{4}{\Delta_x^2}+\frac{4}{\Delta_y^2}$
$\Psi_{11}(\lambda)$	a	-a	-a	a
$\Psi_{12}(\lambda)$	a	a	-a	-a
$\Psi_{21}(\lambda)$	a	-a	a	-a
$\Psi_{22}(\lambda)$	a	a	a	a

where $a = 1/(2\sqrt{\Delta_x\Delta_y})$.
Let us put $L = 2$, $M = 2$ and E_λ, $\Psi_{11}(\lambda)$, $\Psi_{12}(\lambda)$ according to the above table $(\lambda = 1,2,3,4)$. Then the solution of the problem (6) – (7) calculated by the formulas (15) – (16) gives us the same potential values like those given in (17) and

$$\Psi_{21}(1) = \Psi_{21}(3) = a, \qquad \Psi_{21}(2) = \Psi_{21}(4) = -a$$

$$\Psi_{22}(1) = \Psi_{22}(2) = \Psi_{22}(3) = \Psi_{22}(4) = a.$$

Example 2.
Let us put $L = 2$, $M = 2$ and

$$V_{11} = V_{22} = 3(1/\Delta_x^2 + 1/\Delta_y^2), \qquad V_{12} = V_{21} = 1/\Delta_x^2 + 1/\Delta_y^2. \tag{18}$$

Then the solution of the direct problem (6)–(7) is shown in the following table:

λ	1	2	3	4
E_λ	$(4-\sqrt{2})b$	$4b-c$	$4b+c$	$(4+\sqrt{2})b$
$\Psi_{11}(\lambda)$	$\frac{a}{\sqrt{2+\sqrt{2}}}$	$-a\sqrt{1-\frac{b}{c}}$	$-a\sqrt{1+\frac{b}{c}}$	$\frac{a}{\sqrt{2-\sqrt{2}}}$
$\Psi_{12}(\lambda)$	$a\frac{1+\sqrt{2}}{\sqrt{2+\sqrt{2}}}$	$\frac{da}{-b+c}\sqrt{1-\frac{b}{c}}$	$\frac{-da}{b+c}\sqrt{1+\frac{b}{c}}$	$a\frac{1-\sqrt{2}}{\sqrt{2-\sqrt{2}}}$
$\Psi_{21}(\lambda)$	$a\frac{1+\sqrt{2}}{\sqrt{2+\sqrt{2}}}$	$\frac{-da}{-b+c}\sqrt{1-\frac{b}{c}}$	$\frac{da}{b+c}\sqrt{1+\frac{b}{c}}$	$a\frac{1-\sqrt{2}}{\sqrt{2-\sqrt{2}}}$
$\Psi_{22}(\lambda)$	$\frac{a}{\sqrt{2+\sqrt{2}}}$	$a\sqrt{1-\frac{b}{c}}$	$a\sqrt{1+\frac{b}{c}}$	$\frac{a}{\sqrt{2-\sqrt{2}}}$

where $b = 1/\Delta_x^2 + 1/\Delta_y^2$, $c = \sqrt{2(1/\Delta_x^4 + 1/\Delta_y^4)}$, $d = 1/\Delta_x^2 - 1/\Delta_y^2$.
If we put $L = 2$, $M = 2$ and E_λ, $\Psi_{11}(\lambda)$, $\Psi_{12}(\lambda)$, according to the last table

$(\lambda = 1,2,3,4)$ then the solution to the problem (6) – (8) calculated by the formulas (15) – (16) give us the potential values $V_{11} = V_{22} = 3b$, $V_{12} = V_{21} = b$ and $\Psi_{21}(1)$, $\Psi_{21}(2)$, $\Psi_{21}(3)$, $\Psi_{21}(4)$ $\Psi_{22}(1)$, $\Psi_{22}(2)$, $\Psi_{22}(3)$, $\Psi_{22}(4)$ just the same to those given in the last table.

3 PARTICULAR INVERSE PROBLEM FOR SDE IN HIGHER DIMENSIONS

Completing this work, we formulate the particular inverse problem for SDE in the general n dimensional Euclidian space in the parallelepiped domain:

$$-\sum_{j=1}^{n} \frac{\Psi_{i_1i_2\ldots,i_j-1,\ldots i_n}(\lambda) - 2\Psi_{i_1i_2\ldots,i_j,\ldots i_n}(\lambda) + \Psi_{i_1i_2\ldots,i_j+1,\ldots i_n}(\lambda)}{\Delta_{i_j}^2} +$$

$$+ V_{i_1i_2\ldots i_n}\Psi_{i_1i_2\ldots i_n}(\lambda) = E_\lambda\Psi_{i_1i_2\ldots i_n}(\lambda) \tag{19}$$

$$\Psi_{0i_2\ldots i_n}(\lambda) = 0, \quad \Psi_{i_10\ldots i_n}(\lambda) = 0, \quad \ldots, \quad \Psi_{i_1i_2\ldots 0}(\lambda) = 0$$

