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CRITICAL-COMPONENT METHOD FOR SOLVING SYSTEMS OF LINEAR EQUATIONS WITH A TRIDIAGONAL MATRIX OF THE GENERAL FORM

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1. Introduction

The present paper is in essence a connecting link between series of our theoretical and numerical studies [1–6] and final works [10, 11] where we have given the description and characterization of a new library of programs on solving the problems of linear algebra.

Here we suggest new effective methods for inverting tridiagonal matrices of the general form and solving the systems of equations with matrices of that sort.

2. On structures of tridiagonal matrices and their inverses

In this subsection, we derive new types of representations describing the structure of tridiagonal matrices and their inverses of the general form.

Let C be a nonsingular real tridiagonal matrix of the general form

$$C = \begin{bmatrix} q_1 & r_2 & & & \\ p_2 & q_2 & r_3 & & \\ & \ddots & \ddots & \ddots & \\ & & p_{m-1} & q_{m-1} & r_m \\ & & & p_m & q_m \end{bmatrix} \equiv \begin{bmatrix} \left[C_1^{l_1-2}\right] & r_{l_1-1} & & & & & \\ p_{l_1-1} & \begin{bmatrix} q_{l_1-1} & r_{l_1} \\ p_{l_1} & q_{l_1} \end{bmatrix} & r_{l_1+1} & & & & \\ & p_{l_1+1} & \left[C_{l_1+1}^{l_2-2}\right] & r_{l_2-1} & & & \\ & & & \cdots\cdots & & & \\ & & & p_{l_n-1} & \begin{bmatrix} q_{l_n-1} & r_{l_n} \\ p_{l_n} & q_{l_n} \end{bmatrix} & r_{l_n+1} & \\ & & & & p_{l_n+1} & \left[C_{l_n+1}^{m}\right] & \end{bmatrix}, \quad (2.1)$$

where

$$C_{l_k+1}^{l_{k+1}-2} = \begin{bmatrix} q_{l_k+1} & r_{l_k+2} & & & \\ p_{l_k+2} & q_{l_k+2} & r_{l_k+3} & & \\ & \ddots & \ddots & \ddots & \\ & & p_{l_{k+1}-3} & q_{l_{k+1}-3} & r_{l_{k+1}-2} \\ & & & p_{l_{k+1}-2} & q_{l_{k+1}-2} \end{bmatrix}, k = 0, 1, ..., n, l_0 = 0, l_{n+1} - 2 = m, \quad (2.2)$$

$\{p_i \neq 0\}_{i=2}^m$ are subdiagonal elements, $\{r_i \neq 0\}_{i=2}^m$ are off-diagonal ones, $\{q_i\}_{i=1}^m$ are diagonal ones and $C_{l_k+1}^{l_{k+1}-2} = tridiag\{q_{l_k+1}, q_i, p_i, r_i\}_{i=l_k+2}^{l_{k+1}-2}$ are submatrices of matrices $C(2.1)$, respectively.

The right-hand representation in (2.1) is a block decomposition of the tridiagonal matrix C (2.1) in which it is assumed that C_ρ^ν are well-posed submatrices. This block decomposition of the matrix C is necessary for the construction of effective direct numerical methods of the decomposition type of solving the algebraic problems with matrices C (2.1). These problems have previously been considered in [3, 4] where generators*) have been proposed for deriving various representations of $B=C_{-1}$, matrices inverse of C (2.1) and for solving the systems of linear equations $CX=Y$. Thus, for $B=C_{-1}$ we have [3]

Representation 2.1 (of the type $B=\mathring{B}+B_1Z$)

$$B=\underbrace{\begin{bmatrix} [\tilde B_1^{l_1-2}] &&&&&&& \\ & \begin{bmatrix}0&\\&0\end{bmatrix} &&&&&& \\ && [\tilde B_{l_1+1}^{l_2-2}] &&&&& \\ &&& \begin{bmatrix}0&\\&0\end{bmatrix} &&&& \\ &&&& [\tilde B_{l_2+1}^{l_3-2}] &&& \\ &&&&& \cdots && \\ &&&&&& \begin{bmatrix}0&\\&0\end{bmatrix} & \\ &&&&&&& [\tilde B_{l_n+1}^{m}]\end{bmatrix}}_{\mathring{B}}+$$

