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CRITICAL-COMPONENT METHOD SOLUTIONS OF LINEAR ALGEBRAIC EQUATIONS

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1.Introduction

The aim of the present paper is as follows:
Development of new effective methods and algorithms of solving the systems of linear algebraic equations with a two–diagonal matrices. The use of these algorithms and those from ref.[1] in solving the systems of linear algebraic equations with a completely filled quadratic matrix. Construction of complexes of new programs on the basis of developed algorithms.

2. Critical - component method of solving systems of linear equations $C_2X = Y$ with two - diagonal matrix C_2

In this section, the critical - component method [2] of solution of the system of equations $C_3X = Y$ is applied to obtain the method of solution of the system of equations $C_2X = Y$.

So, let C_3 be a nonsingular real tridiagonal matrix of the general form

$$C_3 = \begin{bmatrix} q_1 & r_2 & & & \\ p_2 & q_2 & r_3 & & \\ & \ddots & \ddots & \ddots & \\ & & p_{m-1} & q_{m-1} & r_m \\ & & & p_m & q_m \end{bmatrix}, \qquad (2.1)$$

where $\{p_i \neq 0\}_{i=2}^m$ are subdiagonal elements, $\{r_i \neq 0\}_{i=2}^m$ are off–diagonal ones, $\{q_i\}_{i=1}^m$ are diagonal elements of C_3 (2.1) In ref. [2] for matrix C_3(2.1) we introduced the following generalized sequences $\{\Lambda_i\}$ and $\{\mathrm{G}_i\}$:

$$\begin{cases} \Lambda_{i+1} = q_i - p_i\Lambda_i^{-1}r_i, \ \Lambda_2 = q_1, \ i = 2, ..., m, \\ \text{if } \Lambda_i \neq 0 \text{ for all } 2 \leq i \leq m. \\ \text{If } \Lambda_i = 0 \text{ for any } i \text{ from } (2 \leq i \leq m), \text{ then} \\ \Lambda_{i+1} \text{ is undefined, but } \Lambda_{i+2} = q_{i+1}; \end{cases} \qquad (2.2)$$

$$\begin{cases} \mathrm{G}_{i-1} = q_i - r_{i+1}\mathrm{G}_i^{-1}p_{i+1}, \ \mathrm{G}_{m-1} = q_m, \ i = m-1, ..., 1, \\ \text{if } \mathrm{G}_i \neq 0 \text{ for all } 1 \leq i \leq m-1. \\ \text{If } \mathrm{G}_i = 0 \text{ for any } i \text{ from } (1 \leq i \leq m-1), \text{ then} \\ \mathrm{G}_{i-1} \text{ is undefined, but } \mathrm{G}_{i-2} = q_{i-1}. \end{cases} \qquad (2.3)$$

As it is seen, these sequences are defined both when all leading angular upper (lower) minors differ from zero*), and when some of them vanish**).

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*)By leading upper $\{\Delta_1^i\}_{i=1}^m$ and lower $\{\Delta_i^m\}_{i=1}^m$ angular minors of C_3 (2.1) we understand determinants of its submatrices starting from q_1 and q_m, respectively.

**)In solving algebraic problems with the use of processes (2.2) ÷ (2.3), theoretical and computer zeros, as noted in [2], are equivalent in nature.

In ref. [2], in constructing the solution to the system of linear equations, $C_3X = Y$, we introduced the following matrix processes*):

$$\begin{cases} \tilde C^{l_{k+1}}_{l_k+1} = C^{l_{k+1}}_{l_k+1} - [0,\dots,0,p_{l_k+1}]\left[\tilde C^{l_k}_{l_{k-1}+1}\right]^{-1}[0,\dots,0,r_{l_k+1}]^T, \\ \tilde C^{l_1}_1 = C^{l_1}_1,\ k=1,2,\dots,n; \\ \tilde C^{l_k}_{l_{k-1}+1} = C^{l_k}_{l_{k-1}+1} - [r_{l_k+1},0,\dots,0]\left[\tilde C^{l_{k+1}}_{l_k+1}\right]^{-1}[p_{l_k+1},0,\dots,0]^T, \\ \tilde C^m_{l_n+1} = C^m_{l_n+1},\ k=n,\dots,2,1, \end{cases} \qquad (2.3)'$$

where

$$C^\nu_\rho = \begin{bmatrix} q_\rho & r_{\rho+1} & & & \\ p_{\rho+1} & q_{\rho+1} & r_{\rho+2} & & \\ & \ddots & \ddots & \ddots & \\ & & p_{\nu-1} & q_{\nu-1} & r_\nu \\ & & & p_\nu & q_\nu \end{bmatrix},\ l_0=0,\ l_{n+1}=m. \qquad (2.3)''$$

In this case, matrices $\tilde C^\nu_\rho$ are well-posed and C_3 (2.1) is representable in the form

$$C = \begin{bmatrix} q_1 & r_2 & & & \\ p_2 & q_2 & r_3 & & \\ & \ddots & \ddots & \ddots & \\ & & p_{m-1} & q_{m-1} & r_m \\ & & & p_m & q_m \end{bmatrix} \equiv \begin{bmatrix} \left[C^{l_1}_1\right] & r_{l_1+1} & & \\ p_{l_1+1} & \left[C^{l_2}_{l_1+1}\right] & r_{l_2+1} & \\ & & \dots\dots\dots & \\ & & p_{l_n+1} & \left[C^m_{l_n+1}\right] \end{bmatrix}, \qquad (2.4)$$

where $C^{l_{k+1}}_{l_k+1} = tridiag\{q_{l_k+1}, q_i, p_i, r_i\}^{l_{k+1}}_{i=l_k+2}$ are submatrices of the form (2.3)'' of matrices C_3 (2.1) respectively.

In ref.[2], for x_i – components of the solution of system $C_3X = Y$ the following representation

$$\begin{cases} x_i = \overset{\circ}{x}_i + \bar c_i \gamma_k,\ k=n+1,n,\dots,1, \\ i = l_k, l_k-1,\dots,l_{k-1}+1,\ l_{n+1}=m,\ l_0=0, \end{cases} \qquad (2.5)$$

has been obtained by using (2.2) ÷ (2.4), where

$$\begin{cases} \overset{\circ}{x}_i = \tilde B_{il_k+1}\tilde y_{l_k+1} + \sum\limits_{j=l_k+2}^{l_{k+1}} (\tilde B_{ij} y_j),\ i = l_{k+1},\dots,l_k+1,\ k=n,\dots,1,0; \\ [\gamma_k \equiv x_{l_k+1}] = \overset{\circ}{x}_{l_k+1} + \bar c_{l_k+1}[\gamma_{k+1} \equiv x_{l_{k+1}+1}],\ [\gamma_{n+1} \equiv x_{m+1}] = 0,\ k=n,\dots,2,1; \\ \tilde y_{l_k+1} = y_{l_k+1} + \bar\beta_{l_{k-1}+1}\tilde y_{l_{k-1}+1} + \sum\limits_{j=l_{k-1}+2}^{l_k} (\bar\beta_j y_j),\ \tilde y_1 = y_1,\ k=1,2,\dots,n; \\ \bar\beta_i = -p_{l_k+1}\tilde B_{l_k i},\ \bar c_i = -\tilde B_{il_k} r_{l_k+1},\ i = l_{k-1}+1,\dots,l_k,\ k=1,2,\dots,n. \end{cases} \qquad (2.6)$$

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*)Hereafter T is the transposition sign.

Let us now apply the above-presented results for solving the system $C_2X = Y$ with a two-diagonal matrix C_2.