$$\Psi_{L_1+1,i_2,\ldots,i_n}(\lambda) = 0, \ \Psi_{i_1,L_2+1,\ldots,i_n}(\lambda) = 0, \ \ldots \ \Psi_{i_1i_2\ldots,\ L_n+1}(\lambda) = 0 \tag{20}$$

where Δ_{i_j} $i_j = 1,2,\ldots,L_j$; $j = 1,2,\ldots,n$ are positive equidistant steps and $\lambda = 1,2,\ldots,L_1L_2\ldots L_n$. If eigenvalues E_λ

$$E_1 < E_2 < \ldots < E_{L_1L_2\ldots L_n}$$

and the following components of eigenvectors

$$\Psi_{1i_2\ldots i_n}(\lambda), \tag{21}$$

are given for each $\lambda = 1,2,\ldots,L_1L_2\ldots L_n$; $i_j = 1,2,\ldots,L_j$; $j = 2,3,\ldots,n$ then we have the particular inverse problem (19)–(21) for SDE in n dimensions in the parallelepiped domain. In 2D case the existence of the problem (19)–(21) is tied together with the existence of the direct problem (19) – (20) as well.

In the same way the following statements can be proved

Lemma 3 *For eigenvectors of the problem* (19) – (21) *it holds*

$$\Delta_{i_1}\Delta_{i_2}\ldots\Delta_{i_n}\sum_{i_1=1}^{L_1}\sum_{i_2=1}^{L_2}\cdots\sum_{i_n=1}^{L_n}\Psi_{i_1i_2\ldots i_n}(\lambda)\Psi_{i_1i_2\ldots i_n}(\lambda') = \delta_{\lambda\lambda'} \tag{22}$$

where $\lambda, \lambda' = 1,2,\ldots,L_1L_2\ldots L_n$.

Lemma 4 *If the orthonormal eigenvectors of the problem* (19) – (21) *hold the equation* (22), *they also hold*

$$\Delta_{i_1}\Delta_{i_2}\dots\Delta_{i_n}\sum_{\lambda=1}^{L_1L_2\dots L_n}\Psi_{i_1i_2\dots i_n}(\lambda)\Psi_{i'_1i'_2\dots i'_n}(\lambda)=\delta_{i_1i_2\dots i_n,i'_1i'_2\dots i'_n}. \tag{23}$$

Theorem 2 *The problem* (19) – (21) *for SDE has the following solution*

$$\Psi_{i_1+1,i_2\dots i_n}(\lambda)=$$

$$=\Big[-E_\lambda+\Delta_{i_1}\Delta_{i_2}\dots\Delta_{i_n}\sum_{\lambda=1}^{L_1L_2\dots L_n}E_\lambda[\Psi_{i_1i_2\dots i_n}(\lambda)]^2\Big]\Psi_{i_1i_2\dots i_n}(\lambda)\Delta_{i_1}^2-$$

$$-\Psi_{i_1-1,i_2\dots i_n}(\lambda)-$$

$$-\sum_{j=2}^{n}[\Psi_{i_1i_2\dots,i_j+1,\dots i_n}(\lambda)+\Psi_{i_1i_2\dots,i_j-1,\dots i_n}(\lambda)]\Delta_{i_1}^2/\Delta_{i_j}^2, \tag{24}$$

$$V_{i_1i_2\dots i_n}=-\frac{2}{\Delta_{i_1}^2}-\frac{2}{\Delta_{i_2}^2}\dots-\frac{2}{\Delta_{i_n}^2}+$$

$$+\Delta_{i_1}\Delta_{i_2}\dots\Delta_{i_n}\sum_{\lambda=1}^{L_1L_2\dots L_n}E_\lambda[\Psi_{i_1i_2\dots i_n}(\lambda)]^2. \tag{25}$$

Specially for $n=3$ we give the following example:
Example 3. Let us put $L_1=L_2=L_3=2$, and

$$V_{l_1l_2l_3}=\frac{1}{\Delta_{i_1}^2}+\frac{1}{\Delta_{i_2}^2}+\frac{1}{\Delta_{i_3}^2},\quad l_1,l_2,l_3=1,2 \tag{26}$$

($\Delta_{i_1}<\Delta_{i_2}<\Delta_{i_3}$). Then the solution of the direct problem (19) – (20) is given in the following table