$$+\underbrace{\begin{bmatrix} \begin{bmatrix}(\tilde B_{1l_1-2}r_{l_1-1}) & 0\\ \vdots & \vdots\\ (\tilde B_{l_1-2l_1-2}r_{l_1-1}) & 0\end{bmatrix} & & & \\ \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} & & & \\ \begin{bmatrix}0 & (\tilde B_{l_1+1l_1+1}p_{l_1+1})\\ \vdots & \vdots\\ 0 & (\tilde B_{l_2-2l_1+1}p_{l_1+1})\end{bmatrix} & \begin{bmatrix}(\tilde B_{l_1+1l_2-2}r_{l_2-1}) & 0\\ \vdots & \vdots\\ (\tilde B_{l_2-2l_2-2}r_{l_2-1}) & 0\end{bmatrix} & & \\ & \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} & & \\ & \begin{bmatrix}0 & (\tilde B_{l_2+1l_2+1}p_{l_2+1})\\ \vdots & \vdots\\ 0 & (\tilde B_{l_3-2l_2+1}p_{l_2+1})\end{bmatrix} & \cdots & \begin{bmatrix}(\tilde B_{l_{n-1}+1l_n-2}r_{l_n-1}) & 0\\ \vdots & \vdots\\ (\tilde B_{l_n-2l_n-2}r_{l_n-1}) & 0\end{bmatrix} \\ & & & \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} \\ & & & \begin{bmatrix}0 & (\tilde B_{l_n+1l_n+1}p_{l_n+1})\\ \vdots & \vdots\\ 0 & (\tilde B_{ml_n+1}p_{l_n+1})\end{bmatrix}\end{bmatrix}}_{B_1}\cdot Z, \qquad (2.3)$$

*)In these papers, results have been obtained for block–tridiagonal matrices of the general form C(2.1).

where $\mathring{B}$ is a block–diagonal matrix of the dimension $m\times m$, B_1 is a rectangular matrix of the dimension $m\times 2n$, $\tilde B_{ij}$ are elements*) of submatrices $\tilde B_\rho^\nu=[C_\rho^\nu]^{-1}$, where $[C_\rho^\nu]^{-1}$ is a matrix inverse of C_ρ^ν. It is seen that B_1 contains only the elements of last (first) columns of matrices $\tilde B_\rho^\nu$. The rectangular matrix Z of the dimension $2n\times m$ is a solution to the matrix system of equations

$$\Omega Z=B_2, \qquad (2.4)$$

where Ω is a nonsingular [3] tridiagonal matrix of the dimension $2n\times 2n$ and B_2 is a rectangular matrix of the dimension $2n\times m$, whose general form looks as follows:

$$\Omega=\begin{bmatrix}[Q_{l_1}] & [R_{l_2}] & & \\ [P_{l_2}] & [Q_{l_2}] & [R_{l_3}] & \\ & \cdots\cdots & & \\ & & [P_{l_n}] & [Q_{l_n}]\end{bmatrix}, \text{where } \begin{aligned} Q_{l_i} &= \begin{bmatrix}(q_{l_i-1}-p_{l_i-1}\tilde B_{l_i-2l_i-2}r_{l_i-1}) & (r_{l_i})\\ (p_{l_i}) & (q_{l_i}-r_{l_i+1}\tilde B_{l_i+1l_i+1}p_{l_i+1})\end{bmatrix},\\ R_{l_i} &= \begin{bmatrix}0 & 0\\ (-r_{l_{i-1}+1}\tilde B_{l_{i-1}+1l_i-2}r_{l_i-1}) & 0\end{bmatrix},\\ P_{l_i} &= \begin{bmatrix}0 & (-p_{l_i-1}\tilde B_{l_i-2l_{i-1}+1}p_{l_{i-1}+1})\\ 0 & 0\end{bmatrix},\end{aligned} \qquad (2.5)$$

$$B_2=\begin{bmatrix}[b_{l_1l_1-2}]\begin{bmatrix}-1&0\\0&-1\end{bmatrix}[b_{l_1l_2-2}] & & \\ & [b_{l_2l_2-2}]\begin{bmatrix}-1&0\\0&-1\end{bmatrix}[b_{l_2l_3-2}] & \\ & & \cdots\cdots\cdots \\ & & [b_{l_nl_n-2}]\begin{bmatrix}-1&0\\0&-1\end{bmatrix}[b_{l_nm}]\end{bmatrix}, \qquad (2.6)$$