Thus, let elements $\{p_i \equiv 0\}^m_{i=2}$ of the matrix C_3 (2.1) be zero, i.e. C (2.1) is an upper two-diagonal matrix of the general form

$$C_2 = \begin{bmatrix} q_1 & r_2 & & \\ & q_2 & r_3 & \\ & & \ddots & \ddots \\ & & q_{m-1} & r_m \\ & & & q_m \end{bmatrix}. \qquad (2.7)$$

Accordingly, we represent this matrix in the form

$$C_2 = \begin{bmatrix} q_1 & r_2 & & \\ & q_2 & r_3 & \\ & & \ddots & \ddots \\ & & q_{m-1} & r_m \\ & & & q_m \end{bmatrix} \equiv \begin{bmatrix} \left[C^{l_1}_1\right] & r_{l_1+1} & & \\ & \left[C^{l_2}_{l_1+1}\right] & r_{l_2+1} & \\ & & \dots\dots & \\ & & & \left[C^m_{l_n+1}\right] \end{bmatrix}, \qquad (2.8)$$

where

$$C^{l_{k+1}}_{l_k+1} = \begin{bmatrix} q_{l_k+1} & r_{l_k+2} & & \\ & q_{l_k+2} & r_{l_k+3} & \\ & & \ddots & \ddots \\ & & q_{l_{k+1}-1} & r_{l_{k+1}} \\ & & & q_{l_{k+1}} \end{bmatrix},\ k=0,1,\dots,n,\ l_0=0,\ l_{n+1}=m. \qquad (2.9)$$

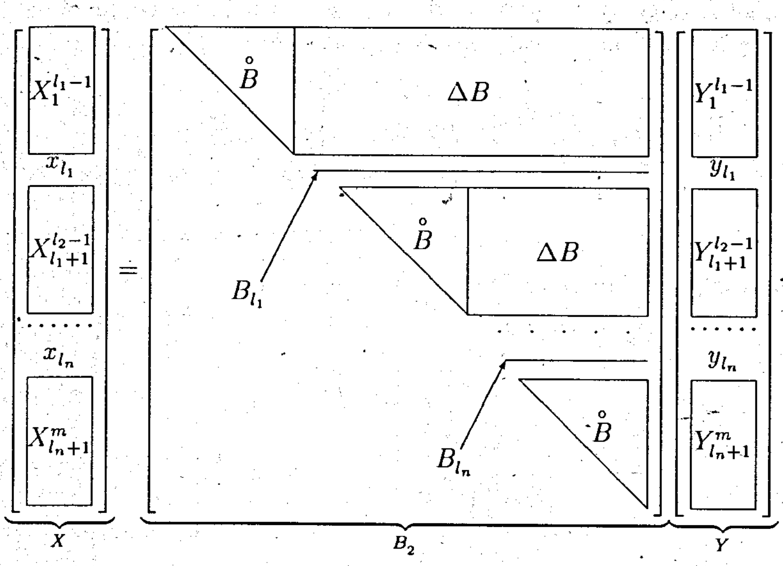
Here we will assume that $C^{l_{k+1}}_{l_k+1} = \{q_{l_k+1}, q_i, r_i\}^{l_{k+1}}_{l_k+2}$ are well-posed (of maximal-possible dimension) submatrices of the matrix C_2(2.7). Then from representation (2.5) ÷ (2.6) we get the following representation for the solution to the system $C_2X = Y$:

$$\begin{cases} x_i = \overset{\circ}{x}_i + \prod\limits_{\xi=i}^{l_k} (-r_{\xi+1} q_\xi^{-1})\gamma_k,\ k=n+1,n,\dots,1, \\ i = l_k, l_k-1,\dots,l_{k-1}+1,\ l_0=0,\ l_{n+1}=m, \end{cases} \qquad (2.10)$$

where

$$\begin{cases} \overset{\circ}{x}_i = q_i^{-1}(y_i + \sum\limits_{j=i+1}^{l_{k+1}-1} \prod\limits_{\xi=i+1}^{j} (-r_\xi q_\xi^{-1}) y_j,\ i = l_{k+1}, l_{k+1}-1,\dots,l_k+1, \\ k=n,\dots,1,0; \\ \left[\gamma_k \equiv x_{l_k+1}\right] = \overset{\circ}{x}_{l_k+1} + \prod\limits_{\xi=l_k+1}^{l_{k+1}} (-r_\xi q_{\xi-1}^{-1})[\gamma_{k+1} \equiv x_{l_{k+1}+1}], \\ \left[\gamma_{n+1} \equiv x_{m+1}\right] = 0,\ k=n,n-1,\dots,1. \end{cases} \qquad (2.11)$$

The graphic scheme corresponding to this representation, (2.10) ÷ (2.11), is of the form



(2.12)

So, the solution to the system $C_2X = Y$ is a sum of two parts: $X = \overset{\circ}{X} + \Delta X$, where $\overset{\circ}{X} = [\overset{\circ}{B}_2]Y$, and $\Delta X = [\Delta B_2 \equiv c \cdot b]Y$. Then, $B_2 = C_2^{-1}$ is of the form:

$$B_2 = \underbrace{\begin{bmatrix} \left[\overset{\circ}{B}{}_1^{l_1}\right] & & & & \\ & \left[\overset{\circ}{B}{}_{l_1+1}^{l_2}\right] & & & \\ & & \left[\overset{\circ}{B}{}_{l_2+1}^{l_3}\right] & & \\ & & & \cdots\cdots & \\ & & & & \left[\overset{\circ}{B}{}_{l_n+1}^{m}\right] \end{bmatrix}}_{\overset{\circ}{B}_2} + \underbrace{\begin{bmatrix} \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}\begin{matrix} c_1 \\ \vdots \\ c_{l_1} \end{matrix} & & & \\ & \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}\begin{matrix} c_{l_1+1} \\ \vdots \\ c_{l_2} \end{matrix} & & \\ & & \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}\begin{matrix} c_{l_2+1} \\ \vdots \\ c_{l_3} \end{matrix} & \\ & & \cdots\cdots & \\ & & & \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} \end{bmatrix}}_{c} \times$$

$$\times \underbrace{\begin{bmatrix} \begin{bmatrix} 0 \end{bmatrix} & & & \\ \begin{bmatrix} b^{l_2}_{l_1+1} \\ 0 \end{bmatrix} & [(c_{l_1}+1)b^{l_3}_{l_2+1}] \cdots & [(\prod_{\xi=1}^{n-1} c_{l_\xi+1})b^m_{l_n+1}] & \\ 0 & \begin{bmatrix} b^{l_3}_{l_2+1} \\ 0 \end{bmatrix} \cdots & [(\prod_{\xi=2}^{n-1} c_{l_\xi+1})b^m_{l_n+1}] & \\ & \cdots\cdots & 0 & \\ & & & \begin{bmatrix} b^m_{l_n+1} \\ 0 \end{bmatrix} \end{bmatrix}}_{b}, \tag{2.13}$$

where

$\overset{\circ}{B}{}^{\nu}_{\rho} = [C^{\nu}_{\rho}]^{-1}$ are submatrices inverse of C^{ν}_{ρ} и $c_i = \prod_{\xi=i+1}^{l_k+1}(-r_\xi q_{\xi-1}^{-1})$,

$i = l_k, l_k - 1, ..., l_{k-1}+1,\ k = n, ..., 2, 1,\ l_0 = 0,\ l_{n+1} = m$;

$\prod_{\xi=i}^{j} c_{l_\xi+1} = c_{l_i} c_{l_{i+1}} \cdots c_{l_j}$; $b^j_i = [\overset{\circ}{B}_{ii}, \overset{\circ}{B}_{ii+1}, ..., \overset{\circ}{B}_{ij}]$.

Remark 1. As follows from (2.12) ÷ (2.13), representation (2.10) ÷ (2.11) may be called (like in [2]) the direct representation of critical components of the solution of the system of linear equations $C_2X = Y$. The reason is that any components x_i of the solution to the system $C_2X = Y$ are nonrecurrent functions of well-posed components ($\gamma_k \equiv x_{l_k+1}$). Ill-posed components x_{l_k} are determined separately and do not participate in recurrence processes of deriving x_i, any components of the solution X. The components x_{l_k} may, therefore, be called the critical components. This method of solution of the system of equations $C_2X = Y$ follows from representation (2.10) ÷ (2.11) and belongs in essence to the class of direct methods of the decomposition type whose generators have been constructed in refs. [3,4].

3. Algorithms of solution of systems $C_2X = Y$ and $AX = Y$

In this section, we present solution algorithms for systems $C_2X = Y$ and $AX = Y$, based on the representations, given in sect.2, for solutions of systems $C_2X = Y$ and $C_3X = Y$.

Algorithm I (solution of the system $C_2X = Y$):

Input data: $\{q_1 \neq 0, q_i \neq 0, r_i; y_1, y_i\}_{i=2}^{m}$ are elements of the matrix C_2 and the right–hand side of the system of equations $C_2X = Y$, respectively; m is the dimension of the matrix C_2; ε is the relative error of computer arithmetic.

The beginning of computations.