λ	1	2	3	4	5	6	7	8
E_λ	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
$\Psi_{111}(\lambda)$	a	-a	a	a	-a	a	a	a
$\Psi_{112}(\lambda)$	a	a	a	-a	-a	-a	a	-a
$\Psi_{121}(\lambda)$	a	-a	-a	-a	-a	a	-a	-a
$\Psi_{122}(\lambda)$	a	a	-a	a	-a	-a	-a	a
$\Psi_{211}(\lambda)$	a	-a	a	a	a	-a	-a	-a
$\Psi_{212}(\lambda)$	a	a	a	-a	a	a	-a	a
$\Psi_{221}(\lambda)$	a	-a	-a	-a	a	-a	a	a
$\Psi_{222}(\lambda)$	a	a	-a	a	a	a	a	-a

where $a=1/\sqrt{8\Delta_{i_1}\Delta_{i_2}\Delta_{i_3}}$ and

$$E_1=2\left(\frac{1}{\Delta_{i_1}^2}+\frac{1}{\Delta_{i_2}^2}+\frac{1}{\Delta_{i_3}^2}\right),\qquad E_2=2\left(\frac{1}{\Delta_{i_1}^2}+\frac{1}{\Delta_{i_2}^2}+\frac{2}{\Delta_{i_3}^2}\right),$$

$$E_3=2\left(\frac{1}{\Delta_{i_1}^2}+\frac{2}{\Delta_{i_2}^2}+\frac{1}{\Delta_{i_3}^2}\right),\qquad E_4=2\left(\frac{1}{\Delta_{i_1}^2}+\frac{2}{\Delta_{i_2}^2}+\frac{2}{\Delta_{i_3}^2}\right),$$

$$E_5=2\left(\frac{2}{\Delta_{i_1}^2}+\frac{1}{\Delta_{i_2}^2}+\frac{1}{\Delta_{i_3}^2}\right),\qquad E_6=2\left(\frac{2}{\Delta_{i_1}^2}+\frac{1}{\Delta_{i_2}^2}+\frac{2}{\Delta_{i_3}^2}\right),$$

$$E_7=2\left(\frac{2}{\Delta_{i_1}^2}+\frac{2}{\Delta_{i_2}^2}+\frac{1}{\Delta_{i_3}^2}\right),\qquad E_8=2\left(\frac{2}{\Delta_{i_1}^2}+\frac{2}{\Delta_{i_2}^2}+\frac{2}{\Delta_{i_3}^2}\right).$$

Let us put $L_1=L_2=L_3=2$, and E_λ, $\Psi_{111}(\lambda)$, $\Psi_{112}(\lambda)$, $\Psi_{121}(\lambda)$, $\Psi_{122}(\lambda)$ according to the above table ($\lambda=1,2,3,4,5,6,7,8$). Then the solution of the problem (19) – (21) calculated by the formulas (24) – (25) give us the same potential values like (26) and $\Psi_{2i_2i_3}(\lambda)$ $i_2,i_3=1,2$ the same like those given in the table.

We used MAPLE V Release 4 for verification of calculated values in **Examles 1., 2.** and **3.**

4 CONCLUSION REMARKS

Apriori information (8) resp. (21) is important in the formulation of the problems (6) – (8) resp. (19) – (21). Having this information the rest of wave functions and all potential values can be determined according to the relativelly easy formulas (15) – (16) resp. (24) – (25). The question how to reduce or eliminate this apriori information is not studied here.

There are other possibilities in formulation of boundary conditions (8) resp. (21). We can consider, for example, eigenvectors components $\Psi_{i1}(\lambda)$ resp. $\Psi_{i_11\dots i_n}(\lambda)$ or other boundary formulation. However, this requires a new formula (15) resp. (24). In the case when $\Psi_{i1}(\lambda)$ resp. $\Psi_{i_11\dots i_n}(\lambda)$ is given instead of (8) resp. (21), we should use the following formula

$$\Psi_{i,j+1}(\lambda)=\Big[-E_\lambda+\Delta_x\Delta_y\sum_{\lambda=1}^{LM}E_\lambda[\Psi_{ij}(\lambda)]^2\Big]\Psi_{ij}(\lambda)\Delta_y^2-\Psi_{i,j-1}(\lambda)-$$

$$-\frac{\Delta_y^2}{\Delta_x^2}[\Psi_{i+1,j}(\lambda)+\Psi_{i-1,j}(\lambda)], \tag{27}$$

resp.

$$\Psi_{i_1,i_2+1,\dots i_n}(\lambda) =$$

$$= \Big[- E_\lambda + \Delta_{i_1}\Delta_{i_2}\dots\Delta_{i_n} \sum_{\lambda=1}^{L_1 L_2 \dots L_n} E_\lambda [\Psi_{i_1 i_2 \dots i_n}(\lambda)]^2 \Big] \Psi_{i_1 i_2 \dots i_n}(\lambda)\Delta_{i_2}^2 -$$

$$-\Psi_{i_1,i_2-1,\dots i_n}(\lambda)-$$

$$- \sum_{j\neq 2} [\Psi_{i_1 i_2 \dots, i_j+1,\dots i_n}(\lambda) + \Psi_{i_1 i_2 \dots, i_j-1,\dots i_n}(\lambda)]\Delta_{i_2}^2/\Delta_{i_j}^2. \qquad (28)$$

The formula for potential values (16) resp. (25) is still the same.

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