где

$$\begin{aligned} b_{l_il_i-2} &= \begin{bmatrix}(p_{l_i-1}\tilde B_{l_i-2l_{i-1}+1})\cdots(p_{l_i-1}\tilde B_{l_i-2l_i-2})\\ 0 \quad \cdots \quad 0\end{bmatrix},\\ b_{l_il_{i+1}-2} &= \begin{bmatrix}0 \quad \cdots \quad 0\\ (r_{l_i+1}\tilde B_{l_i+1l_i+1})\cdots(r_{l_i+1}\tilde B_{l_i+1l_{i+1}-2})\end{bmatrix}, \quad i=1,2,\ldots n.\end{aligned} \qquad (2.7)$$

In (2.3) ÷ (2.7), the elements $(\tilde B_\rho^\nu)_{ij}$ possess [1,2,9] (when C_ρ^ν have no zeroth minors**)) the following***) representations:

Representation 2.2

$$\tilde B_{ij}(\Lambda)=\begin{cases}\tilde B_{ij+1}^{(\Lambda)}\beta_{j+1}, & l_k+1\le j<i\le l_{k+1}-2,\\ c_{i+1}\tilde B_{i+1j}^{(\Lambda)}, & l_k+1\le i<j\le l_{k+1}-2, k=0,1,\ldots,n,\end{cases} \qquad (2.8)$$

*)Hereafter, B-matrix elements will be denoted by B_{ij}, in contrast to $\tilde B_{ij}$ elements of submatrices $\tilde B_\rho^\nu=[C_\rho^\nu]^{-1}$.

**)This restriction is not principal and will be removed in what follows.

***)Note that $(\tilde B_\rho^\nu)_{ij}$ –elements can be calculated by any other algebraic method [3]. In the representations 2.2 ÷ 2.7, we use, for simplicity, the notation $\tilde B_{ij}$ instead of $(\tilde B_\rho^\nu)_{ij}$.

where $\tilde{B}_{ii}(\Lambda) = \Lambda_{i+1}^{-1} + c_{i+1}\tilde{B}^{(\Lambda)}_{i+1i+1}\beta_{i+1}, \tilde{B}^{(\Lambda)}_{l_{k+1}l_{k+1}} = \Lambda^{-1}_{l_{k+1}-1}, \; i = l_{k+1}-3, ..., l_k+1;$

Representation 2.3

$$\tilde{B}_{ij}(\mathrm{G}) = \begin{cases} \hat{c}_i \tilde{B}^{(\mathrm{G})}_{i-1j}, \; l_k+1 \le j < i \le l_{k+1}-2, \\ \tilde{B}^{(\mathrm{G})}_{ij-1}\hat{\beta}_j, \; l_k+1 \le i < j \le l_{k+1}-2, \; k = 0, 1, ..., n, \end{cases} \tag{2.9}$$

where $\tilde{B}_{ii}(\mathrm{G}) = \mathrm{G}_{i-1}^{-1} + \hat{c}_i \tilde{B}^{(\mathrm{G})}_{i-1i-1}\hat{\beta}_i, \tilde{B}^{(\mathrm{G})}_{l_k+1l_k+1} = \mathrm{G}^{-1}_{l_k}, \; i = l_k+2, ..., l_{k+1}-2;$

Representation 2.4

$$\tilde{B}_{ij}(\Lambda, \mathrm{G}) = \begin{cases} \tilde{B}_{ii}(\Lambda, \mathrm{G}) \prod\limits_{\xi=j+1}^{i} \beta_\xi, \; l_k+1 \le j < i \le l_{k+1}-2, \\ \prod\limits_{\xi=i+1}^{j} c_\xi \tilde{B}_{jj}(\Lambda, \mathrm{G}), \; l_k+1 \le i < j \le l_{k+1}-2, \\ k = 0, 1, ..., n; \end{cases} \tag{2.10}$$