Assignments:

$$k = 0; \; i = m.$$

Computations:

$$x_m = y_m/q_m,$$

$$i = i - 1. \tag{3.1}$$

Computation of well–posed x_i, separation and computation of critical components x_{l_k} of the solution X.

$$x_i = (y_i - r_{i+1}x_{i+1})/q_i.$$

If $i > 1$, the algorithm is continued from (3.2), and if $i \leq 1$,, the computation stops.

Assignment:

$$F_i = |y_i| \text{ and } f_i = 1, \text{ if } |y_i| \leq 1, \tag{3.2}$$

or

$$F_i = 1 \text{ and } f_i = |y_i|, \text{ if } |y_i| > 1.$$

Check of the condition of "extended discrepancy":

$$|F_i - |q_ix_i + r_{i+1}x_{i+1}|/f_i| < 2\varepsilon.$$

If the condition is valid, one returns to (3.1), otherwise, computations are to be made:

$$k = k + 1, \; l_k = i; \tag{3.3}$$

$$\overset{\circ}{x}_{l_k} = y_{l_k}, c_{l_k} = -r_{l_k+1};$$

$$x_i = (\overset{\circ}{x}_i + c_i x_{l_k+1})/q_i. \tag{3.4}$$

If $i > 1$, the algorithm is continued from (3.5), and if $i \leq 1$,, then computations are completed.

Computations:

$$i = i - 1, \tag{3.5}$$

$$c_i = (-r_{i+1}/q_i)c_{i+1}.$$

Check of the condition of well–posedness) of submatrices C_ρ^ν:*

$$\varepsilon < |c_i| < \varepsilon^{-1}. \tag{3.6}$$

If this condition is not fulfilled, then one goes back to (3.3), otherwise, one performs computations

$$\overset{\circ}{x}_i = y_i - (r_{i+1}/q_{i+1})\, \overset{\circ}{x}_{i+1}$$

and comes back to (3.4).

The end of computations.

Algorithm II (solution of the system $AX = Y$):

Case 1. $A = A^T$ is a symmetric matrix.

The matrix A and vector from the r.h.s. Y are scaled according to the algorithms of programs MSCL from ref.[5].

The scaled matrix A is represented in the form

$$Q^T AQ = (C_3 = C_3^T)$$

with the use of the reflection algorithm (programs BTD from ref. [5]).

Formation of the vector $L = Q^TY$.

Solution of the system $C_3U = L$ with the help of the algorithm *II* from ref.[1].

Derivation of the solution $X = QU$.

The end of computations.

Case 2. $A \neq A^T$ is a nonsymmetric matrix.

Process of scaling of the matrix A and the vector in the r.h.s. Y in accordance with the algorithms of programs-MSCL from ref. [5].

Representation of the scaled matrix A in the form

$$PAQ = C_2$$

with the use of the reflection algorithm(programs BTD from ref. [5]).

Formation of the vector $L = PY$.

Solution of the system $C_2U = L$ by using the algorithm *I*.

Derivation of the solution $X = QU$.

The end of calculations.

*) In essence, the condition (3.6) implies $\varepsilon < |\prod(r_{\xi+1}/q_\xi)| < \varepsilon^{-1}$.

4. Description of the solution programs of systems of linear equations $C_2X = Y$, $C_3X = Y$ and $AX = Y$

In this section, we describe the program DCSOL on Fortran - 77, constructed on the basis of algorithms I and II given in sect. 3 and algorithms from ref. [1]. In Fig.1, we draw the block-scheme of that program.

Call to the subprogram DCSOL:

CALL DCSOL(M,A,INF,IDIM,B,R)

Here:

M – (integer) the dimension of the quadratic matrix A;

A – (real*8) is a two-dimensional array*), that contains the initial matrix A at the input; whereas P и Q at the output [5]. Matrices, for instance A, are stored in the form

$$A = (a_{11}, a_{21}, ..., a_{M1}; a_{12}, a_{22}, ..., a_{M2}; a_{13}, ..., a_{MM}); \qquad (*)$$

INF – (integer)

At the input (the value of INF is assigned prior to the call to DCSOL):

INF= 0, if one solves the system with a two-diagonal matrix;

INF= 1, if one solves the system with a symmetric tridiagonal matrix $C_3 = C_3^T$;

INF= 2, if one solves (without scaling) the system with a filled matrix $A \neq A^T$;

INF= −2, if the system with $A \neq A^T$ is solved with scaling;

INF= 3, if one solves (without scaling) the system with filled matrix $A = A^T$;

INF= −3, if the system with $A = A^T$ is solved with scaling.

At the output:

INF= 0, the normal completion of the work of subprograms;

INF= −1, the initial matrix is singlet;

IDIM – (integer) the number of right-hand sides given in B;

*)Dimension of the array A: M×M, if at the input |INF| > 1; M×3, if at the input INF=1; 1 × 1(or A is a real variable), if at the input INF=0.

B – (real*8) a two-dimensional array of dimension M×IDIM(can be one-dimensional if IDIM=1) containing:

at the input, the matrix of r.h.s. in the form (*);

at the output, the matrix of solutions in the form (*);

R – (real*8) one-dimensional working array of the dimension 2M. At the input, it contains (at INF=0 and INF=1) two-diagonal or, respectively, symmetric tridiagonal matrices in the following form

$$R = [0, q_1, r_2, q_2, r_3, ..., q_{M-2}, r_{M-1}, q_{M-1}, r_M, q_M].$$

Remark 2. The program DCSOL uses the subprograms*) INIT, MSCL, GTEXP, ČHEXP [5] and the block COMMON Q1, DET. Upon completing the work of programs in Q1 one determines the relative error of computer arithmetic**) $\varepsilon = \beta^t$ and at DET – the value of the determinant of matrix A. The subprogram***) INIT defines constants of the IBM real arithmetic; MSCL realizes the scaling decomposition of a real matrix and multiplication of the real matrix by the degree of base of computer real numbers; GTEXP selects mantissas and exponents of a real number; CHEXP restores real numbers from a mantissa and exponent.

If DCSOL is already included into the library of a given IBM, the subprograms INIT, GTEXP and CHEXP will also be included there.

At the same time, if DCSOL is only being included into the library of a particular IBM, then it is necessary to choose a specific version of those subprograms [5] in accordance with the type of a given IBM.

*)The subprograms from [5] can be also employed independently.

**)The value of ε : $\varepsilon = 16^{-13}$ – at IBM series EC; $\varepsilon = 2^{-55}$ – at IBM type CM-4; $\varepsilon = 2^{-52}$ – at IBM PC.