Representation 2.5

$$\tilde{B}_{ij}(\Lambda, \mathrm{G}) = \begin{cases} \prod\limits_{\xi=j+1}^{i} \hat{c}_\xi \tilde{B}_{jj}(\Lambda, \mathrm{G}), \; l_k+1 \le j < i \le l_{k+1}-2, \\ \tilde{B}_{ii}(\Lambda, \mathrm{G}) \prod\limits_{\xi=i+1}^{j} \hat{\beta}_\xi, \; l_k+1 \le i < j \le l_{k+1}-2, \\ k = 0, 1, ..., n; \end{cases} \tag{2.11}$$

Representation 2.6

$$\tilde{B}_{ij}(\Lambda, \mathrm{G}) = \begin{cases} \tilde{B}_{ii}(\Lambda, \mathrm{G}) \prod\limits_{\xi=j+1}^{i} \beta_\xi, \; l_k+1 \le j < i \le l_{k+1}-2, \\ \tilde{B}_{ii}(\Lambda, \mathrm{G}) \prod\limits_{\xi=i+1}^{j} \hat{\beta}_\xi, \; l_k+1 \le i < j \le l_{k+1}-2, \\ k = 0, 1, ..., n; \end{cases} \tag{2.12}$$

Representation 2.7

$$\tilde{B}_{ij}(\Lambda, \mathrm{G}) = \begin{cases} \prod\limits_{\xi=j+1}^{i} \hat{c}_\xi \tilde{B}_{jj}(\Lambda, \mathrm{G}), \; l_k+1 \le j < i \le l_{k+1}-2, \\ \prod\limits_{\xi=i+1}^{j} c_\xi \tilde{B}_{jj}(\Lambda, \mathrm{G}), \; l_k+1 \le i < j \le l_{k+1}-2, \\ k = 0, 1, ..., n. \end{cases} \tag{2.13}$$

Remark 1. In the above representations we took advantage of the notation $\tilde{B}_{ij}(\Lambda), \tilde{B}_{ij}(\mathrm{G})$ and $\tilde{B}_{ij}(\Lambda, \mathrm{G})$ for elements $\tilde{B}_{ij}$ in order to emphasize which sequences, $\{\Lambda\}, \{\mathrm{G}\}$ or $\{\Lambda, \mathrm{G}\}$, make the basis for computing the elements $\tilde{B}_{ij}$.

In (2.10) ÷ (2.13), $\tilde{B}_{ii}$, diagonal elements of the submatrices $\tilde{B}^\nu_\rho = [C^\nu_\rho]^{-1}$ are defined in the following manner:

$$\begin{cases} [\tilde{B}_{ii} = \tilde{B}_{ii}(\Lambda, \mathrm{G})] = (\Lambda_{i+1} + \mathrm{G}_{i-1} - q_i)^{-1}, \; i = l_k+1, ..., l_{k+1}-2, \\ \tilde{B}_{l_k+1l_k+1} = \mathrm{G}^{-1}_{l_k}, \; \tilde{B}_{l_{k+1}-2l_{k+1}-2} = \Lambda^{-1}_{l_{k+1}-1}, \; k = 0, 1, ..., n, \end{cases} \tag{2.14}$$

where elements of the sequences $\{\Lambda_i\}$ and $\{\mathrm{G}_i\}$ are as follows:

$$\begin{cases} \Lambda_{i+1} = q_i - p_i\Lambda_i^{-1}r_i, \Lambda_{l_k+2} = q_{l_k+1} \text{ and} \\ l_k+2 \le i \le l_{k+1}-2. \\ \text{Here } \beta_i = -p_i\Lambda_i^{-1}, c_i = -\Lambda_i^{-1}r_i \\ \text{for all } i = l_k+2, ..., l_{k+1}-2, \; k = 0, 1, ..., n; \end{cases} \tag{2.15}$$

$$\begin{cases} \mathrm{G}_{i-1} = q_i - r_{i+1}\mathrm{G}_i^{-1}p_{i+1}, \mathrm{G}_{l_{k+1}-3} = q_{l_{k+1}-2} \text{ and} \\ l_k+1 \le i \le l_{k+1}-3. \\ \text{Here } \hat{\beta}_{i+1} = -r_{i+1}\mathrm{G}_i^{-1}, \hat{c}_{i+1} = -\mathrm{G}_i^{-1}p_{i+1} \\ \text{for all } i = l_{k+1}-3, ..., l_k+1, \; k = n, n-1, ..., 0. \end{cases} \tag{2.16}$$

As it is seen from (2.15) ÷ (2.16), these processes are determined when C^ν_ρ has no zeroth minors. In ref. [3] we introduced the generalized sequences $\{\Lambda_i\}$ and $\{\mathrm{G}_i\}$ in the case when the whole matrix $C(2.1)$ and its submatrices C^ν_ρ contain some zeroth leading upper (lower) angular minors*):

$$\begin{cases} \Lambda_{i+1} = q_i - p_i\Lambda_i^{-1}r_i, \; \Lambda_2 = q_1, \; i = 2, ..., m, \text{ when } \Lambda_i \ne 0 \\ \text{for all } 2 \le i \le m. \\ \text{If } \Lambda_i = 0 \text{ for any } i \text{ from } (2 \le i \le m), \text{then} \\ \Lambda_{i+1} \text{is nondefined, but} \Lambda_{i+2} = q_{i+1}; \end{cases} \tag{2.17}$$

$$\begin{cases} \mathrm{G}_{i-1} = q_i - r_{i+1}\mathrm{G}_i^{-1}p_{i+1}, \; \mathrm{G}_{m-1} = q_m, \; i = m-1, ..., 1, \\ \text{when } \mathrm{G}_i \ne 0 \text{ for all } 1 \le i \le m-1. \\ \text{If } \mathrm{G}_i = 0 \text{ for any } i \text{ from } (1 \le i \le m-1), \text{then} \\ \mathrm{G}_{i-1} \text{is nondefined, but } \mathrm{G}_{i-2} = q_{i-1}. \end{cases} \tag{2.18}$$

Remark 2. If in (2.17), (2.18) $q_1 = 0 (q_{i+1} = 0)$ or $q_m = 0 (q_{i-1} = 0)$, then we first choose a new block decomposition [2,7] for the initial matrix $C(2.1)$. Really:

1. If $[\Delta^1_1 \ne 0]$ (i.e. $[\Lambda_2 = q_1] \ne 0$), but if for any fixed i from the interval $3 \le i \le m-1$ there holds the inequality

$$[\Lambda_i = 0], \text{ i.e. } [\Delta_1^{i-1} = 0] \text{ provided that } q_{i+1} = 0, \tag{2.19}$$

*)By leading upper $\{\Delta^i_1\}^m_{i=1}$ and lower $\{\Delta^m_i\}^m_{i=1}$ angular minors of matrix $C(2.1)$ we mean the determinants if its submatrices starting from q_1 and q_m respectively.

then one should perform a new decomposition of the initial*) of matrix $C(2.1)$:

$$\bar{q}_j = \begin{cases} q_j, \text{ when } 1 \le j \le i \text{ and } i+3 \le j \le m, \\ \begin{bmatrix} q_{i+1} & r_{i+2} \\ p_{i+2} & q_{i+2} \end{bmatrix}, \text{ when } j = i+1; \end{cases}$$
$$\bar{r}_j = \begin{cases} r_j, \text{ when } 2 \le j \le i \text{ and } i+4 \le j \le m, \\ [r_{i+1},\ 0], \text{ when } j = i+1, \\ [0,\ r_{i+3}]^T, \text{ when } j = i+2; \end{cases} \tag{2.20}$$
$$\bar{p}_j = \begin{cases} p_j, \text{ when } 2 \le j \le i \text{ and } i+4 \le j \le m, \\ [p_{i+1},\ 0]^T, \text{ when } j = i+1, \\ [0,\ p_{i+3}], \text{ when } j = i+2. \end{cases}$$

For $[\Delta_1^1 = 0]$ (i.e. $[\Lambda_2 = q_1] = 0$), the initial decomposition looks as follows:

$$\begin{cases} \bar{q}_1 = \begin{bmatrix} q_1 & r_2 \\ p_2 & q_2 \end{bmatrix}, \ \bar{r}_2 = [0,\ r_3]^T, \ \bar{p}_2 = [0,\ p_3], \\ \bar{q}_j = q_{j+1}, \text{ when } 2 \le j \le m-1, \\ \bar{r}_j = r_{j+1}, \ \bar{p}_j = p_{j+1}, \text{ when } 3 \le j \le m-1. \end{cases} \tag{2.21}$$

This decomposition is employed again for analysing (2.19) and, if necessary, the new decomposition (2.20) is performed. In so doing, in accordance with Lemma 10 [2,7] and owing to the performed decomposition, the conditions $\det(\bar{q}_1) \ne 0$ in (2.21) and $\det(\bar{q}_{i+1}) \ne 0$ in (2.20) are always fulfilled.