***)A detailed description of these programs can be found in [5].

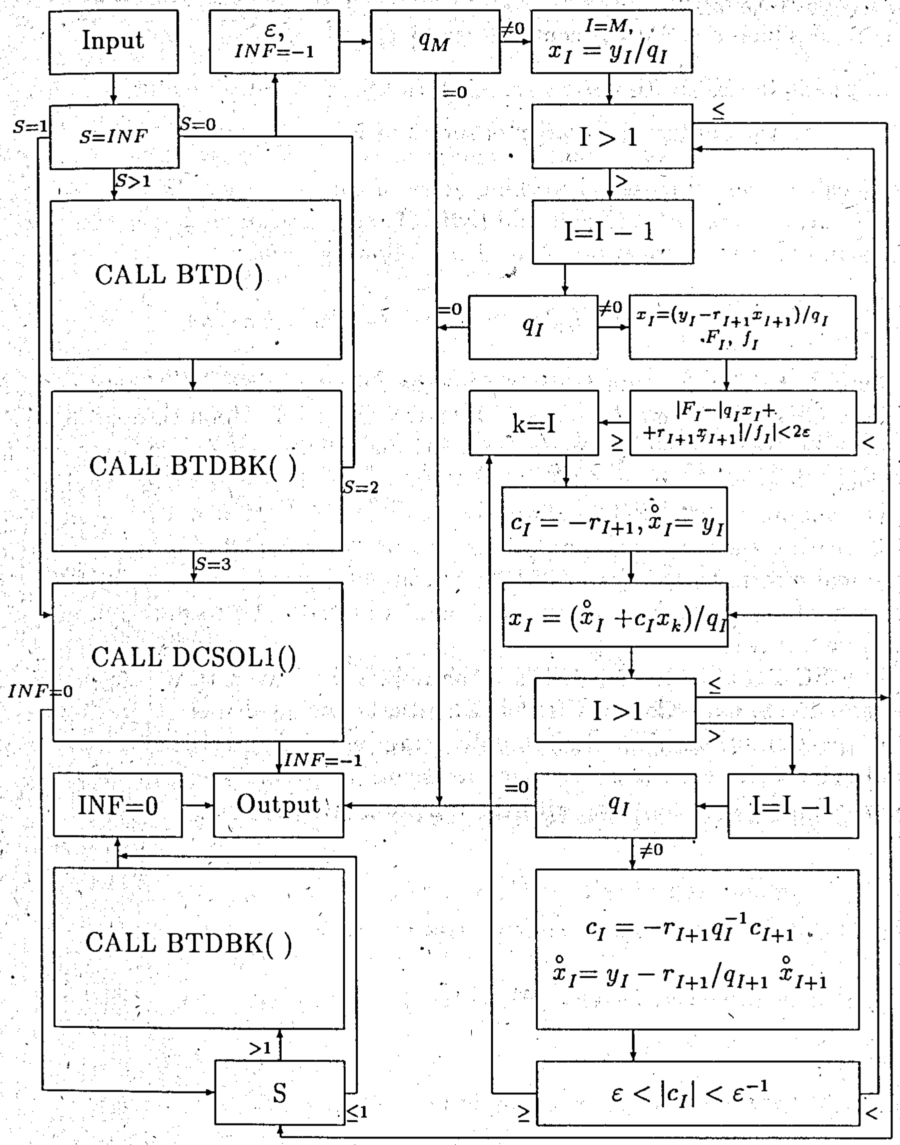


Fig.1: Block-scheme of the program DCSOL of algorithms I and II.

5. Results of numerical experiments and their analysis

In this sect., we report and discuss results of numerical experiments on testing the above programs of solution of the systems of linear equations $C_2X = Y$, $C_3X = Y$ and $AX = Y$ for various characteristic examples taken, in particular, from [5,6,7]. To start with, we explain the notation and abbreviations adopted in Tables 1 ÷ 6:

- M – the dimension of matrices C_2, C_3 and A;
- Cond(A) – the condition number of the matrix A;
- D1=$\|Y-A\tilde{X}\|_E$ – the Euclidean norm of discrepancy;
- D2= $\left|\|X\|_E-\|\tilde{X}\|_E\right|$ – the difference modulus of Euclidean norms of the exact X and approximate solution $\tilde{X}$;
- D1+D2 – the sum of discrepancy norm and difference modulus of norms of X and $\tilde{X}$;
- $\|\tilde{X}\|_E$ – the Euclidean norm of $\tilde{X}$.

Notation for subprograms in Tables (OS and OU – our subprograms):

OS – subprogram of DCSOL(at $A = A^T$);

OU – subprogram of DCSOL(for $A \neq A^T$);

GS – subprogram of DBEQN in Tables 1,2 and subprogram of DEQN in Tables 3 ÷ 6 from CERNLIB library;

QR – subprograms of F01AXF and F04AHF from NAGLIB library;

ML – subprogram–function of PSOL from LINA [5];

TH – subprogram of SLAY from LIBJINR library

Here: CERNLIB – the library of subprograms at CERN [8]; NAGLIB – the package of mathematical programs (Numerical Algorithms Group, Oxford) [9]; LIBJINR –the JINR library of programs [10]; LINA – the program package [5].

In our subprograms of DCSOL we realized algorithms suggested in the given series of papers.

The subprograms of DBEQN and DEQN are based on modified algorithms of the Gauss exclusion method.

In subprograms of F01AXF and F04AHF, algorithms of the QR –method are employed.

The subprogram of PSOL uses the algorithms of the method of singular expansion with the use of exhaustion.

The subprogram of SLAY realizes algorithms of the Tikhonov regularization method.

Note also that the Euclidean norms $\|\cdot\|_E$ were calculated with our subprograms TOCHS3 and TOCHSM. Computation time was found by using the subprograms of TIMEST and TIMEX from CERNLIB. When calculating Cond(A), we used the subprogram of PSOL from the package LINA. When Cond(A)> $1/\varepsilon$, where ε is the relative accuracy of computer arithmetic, in Tables, [5] Cond(A)= 0 is listed.

In Tables, no values for Comtime above 100 sec. are given.

The subprogram of SLAY worked up to M=100.

Below we report: in Tables 1,2 – the results of solution of the systems $C_2X = Y$; in Table 3 – the same for the systems $AX = Y$, with $A \neq A^T$; in Tables 4,6 – the same for the systems $AX = Y$, with $A = A^T$ and $A \neq A^T$. Note here that our subprograms OS and OU are working with symmetric $(A = A^T)$ and $(A = A^T)$ asymmetric matrices, respectively.

All the programs indicated above have been tested for the specific examples cited below. Examples 1,3; 8,9,10; 12,14 are taken from [5,6,7]. They are usually used for testing the programs of solution of the systems $C_2X = Y, AX = Y$. As further examples, we also propose testing examples 2,4,5; 6,7; 11,13,15. In tables $1 \div 6$, we will, for comparison, present the results of calculation on the basis of all the programs given above for the most ill–posed system of equations.

Solutions of the systems of equations $AX = Y$ and $C_2X = Y$ derived by our methods obey [3] the conditions: $\min\limits_X \|WX - Y\|$ and $\min\limits_X \|X\|$, where $W:$ A or C_2.

I. Testing examples of the systems of equations $C_2X = Y$ with two–diagonal matrices of the general form:

System 1

$$C_2 = \begin{bmatrix} 1\ 2 & & & & \\ & 1\ 2 & & & \\ & & \cdots & & \\ & & & 1\ 2 & \\ & & & & 1 \end{bmatrix}, \quad \begin{aligned} &x_i = 1/i,\ i = 1,2,...,M, \\ &y_i = \tfrac{3i+1}{i(i+1)},\ i = 1,2,...,M-1, \\ &y_M = 1/M. \end{aligned}$$

System 2

$$C_2 = \begin{bmatrix} \varepsilon_0\ r & & & & \\ & \varepsilon_0\ r & & & \\ & & \cdots & & \\ & & & \varepsilon_0\ r & \\ & & & & \varepsilon_0 \end{bmatrix}, \quad \begin{aligned} &x_i = 1/(2i+1),\ i = 1,2,...,M, \\ &y_i = \tfrac{2i+3\varepsilon_0}{(2i+\varepsilon_0)(2i+\varepsilon_0+2)},\ i = 1,2,...,M-1, \\ &y_M = \varepsilon_0/(2M+\varepsilon_0),\ \text{где } r = 1-\varepsilon_0,\ \varepsilon_0 = 0,01. \end{aligned}$$

System 3

$$C_2 = \begin{bmatrix} \frac{7}{5}\ \frac{11}{3} & & & & \\ & \frac{7}{5}\ \frac{11}{3} & & & \\ & & \cdots & & \\ & & & \frac{7}{5}\ \frac{11}{3} & \\ & & & & \frac{7}{5} \end{bmatrix}, \quad \begin{aligned} &x_i = 1/(2i+1),\ i = 1,2,...,M, \\ &y_i = \tfrac{152i+118}{15(2i+1)(2i+3)},\ i = 1,2,...,M-1, \\ &y_M = 1/(2M+1). \end{aligned}$$

System 4

$$C_2 = \begin{bmatrix} \varepsilon_0\ 2 & & & & & & & & \\ -1\ 2 & & & & & & & & \\ & \cdots & & & & & & & \\ & & -1\ 2 & & & & & & \\ & & & \varepsilon_1\ 2 & & & & & \\ & & & & -1\ 2 & & & & \\ & & & & & \cdots & & & \\ & & & & & & -1\ 2 & & \\ & & & & & & & \varepsilon_1 & \end{bmatrix}, \quad \begin{aligned} &x_i = (-1)^{i+1}a,\ i = 1,2,...,M, \\ &y_1 = (\varepsilon_0 - 2)a,\ y_i = (-1)^{i+2}a, \\ &i = 2,3,...,k-1,k+1,...,M-1, \\ &y_k = (-1)^k(2-\varepsilon_1)a,\ y_M = (-1)^{M+1}\varepsilon_1 a, \\ &\text{where } a = 1+\varepsilon_0,\ \varepsilon_0 = 0,0000001,\ \varepsilon_1 = 0,0001. \end{aligned}$$

System 5

$$C_2 = \begin{bmatrix} 3\ 7 & & & & \\ & 3\ 7 & & & \\ & & \cdots & & \\ & & & 3\ 7 & \\ & & & & 3 \end{bmatrix}, \quad \begin{aligned} &x_i = 1,\ i = 1,2,...,M, \\ &y_i = 10,\ i = 1,2,...,M-1, \\ &y_M = 3. \end{aligned}$$

Table 1(Results of computation of system 3)

	No pr.	Com.time (sec.)	D1= $\|\|Y - C\tilde{X}\|\|$	D2= $\|\|X\|\| - \|\|\tilde{X}\|\|$	D2+D1	$\|\|\tilde{X}\|\|$
M=10	OU	0.0005	0.227E-15	0.209E-13	0.211E-13	0.459E00
Cond(C)	GS	0.0007	0.227E-15	0.209E-13	0.211E-13	0.459E00
=.243E05	QR	0.0035	0.485E-15	0.394E-14	0.443E-14	0.459E00
	ML	0.1165	0.227E-15	0.209E-13	0.211E-13	0.459E00
	TH	0.0238	0.134E01	0.150E03	0.151E03	0.150E03
M=15	OU	0.0007	0.229E-15	0.852E-12	0.852E-12	0.467E00
Cond(C)	GS	0.0008	0.229E-15	0.852E-12	0.852E-12	0.467E00
=.301E07	QR	0.0070	0.466E-15	0.129E-10	0.129E-10	0.467E00
	ML	0.2579	0.229E-15	0.852E-12	0.852E-12	0.467E00
	TH	0.0620	0.138E01	0.127E05	0.127E05	0.127E05
M=20	OU	0.0008	0.230E-15	0.435E-10	0.435E-10	0.471E00
Cond(C)	GS	0.0010	0.230E-15	0.435E-10	0.435E-10	0.471E00
=.372E09	QR	0.0125	0.885E-15	0.515E-09	0.515E-09	0.471E00
	ML	0.4336	0.230E-15	0.435E-10	0.435E-10	0.471E00
	TH	0.1316	0.141E01	0.118E07	0.118E07	0.118E07
M=25	OU	0.0009	0.234E-15	0.232E-07	0.232E-07	0.473E00
Cond(C)	GS	0.0012	0.234E-15	0.232E-07	0.232E-07	0.473E00
=.459E11	QR	0.0202	0.486E-15	0.114E-06	0.114E-06	0.473E00
	ML	0.6873	0.234E-15	0.232E-07	0.232E-07	0.473E00
	TH	0.2375	0.142E01	0.118E09	0.118E09	0.118E09
M=30	OU	0.0010	0.238E-15	0.130E-05	0.130E-05	0.475E00
Cond(C)	GS	0.0013	0.238E-15	0.130E-05	0.130E-05	0.475E00
=.566E13	QR	0.0309	0.652E-15	0.292E-04	0.292E-04	0.475E00
	ML	0.9776	0.238E-15	0.130E-05	0.130E-05	0.475E00
	TH	0.4065	0.143E01	0.121E11	0.121E11	0.121E11
M=35	OU	0.0012	0.237E-15	0.140E-03	0.140E-03	0.476E00
Cond(C)	GS	0.0016	0.237E-15	0.140E-03	0.140E-03	0.476E00
=.692E15	QR	0.0448	0.608E-15	0.377E-02	0.377E-02	0.472E00
	ML	1.2884	0.237E-15	0.140E-03	0.140E-03	0.476E00
	TH	0.6046	0.144E01	0.129E13	0.129E13	0.129E13
M=40	OU	0.0013	0.242E-15	0.121E-01	0.121E-01	0.465E00
Cond(C)	GS	0.0017	0.242E-15	0.121E-01	0.121E-01	0.465E00
=.000E00	QR	0.0630	0.148E-14	0.601E00	0.601E00	0.108E01
	ML	1.6622	0.622E01	0.144E01	0.766E01	0.191E01
	TH	0.8813	0.143E01	0.149E15	0.149E15	0.149E15
M=45	OU	0.0014	0.459E-15	0.173E01	0.173E01	0.221E01
Cond(C)	GS	0.0019	0.567E-13	0.492E02	0.492E02	0.496E02
=.000E00	QR	0.0845	0.116E-13	0.190E02	0.190E02	0.194E02
	ML	2.1625	0.625E01	0.144E01	0.769E01	0.192E01
	TH	1.2821	0.329E02	0.103E18	0.103E18	0.103E18

Table 2 (Results of computation of system 4)

	No pr.	Com.time (sec.)	D1= $\|\|Y - C\tilde{X}\|\|$	D2= $\|\|X\|\| - \|\|\tilde{X}\|\|$	D2+D1	$\|\|\tilde{X}\|\|$
M=5	OU	0.0004	0.203E-19	0.157E-03	0.157E-03	0.224E01
Cond(C)	GS	0.0005	0.203E-19	0.157E-03	0.157E-03	0.224E01
=.000E00	QR	0.0015	0.188E-14	0.789E-03	0.789E-03	0.224E01
	ML	0.0290	0.162E02	0.286E01	0.191E02	0.510E01
	TH	0.0056	0.714E01	0.160E17	0.160E17	0.160E17
M=7	OU	0.0005	0.203E-19	0.540E-03	0.540E-03	0.265E01
Cond(C)	GS	0.0006	0.203E-19	0.540E-03	0.540E-03	0.265E01
=.000E00	QR	0.0021	0.200E-14	0.600E-03	0.600E-03	0.265E01
	ML	0.0541	0.235E02	0.399E01	0.275E02	0.663E01
	TH	0.0114	0.911E01	0.640E17	0.640E17	0.640E17
M=10	OU	0.0006	0.203E-19	0.362E-02	0.362E-02	0.317E01
Cond(C)	GS	0.0007	0.203E-19	0.362E-02	0.362E-02	0.317E01
=.000E00	QR	0.0035	0.261E-14	0.298E-02	0.298E-02	0.316E01
	ML	0.1263	0.314E02	0.526E01	0.366E02	0.843E01
	TH	0.0235	0.114E02	0.512E18	0.512E18	0.512E18
M=12	OU	0.0006	0.203E-19	0.134E-01	0.134E-01	0.348E01
Cond(C)	GS	0.0008	0.203E-19	0.134E-01	0.134E-01	0.348E01
=.000E00	QR	0.0048	0.326E-14	0.120E-01	0.120E-01	0.345E01
	ML	0.1609	0.357E02	0.597E01	0.416E02	0.943E01
	TH	0.0364	0.128E02	0.205E19	0.205E19	0.205E19
M=13	OU	0.0007	0.203E-19	0.241E-01	0.241E-01	0.358E01
Cond(C)	GS	0.0008	0.203E-19	0.241E-01	0.241E-01	0.358E01
=.000E00	QR	0.0055	0.343E-14	0.365E-01	0.365E-01	0.361E01
	ML	0.1656	0.376E02	0.629E01	0.439E02	0.990E01
	TH	0.0429	0.134E02	0.410E19	0.410E19	0.410E19
M=15	OU	0.0007	0.203E-19	0.776E-01	0.776E-01	0.380E01
Cond(C)	GS	0.0009	0.203E-19	0.776E-01	0.776E-01	0.380E01
=.000E00	QR	0.0070	0.381E-14	0.142E-01	0.142E-01	0.386E01
	ML	0.2159	0.413E02	0.690E01	0.482E02	0.108E02
	TH	0.0626	0.145E02	0.164E20	0.164E20	0.164E20
M=17	OU	0.0007	0.222E-15	0.973E-01	0.973E-01	0.403E01
Cond(C)	GS	0.0009	0.222E-15	0.973E-01	0.973E-01	0.403E01
=.000E00	QR	0.0090	0.457E-14	0.202E01	0.202E01	0.614E01
	ML	0.3070	0.446E02	0.745E01	0.521E02	0.116E02
	TH	0.0848	0.156E02	0.655E20	0.655E20	0.655E20
M=18	OU	0.0007	0.203E-19	0.144E01	0.144E01	0.568E01
Cond(C)	GS	0.0010	0.203E-19	0.144E01	0.144E01	0.568E01
=.000E00	QR	0.0098	0.447E-14	0.667E00	0.667E00	0.491E01
	ML	0.3621	0.462E02	0.772E01	0.539E02	0.120E02
	TH	0.0977	0.161E02	0.131E21	0.131E21	0.131E21

From the general analysis of the results of testing (in particular listed in Table 1 and 2) it follows that our program OU (DCSOL) possesses better characteristics of time and accuracy.