2. If $[\Delta_m^m \ne 0]$ (i.e. $[G_{m-1} = q_m] \ne 0$), but if for any fixed i from the interval $3 \le i \le m-1$, valid is the inequality

$$[G_i = 0], \text{ i.e. } [\Delta_{i+1}^m = 0] \text{ and in this case } q_{i-1} = 0, \tag{2.22}$$

then one should make a new decomposition of the initial matrix $C(2.1)$:

$$\bar{q}_j = \begin{cases} q_j, \text{ when } 1 \le j \le i-3 \text{ and } i \le j \le m, \\ \begin{bmatrix} q_{i-2} & r_{i-1} \\ p_{i-1} & q_{i-1} \end{bmatrix}, \text{ when } j = i-1; \end{cases}$$
$$\bar{r}_j = \begin{cases} r_j, \text{ when } 2 \le j \le i-3 \text{ and } i+1 \le j \le m, \\ [0,\ r_i]^T, \text{ when } j = i, \\ [r_{i-2},\ 0], \text{ when } j = i-1; \end{cases} \tag{2.23}$$
$$\bar{p}_j = \begin{cases} p_j, \text{ when } 2 \le j \le i-3 \text{ and } i+1 \le j \le m, \\ [0,\ p_i], \text{ when } j = i, \\ [p_{i-2},\ 0]^T, \text{ when } j = i-1. \end{cases}$$

*)Hereafter T is the symbol of transposition.

In case $[\Delta_m^m = 0]$ (i.e. $[G_{m-1} = q_m] = 0$), the initial decomposition is of the form

$$\begin{cases} \bar{q}_m = \begin{bmatrix} q_{m-1} & r_m \\ p_m & q_m \end{bmatrix}, \ \bar{r}_m = [r_{m-1},\ 0], \ \bar{p}_m = [p_{m-1},\ 0]^T, \\ \bar{q}_j = q_{j-1}, \text{ when } 2 \le j \le m-1, \\ \bar{r}_j = r_{j+1}, \ \bar{p}_j = p_{j+1}, \text{ when } 3 \le j \le m-1. \end{cases} \tag{2.24}$$

With this decomposition we again analyze (2.22) and make a new decomposition (2.23), if necessary. In this case, according to Lemma 10 [2,7] and in view of the performed decomposition, there always hold the condition $\det(\bar{q}_m) \ne 0$ in (2.24) and $\det(\bar{q}_{i-1}) \ne 0$ in (2.23).

Finally, let us present a new type of representations*) for matrix $C(2.1)$, by making use of representations 2.3 and 2.4 from [5,8].

Representation 2.9($\Delta_1^{\xi-1} = 0$ for any ξ from $(2 \le \xi \le m)$)

$$C = \begin{bmatrix} 1 \\ -\beta_2 \ 1 \\ \ddots \\ -\beta_{k-2} \ 1 \\ -\beta_{k-1} \begin{bmatrix} 1 & \\ 0 & 1 \end{bmatrix} \\ -[\beta_{k+1}] \ 1 \\ -\beta_{k+2} \ 1 \\ \ddots \\ -\beta_{l-2} \ 1 \\ -\beta_{l-1} \begin{bmatrix} 1 & \\ 0 & 1 \end{bmatrix} \\ -[\beta_{l+1}] \ 1 \\ -\beta_{l+2} \ 1 \\ \ddots \\ -\beta_m \ 1 \end{bmatrix} \begin{bmatrix} \Lambda_2 \\ \Lambda_3 \\ \ddots \\ \Lambda_{k-1} \\ \begin{bmatrix} \Lambda_k & r_k \\ p_k & q_k \end{bmatrix} \\ \Lambda_{k+2} \\ \ddots \\ \Lambda_{l-1} \\ \begin{bmatrix} \Lambda_l & r_l \\ p_l & q_l \end{bmatrix} \\ \Lambda_{l+2} \\ \Lambda_{l+3} \\ \ddots \\ \Lambda_{m+1} \end{bmatrix} \times$$