II. Testing examples of the systems of equations $AX = Y$ with $A \neq A^T$ – filled matrices of the general form:

System 6

$$A = \begin{bmatrix} M & M-1 & M-2 & \cdots & 3 & 2 & 333 \\ M-1 & M-1 & M-2 & \cdots & 3 & 2 & 1 \\ M-2 & M-2 & M-2 & \cdots & 3 & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 3 & 3 & 3 & \cdots & 3 & 2 & 1 \\ 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ \varepsilon_0 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix},$$

$x_i = 1/i, \; i = 1, 2, ..., M,$

$y_1 = \sum_{k=1}^{M-1} \frac{M-k+1}{k} + \frac{333}{M}, \; y_M = \sum_{k=2}^{M} \frac{1}{k} + \varepsilon_0,$

$y_i = (M-i+1) \sum_{k=1}^{i} \frac{1}{k} + \sum_{k=i+1}^{M} \frac{M-k+1}{k},$

$i = 2, 3, ..., M-1,$

where $\varepsilon_0 = 0,0000001.$

System 7

$A = (a_{ij}), \; a_{ij} = \frac{1}{i+j-1},$
$i = 1, 2, ..., M-1, \; j = 1, 2, ..., M,$
$a_{M1} = 333,$
$a_{M1} = \frac{1}{M+j-1}, \; j = 2, 3, ..., M,$

$x_i = 1/(2i+1), \; i = 1, 2, ..., M,$
$y_i = \sum_{k=1}^{M} \frac{1}{(2k+1)(i+k-1)},$
$i = 1, 2, ..., M-1,$
$y_M = \sum_{k=2}^{M} \frac{1}{(2k+1)(i+k-1)} + 111.$

System 8

$A = (a_{ij}), \; a_{1j} = a_{j1} = \frac{1}{M-j+1},$
$j = 1, 2, ..., M-1,$
$a_{1M} = 1 + \varepsilon_0, \; a_{M1} = 1 - \varepsilon_0,$
$a_{ij} = a_{ji} = \frac{1}{M-i+1},$
$i = 2, 3, ..., M, \; j = 2, 3, ..., i,$

$x_i = 1 - \varepsilon_0, \; i = 1, 2, ..., M,$
$y_1 = (1-\varepsilon_0)(\sum_{k=1}^{M-1} \frac{1}{M-k+1} + 1 + \varepsilon_0),$
$y_i = (1-\varepsilon_0)(\frac{i}{M-i+1} + \sum_{k=1}^{M} \frac{1}{M-k+1}),$
$i = 2, 3, ..., M-1,$
$y_M = (1-\varepsilon_0)(1 - \varepsilon_0 + \sum_{k=2}^{M} \frac{1}{M-k+1})$
where $\varepsilon_0 = 0,00001.$

System 9

$$A = \begin{bmatrix} M & M-1 & & & & \\ M-1 & M-1 & M-2 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ 3 & 3 & 3 & \cdots & 3 & 2 \\ 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix},$$

$x_i = (-1)^i/i, \; i = 1, 2, ..., M,$
$y_i = (1 + \frac{(-1)^i(i-M)}{i+1}) \sum_{k=1}^{i} \frac{(-1)^k}{k},$
$i = 1, 2, ..., M-1,$
$y_M = \sum_{k=1}^{M} (-1)^k/k.$

System 10

$A = (a_{ij}), \; a_{ij} = \frac{1}{i-j+M},$
$i = 1, 2, ..., M, \; j = 1, 2, ..., M,$

$x_i = 1/i, \; i = 1, 2, ..., M,$
$y_i = \sum_{k=1}^{M} \frac{1}{k(i-k+M)},$
$i = 1, 2, ..., M.$

Table 3(Results of computation of the system 8)

	No pr.	Com.time (sec.)	D1= $\|\|Y - C\tilde{X}\|\|$	D2= $\|\|\|X\|\| - \|\|\tilde{X}\|\|\|$	D2+D1	$\|\|\tilde{X}\|\|$
	OU	1.1885	0.821E-13	0.422E-14	0.863E-13	0.707E01
M=50	GS	0.1063	0.129E-12	0.115E-13	0.140E-12	0.707E01
Cond(A).	QR	0.1807	0.148E-12	0.955E-14	0.158E-12	0.707E01
=.956E05	ML	3.3575	0.778E-13	0.355E-14	0.813E-13	0.707E01
	TH	2.0613	0.768E-11	0.382E-13	0.771E-11	0.707E01
	OU	7.8580	0.297E-12	0.999E-14	0.307E-12	0.100E02
M=100	GS	0.7806	0.400E-12	0.213E-13	0.421E-12	0.100E02
Cond(A)	QR	1.3340	0.452E-12	0.155E-13	0.468E-12	0.100E02
=.563E06	ML	16.3126	0.291E-12	0.955E-14	0.301E-12	0.100E02
	TH	16.0278	0.514E-10	0.628E-12	0.521E-10	0.100E02
	OU	28.7184	0.477E-12	0.120E-13	0.489E-12	0.122E02
M=150	GS	2.6359	0.698E-12	0.269E-13	0.725E-12	0.122E02
Cond(A)	QR	4.5101	0.782E-12	0.200E-13	0.802E-12	0.122E02
=.157E07	ML	44.1513	0.465E-12	0.120E-13	0.477E-12	0.122E02
	TH					
	OU	63.9960	0.741E-12	0.113E-13	0.752E-12	0.141E02
M=200	GS	6.2067	0.108E-11	0.318E-13	0.111E-11	0.141E02
Cond(A)	QR	10.5183	0.109E-11	0.244E-13	0.112E-11	0.141E02
=.323E07	ML	90.4003	0.744E-12	0.118E-13	0.756E-12	0.141E02
	TH					
	OU		0.803E-12	0.349E-13	0.838E-12	0.158E02
M=250	GS	12.1114	0.145E-11	0.342E-13	0.148E-11	0.158E02
Cond(A)	QR	20.4187	0.220E-11	0.822E-13	0.228E-11	0.158E02
=.565E07	ML		0.796E-12	0.369E-13	0.833E-12	0.158E02
	TH					
	OU		0.101E-11	0.355E-12	0.156E-11	0.173E02
M=300	GS	21.1864	0.402E-11	0.128E-12	0.415E-11	0.173E02
Cond(A)	QR	35.4692	0.435E-11	0.135E-12	0.448E-11	0.173E02
=.891E07	ML		0.100E-11	0.355E-12	0.156E-11	0.173E02
	TH					
	OU		0.163E-10	0.465E-12	0.167E-10	0.200E02
M=400	GS	57.0168	0.960E-10	0.313E-12	0.964E-10	0.200E02
Cond(A)	QR	82.8388	0.981E-10	0.313E-12	0.985E-10	0.200E02
=.183E08	ML		0.162E-10	0.465E-12	0.167E-10	0.200E02
	TH					
	OU		0.232E-10	0.551E-12	0.237E-10	0.224E02
M=500	GS		0.154E-10	0.437E-12	0.158E-10	0.224E02
Cond(A)	QR		0.154E-10	0.387E-12	0.158E-10	0.224E02
=.318E08	ML		0.232E-10	0.554E-12	0.238E-10	0.224E02
	TH					

The analysis of results of testing given in Table 3 shows that our program OU (DCSOL) have certainly better characteristics of accuracy but is worse in time characteristics as compared to GS and QR. This is due to a large time comsumption for the reduction of an asymmetric matrix to a two–diagonal one. We hope to considerably reduce this drawback in further optimization of the procedure.

III. Testing examples for the systems of equations $AX = Y$ with $A = A^T$ – filled matrices of the general form:

System 11

$$A = (a_{ij}),\ a_{1j} = a_{j1} = \frac{1}{M-j+1},\quad j = 1,2,...,M-1,\quad a_{M1} = a_{1M} = M + \varepsilon_0,\quad a_{ij} = a_{ji} = \frac{1}{M-i+1},\quad i = 2,3,...,M,\ j = 2,3,...,i,\quad x_i = (i+1)/i,\ i = 1,2,...,M,$$

$$y_1 = \sum_{k=1}^{M-1} \frac{k+1}{k(M-k+1)} + \frac{(M+\varepsilon_0)(M+1)}{M},\quad y_i = \frac{1}{M-i+1}\sum_{k=1}^{i} \frac{k+1}{k} + \sum_{k=i+1}^{M} \frac{k+1}{k(M-k+1)},\quad y_M = \sum_{k=2}^{M} \frac{k+1}{k} + 2(M+\varepsilon_0),\quad i = 1,2,...,M-1, \text{ where } \varepsilon_0 = 10^{-7}.$$

System 12(A – the Hilbert matrix)

$$A = (a_{ij}),\ a_{ij} = \frac{1}{i+j-1},\quad i = 1,2,...,M,\ j = 1,2,...,M,\qquad x_i = 1/i,\ i = 1,2,...,M,\quad y_i = \sum_{k=1}^{M} \frac{1}{k(i+k-1)},\ i = 1,2,...,M.$$

System 13

$$A = (a_{ij}),\ a_{1j} = a_{j1} = \frac{1}{i+j-1},\quad j = 1,2,...,M-1,\quad a_{M1}=a_{1M}=333,\ a_{ij}=a_{ji}=\frac{1}{i+j-1},\quad i = 2,3,...,M,\ j = 2,3,...,i,\quad x_i = (-1)^i/i,\ i = 1,2,...,M,$$

$$y_1 = \sum_{k=1}^{M-1} \frac{(-1)^k}{k^2} + \frac{(-1)^M 333}{M},\quad y_{i-1} = \sum_{k=1}^{M} \frac{(-1)^k}{k(k+i-2)},\ i = 3,4,...,M,\quad y_M = \sum_{k=2}^{M} \frac{(-1)^k}{k(k+M-1)} + 333.$$

System 14

$$A = \begin{bmatrix} \varepsilon_0 & M-1 & M-2 & \cdots & 3 & 2 & M \\ M-1 & M-1 & M-2 & \cdots & 3 & 2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ M & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix},\qquad \begin{aligned} & x_i = 1-\varepsilon_0,\ i = 1,2,...,M, \\ & y_1 = \frac{(1-\varepsilon_0)(M^2-M+2\varepsilon_0)}{2}, \\ & y_i = \frac{(1-\varepsilon_0)(M-i)(i+M-3)}{2}, \\ & y_M = (2M-1)(1-\varepsilon_0), \\ & i = 2,3,...,M-1, \\ & \text{where } \varepsilon_0 = 0,0000001. \end{aligned}$$

System 15(det(A)=0)

$$A = \begin{bmatrix} a & b & b & \cdots & b & a \\ b & a & b & \cdots & b & b \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ b & b & b & \cdots & a & b \\ a & b & b & \cdots & b & a \end{bmatrix},\qquad \begin{aligned} & x_i = \frac{(-1)^i}{2i+1},\ i = 1,2,...,M, \\ & y_1 = y_M = b\sum_{k=2}^{M-1}\frac{(-1)^k}{2k+1} + a\left(\frac{(-1)^M}{2M+1} - \frac{1}{3}\right); \\ & y_i = b\sum_{k=1}^{M}\frac{(-1)^k}{2k+1} - \frac{(-1)^i a}{2i+1}, \\ & i = 2,3,...,M-1, \text{ where } a = 1-\varepsilon_0, \\ & b = 1+\varepsilon_0,\ \varepsilon_0 = 1\cdot 10^{-11}. \end{aligned}$$

Table 4(Computation results for system 12)

	No pr.	Com.time (sec.)	D1= $\|\|Y - C\tilde{X}\|\|$	D2= $\|\|X\|\| - \|\|\tilde{X}\|\|$	D2+D1	$\|\|\tilde{X}\|\|$
	OS	0.0073	0.916E-15	0.444E-15	0.136E-14	0.123E01
M=7	OU	0.0085	0.119E-14	0.111E-14	0.230E-14	0.123E01
	GS	0.0011	0.144E-14	0.111E-14	0.255E-14	0.123E01
Cond(A)	QR	0.0022	0.125E-14	0.111E-14	0.236E-14	0.123E01
=.475E09	ML	0.0600	0.119E-14	0.111E-14	0.230E-14	0.123E01
	TH	0.0126	0.195E-11	0.104E-06	0.104E-06	0.123E01
	OS	0.0095	0.196E-14	0.133E-14	0.330E-14	0.124E01
M=8	OU	0.0110	0.233E-14	0.600E-14	0.833E-14	0.124E01
	GS	0.0013	0.146E-14	0.888E-15	0.235E-14	0.124E01
Cond(A)	QR	0.0027	0.739E-15	0.666E-15	0.141E-14	0.124E01
=.153E11	ML	0.0754	0.233E-14	0.600E-14	0.833E-14	0.124E01
	TH	0.0166	0.242E-12	0.164E-05	0.164E-05	0.124E01
	OS	0.0130	0.158E-14	0.555E-11	0.555E-11	0.124E01
M=9	OU	0.0150	0.259E-14	0.906E-11	0.906E-11	0.124E01
	GS	0.0016	0.148E-14	0.199E-11	0.200E-11	0.124E01
Cond(A)	QR	0.0033	0.149E-14	0.115E-11	0.115E-11	0.124E01
=.493E12	ML	0.0884	0.259E-14	0.906E-11	0.906E-11	0.124E01
	TH	0.0212	0.145E-09	0.372E02	0.372E02	0.385E02
	OS	0.0157	0.199E-14	0.142E-09	0.142E-09	0.124E01
M=10	OU	0.0181	0.164E-14	0.938E-09	0.938E-09	0.124E01
	GS	0.0019	0.216E-14	0.992E-09	0.992E-09	0.124E01
Cond(A)	QR	0.0039	0.244E-14	0.603E-09	0.603E-09	0.124E01
=.160E14	ML	0.1140	0.164E-14	0.938E-09	0.938E-09	0.124E01
	TH	0.0273	0.306E-09	0.263E04	0.263E04	0.263E04
	OS	0.0201	0.159E-14	0.821E-07	0.821E-07	0.125E01
M=11	OU	0.0225	0.183E-14	0.483E-07	0.483E-07	0.125E01
	GS	0.0023	0.248E-14	0.201E-05	0.201E-05	0.125E01
Cond(A)	QR	0.0046	0.212E-14	0.304E-05	0.304E-05	0.125E01
=.518E15	ML	0.1322	0.184E-14	0.483E-07	0.483E-07	0.125E01
	TH	0.0342	0.465E-08	0.130E07	0.130E07	0.130E07
	OS	0.0250	0.168E-14	0.245E-03	0.245E-03	0.125E01
M=12	OU	0.0285	0.185E-14	0.237E-01	0.237E-01	0.127E01
	GS	0.0028	0.265E-14	0.105E-01	0.105E-01	0.126E01
Cond(A)	QR	0.0054	0.168E-14	0.179E-01	0.179E-01	0.127E01
=.000E00	ML	0.1381	0.175E01	0.954E00	0.271E01	0.220E01
	TH	0.0421	0.117E-07	0.630E08	0.630E08	0.630E08
	OS	0.0307	0.268E-14	0.346E01	0.346E01	0.471E01
M=13	OU	0.0336	0.320E-14	0.140E01	0.140E01	0.152E01
	GS	0.0033	0.277E-14	0.463E01	0.463E01	0.588E01
Cond(A)	QR	0.0062	0.237E-14	0.872E01	0.872E01	0.997E01
=.000E00	ML	0.1503	0.181E01	0.974E00	0.279E01	0.223E01
	TH	0.0508	0.123E-03	0.796E12	0.796E12	0.796E12

Table 5(Computation results for system 15)