*)Remark 3. These representations hold valid also for nonzero $\Delta_1^{\xi-1}$ and $\Delta_{\xi+1}^m$ minors, where $\xi = k, l$. In particular, they will be valid for small values of $\Lambda_k, \Lambda_l, G_l$ and G_k of an order of computer zero.

$$
\times\begin{bmatrix}
1 \ -c_2 & & & & & & & & \\
& \cdots & & & & & & & \\
& & 1 \ -c_{k-1} & & & & & & \\
& & & \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix}\begin{bmatrix} \\ -c_{k+1} \end{bmatrix} & & & & & \\
& & & & 1 \ -c_{k+2} & & & & \\
& & & & & \cdots & & & \\
& & & & & & 1 \ -c_{l-1} & & \\
& & & & & & & \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix}\begin{bmatrix} \\ -c_{l+1} \end{bmatrix} & \\
& & & & & & & & 1 \ -c_{l+2} \\
& & & & & & & & \cdots \\
& & & & & & & & 1 \ -c_m \\
& & & & & & & & 1
\end{bmatrix}, \tag{2.25}
$$

where

$$
\left\{
\begin{array}{l}
\beta_i = -p_i\Lambda_i^{-1}, \ c_i = -\Lambda_i^{-1} r_i, \\
i = 2, ..., k-1, k+2, ..., l-1, l+2, ..., m; \\
[\beta_{\xi+1}] = \frac{p_{\xi+1}}{d} \cdot [p_\xi, \ -\Lambda_\xi], \ [c_{\xi+1}] = [r_\xi, \ -\Lambda_\xi]^{\mathrm{T}} \cdot \frac{r_{\xi+1}}{d}, \\
\Lambda_{\xi+2} = q_{\xi+1} - p_{\xi+1}\frac{\Lambda_\xi}{d} r_{\xi+1}, \ d = \Lambda_\xi q_\xi - p_\xi r_\xi, \ \xi = k, l.
\end{array}
\right. \tag{2.26}
$$

Representation 2.10($\Delta_{\xi+1}^m = 0$ for any ξ from $(1 \le \xi \le m-1)$)

$$
C = \begin{bmatrix}
1 \ -\hat\beta_2 & & & & & & & & & & \\
& \cdots & & & & & & & & & \\
& & 1 \ -\hat\beta_{k-1} & & & & & & & & \\
& & 1 \ [-\hat\beta_k] & & & & & & & & \\
& & & \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix}_{-\hat\beta_{k+2}} & & & & & & & \\
& & & & 1 \ -\hat\beta_{k+3} & & & & & & \\
& & & & & \cdots & & & & & \\
& & & & & & 1 \ -\hat\beta_{l-1} & & & & \\
& & & & & & 1 \ [-\hat\beta_l] & & & & \\
& & & & & & & \begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix}_{-\hat\beta_{l+2}} & & & \\
& & & & & & & & 1 \ -\hat\beta_{l+3} & & \\
& & & & & & & & & \cdots & \\
& & & & & & & & & & 1 \ -\hat\beta_m \\
& & & & & & & & & & 1
\end{bmatrix}
\begin{bmatrix}
\mathrm{G}_0 & & & & & & & & & & \\
& \cdots & & & & & & & & & \\
& & \mathrm{G}_{k-3} & & & & & & & & \\
& & & \mathrm{G}_{k-2} & & & & & & & \\
& & & & \begin{bmatrix} q_k & r_{k+1} \\ p_{k+1} & \mathrm{G}_k \end{bmatrix} & & & & & & \\
& & & & & \mathrm{G}_{k+1} & & & & & \\
& & & & & & \cdots & & & & \\
& & & & & & & \mathrm{G}_{l-3} & & & \\
& & & & & & & & \mathrm{G}_{l-2} & & \\
& & & & & & & & & \begin{bmatrix} q_l & r_{l+1} \\ p_{l+1} & \mathrm{G}_l \end{bmatrix} & \\
& & & & & & & & & & \mathrm{G}_{l+1} \\
& & & & & & & & & & \cdots \\
& & & & & & & & & & \mathrm{G}_{m-2} \\
& & & & & & & & & & \mathrm{G}_{m-1}
\end{bmatrix} \times
$$