	No pr.	Com.time (sec.)	D1= $\|Y - C\tilde{X}\|$	D2= $\|\|X\| - \|\tilde{X}\|\|$	D2+D1	$\|\tilde{X}\|$
	OS	0.0041	0.126E-15	0.549E-01	0.549E-01	0.493E00
M=5	OU	0.0044	0.821E-16	0.131E-01	0.131E-01	0.425E00
	GS	0.0007	0.547E-11	0.123E05	0.123E05	0.123E05
Cond(A)	QR	0.0015	0.352E-05	0.282E11	0.282E11	0.282E11
=.000E00	ML	0.0224	0.229E01	0.134E00	0.242E01	0.572E00
	TH	0.0067	0.963E00	0.631E16	0.631E16	0.631E16
	OS	0.0166	0.512E-15	0.217E00	0.217E00	0.676E00
M=10	OU	0.0183	0.107E-15	0.832E-01	0.832E-01	0.376E00
	GS	0.0018	0.202E-10	0.247E05	0.247E05	0.247E05
Cond(A)	QR	0.0038	0.302E-03	0.502E12	0.502E12	0.502E12
=.000E00	ML	0.0467	0.546E01	0.148E00	0.561E01	0.607E00
	TH	0.0271	0.220E01	0.372E16	0.372E16	0.372E16
	OS	0.0435	0.458E-15	0.264E-02	0.264E-02	0.470E00
M=15	OU	0.0467	0.384E-15	0.485E-01	0.485E-01	0.419E00
	GS	0.0043	0.432E-10	0.145E05	0.145E05	0.145E05
Cond(A)	QR	0.0083	0.346E-03	0.204E12	0.204E12	0.204E12
=.000E00	ML	0.0899	0.125E02	0.425E00	0.129E02	0.892E00
	TH	0.0734	0.466E01	0.282E16	0.282E16	0.282E16
	OS	0.0915	0.317E-15	0.424E-01	0.424E-01	0.429E00
M=20	OU	0.0984	0.125E-14	0.300E-01	0.300E-01	0.441E00
	GS	0.0087	0.538E-10	0.114E05	0.114E05	0.114E05
Cond(A)	QR	0.0183	0.792E-03	0.168E12	0.168E12	0.168E12
=.000E00	ML	0.1477	0.172E02	0.436E00	0.177E02	0.907E00
	TH	0.1578	0.678E01	0.409E16	0.409E16	0.409E16
	OS	0.1624	0.172E-14	0.251E00	0.251E00	0.725E00
M=25	OU	0.1731	0.188E-14	0.269E-02	0.269E-02	0.476E00
	GS	0.0151	0.106E-09	0.140E05	0.140E05	0.140E05
Cond(A)	QR	0.0281	0.605E-03	0.407E12	0.407E12	0.407E12
=.000E00	ML	0.2361	0.269E02	0.648E00	0.276E02	0.112E01
	TH	0.2882	0.946E02	0.189E17	0.189E17	0.189E17
	OS	0.2589	0.218E-14	0.322E00	0.322E00	0.797E00
M=30	OU	0.2707	0.230E-14	0.200E-01	0.200E-01	0.455E00
	GS	0.0246	0.130E-09	0.107E05	0.107E05	0.107E05
Cond(A)	QR	0.0456	0.102E-02	0.729E12	0.729E12	0.729E12
=.000E00	ML	0.3524	0.328E02	0.656E00	0.335E02	0.113E01
	TH	0.4747	0.592E02	0.726E16	0.726E16	0.726E16
	OS	0.4091	0.725E-14	0.633E00	0.633E00	0.111E01
M=35	OU	0.4310	0.771E-15	0.159E00	0.159E00	0.635E00
	GS	0.0381	0.181E-09	0.138E05	0.138E05	0.138E05
Cond(A)	QR	0.0689	0.134E-02	0.542E12	0.542E12	0.542E12
=.000E00	ML	0.5172	0.446E02	0.834E00	0.454E02	0.131E01
	TH	0.7407	0.116E02	0.155E16	0.155E16	0.155E16

Table 6(Computation results for system 15 with a perturbed(≈ 10%) right-hand side)

	No pr.	Com.time (sec.)	D1= $\|Y - C\tilde{X}\|$	D2= $\|\|X\| - \|\tilde{X}\|\|$	D2+D1	$\|\tilde{X}\|$
	OS	0.0029	0.747E-16	0.103E+00	0.103E+00	0.541E+00
M=5	OU	0.0035	0.196E-16	0.266E-01	0.266E-01	0.465E+00
	GS	0.0007	0.825E-11	0.208E+05	0.208E+05	0.208E+05
Cond(A)	QR	0.0015	0.493E-05	0.825E+11	0.825E+11	0.825E+11
=.000E00	ML	0.0205	0.252E+01	0.191E+00	0.271E+01	0.630E+00
	TH	0.0067	0.938E+01	0.194E+17	0.194E+17	0.194E+17
	OS	0.0126	0.549E-15	0.248E+00	0.248E+00	0.707E+00
M=10	OU	0.0147	0.343E-15	0.485E-01	0.485E-01	0.411E+00
	GS	0.0018	0.178E-10	0.255E+05	0.255E+05	0.255E+05
Cond(A)	QR	0.0038	0.113E-02	0.888E+12	0.888E+12	0.888E+12
=.000E00	ML	0.0480	0.601E+01	0.208E+00	0.622E+01	0.668E+00
	TH	0.0271	0.207E+06	0.266E+22	0.266E+22	0.266E+22
	OS	0.0355	0.285E-14	0.226E+01	0.226E+01	0.273E+01
M=15	OU	0.0398	0.645E-15	0.617E-02	0.617E-02	0.461E+00
	GS	0.0043	0.370E-10	0.197E+05	0.197E+05	0.197E+05
Cond(A)	QR	0.0085	0.422E-03	0.195E+12	0.195E+12	0.195E+12
=.000E00	ML	0.0908	0.137E+02	0.514E+00	0.142E+02	0.981E+00
	TH	0.0739	0.524E+06	0.274E+21	0.274E+21	0.274E+21
	OS	0.0810	0.465E-15	0.339E-01	0.339E-01	0.437E+00
M=20	OU	0.0891	0.566E-15	0.155E+00	0.155E+00	0.626E+00
	GS	0.0086	0.581E-10	0.212E+05	0.212E+05	0.212E+05
Cond(A)	QR	0.0163	0.694E-03	0.209E+12	0.209E+12	0.209E+12
=.000E00	ML	0.1493	0.189E+02	0.526E+00	0.195E+02	0.997E+00
	TH	0.1551	0.784E+03	0.115E+18	0.115E+18	0.115E+18
	OS	0.1414	0.231E-14	0.361E+00	0.361E+00	0.835E+00
M=25	OU	0.1538	0.697E-15	0.953E-01	0.953E-01	0.569E+00
	GS	0.0151	0.787E-10	0.100E+05	0.100E+05	0.100E+05
Cond(A)	QR	0.0280	0.650E-03	0.436E+12	0.436E+12	0.436E+12
=.000E00	ML	0.2333	0.296E+02	0.760E+00	0.304E+02	0.123E+01
	TH	0.2909	0.316E+01	0.699E+15	0.699E+15	0.699E+15
	OS	0.2297	0.309E-14	0.413E+00	0.413E+00	0.889E+00
M=30	OU	0.2459	0.101E-14	0.137E+00	0.137E+00	0.612E+00
	GS	0.0250	0.116E-09	0.201E+05	0.201E+05	0.201E+05
Cond(A)	QR	0.0455	0.891E-03	0.816E+12	0.816E+12	0.816E+12
=.000E00	ML	0.3497	0.361E+02	0.769E+00	0.369E+02	0.124E+01
	TH	0.4876	0.125E+03	0.161E+17	0.161E+17	0.161E+17
	OS	0.3529	0.683E-14	0.743E+00	0.743E+00	0.122E+01
M=35	OU	0.3719	0.139E-15	0.248E+00	0.248E+00	0.724E+00
	GS	0.0391	0.162E-09	0.143E+05	0.143E+05	0.143E+05
Cond(A)	QR	0.0688	0.144E-02	0.698E+12	0.698E+12	0.698E+12
=.000E00	ML	0.4981	0.490E+02	0.966E+00	0.500E+02	0.144E+01
	TH	0.7508	0.119E+02	0.167E+16	0.167E+16	0.167E+16

From the general analysis of the results of testing (in particular, listed in Tables 4-6) it follows that our programs OS and OU (DCSOL) provide better characteristics of accuracy though are not in time as good as the programs GS and QR. This is again due to the procedure of reduction of $A = A^T$ to C_3 in the program OS and to C_2in the program OU. We hope to diminish this drawback via further optimization of the programs.

Finally, it is to be noted that for $A = A^T$, out of programs OS and OU, the program OS has better characteristics, which uses the procedure of reduction of $(A = A^T)$ to $(C_3 = C_3^T)$, and in the program OU,$(A = A^T)$ is reduced to C_2.

In Table 6, we report the results of numerical calculations for solution of the perturbed ($\approx 10\%$ in Y) system $AX = Y$. As follows from the analysis of testing results, our methods are the most stable also with respect to perturbation of the right-hand side of systems of equations.

6. Conclusion

We have developed the new effective method and algorithm for solving systems of linear algebraic equations with a two-diagonal matrix. The programs constructed on their basis surpass all the known modern programs of a similar type in basic characteristics.

The new algorithms have also been employed for solving systems of linear algebraic equations with a completely filled square matrix.

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