$$
\times\begin{bmatrix}
1 & & & & & & & & \\
-\hat c_2 \ 1 & & & & & & & & \\
& \cdots & & & & & & & \\
& & -\hat c_{k-1} \ 1 & & & & & & \\
& & & \begin{bmatrix} \\ -\hat c_k \end{bmatrix}\begin{bmatrix} 1 & \\ 0 & 1 \end{bmatrix} & & & & & \\
& & & & -\hat c_{k+2} \ 1 & & & & \\
& & & & & \cdots & & & \\
& & & & & & -\hat c_{l-1} \ 1 & & \\
& & & & & & & \begin{bmatrix} \\ -\hat c_l \end{bmatrix}\begin{bmatrix} 1 & \\ 0 & 1 \end{bmatrix} & \\
& & & & & & & & -\hat c_{l+2} \ 1 \\
& & & & & & & & \cdots \\
& & & & & & & & -\hat c_m \ 1
\end{bmatrix}, \tag{2.27}
$$

where

$$
\left\{
\begin{array}{l}
\hat\beta_{i+1} = -r_{i+1}\mathrm{G}_i^{-1}, \ \hat c_{i+1} = -\mathrm{G}_i^{-1} p_{i+1}, \\
i = m-1, ..., l+1, l-2, ..., k+1, k-2, ..., 1; \\
[\hat\beta_\xi] = \frac{r_\xi}{d} \cdot [-\mathrm{G}_\xi, \ r_{\xi+1}], \ [\hat c_\xi] = [-\mathrm{G}_\xi, \ p_{\xi+1}]^{\mathrm{T}} \cdot \frac{p_\xi}{d}, \\
\mathrm{G}_{\xi-2} = q_{\xi-1} - r_\xi \frac{\mathrm{G}_\xi}{d} p_\xi, d = \mathrm{G}_\xi q_\xi - r_\xi p_\xi, \ \xi = l, k.
\end{array}
\right. \tag{2.28}
$$

Remark 4. Since in what follows we shall use $(2.25) \div (2.28)$ as the basis for constructing new effective methods and algorithms of the solution of algebraic problems both with matrices $C(2.1)$ and with the completely filled matrices A of the general form, the following note is the point:

First, it is reasonable to call decompositions 2.9 and 2.10 the generalized LR– and RL–decompositions of matrix $C(2.1)$ as they are constructed on the generalized sequences $\{\Lambda_i\}$ (2.17) and $\{\mathrm{G}_i\}$ (2.18) which are in the way of computation invariant with respect to the representation of zero at a computer. Therefore, decompositions $(2.25) \div (2.28)$ are also invariant with respect to the order of small quantities $\Lambda_k, \Lambda_l, \mathrm{G}_l$ and G_k;

Second, an equivalent perturbation of the matrix C connected with its decomposition of the form $(2.25) \div (2.28)$ and round–off errors at a computer when solving the algebraic problems with matrices C (2.1) will further be caused only by the difference between theoretical 0 and computational 0.

In view of the above said, we will derive a number of results whose practical application does not distinguish between those zeros. The theorem presented below*) takes place both for the whole matrix C (2.1) and for any its submatrices C_ρ^ν of the same shape.

*)As it is seen from decompositions $(2.25) \div (2.28)$, upper (lower) angular minors of matrix $C(2.1)$ of any order can be easily obtained as products of diagonal elements and d– determinants of blocks of the corresponding matrices, cofactors.

Theorem 2.1. Let C be a nonsingular tridiagonal matrix of the form (2.1), whose some leading angular minors vanish (or turn into the computer zero), i.e. $\Delta_1^i = 0$ and /or $\Delta_j^m = 0$, for any i from $1 \le i < m$, and j from $1 < j \le m$. Then, off-diagonal elements B_{ij} of the inverse matrix $B = C^{-1}$ possess the following representations:

Representation 2.11 (of multiplicative type)

$$B_{ij} = \begin{cases} \omega_i \prod\limits_{\xi=j+1}^{i} \beta_\xi, \text{ if } 1 \le j < i \le m, \\ 0 \text{ for all } i \text{ from } j < i \le m, \text{ if } \Lambda_j = 0, \\ 0 \text{ for all } j \text{ from } 1 \le j < i, \text{ if } \mathrm{G}_i = 0, \\ \omega_i \prod\limits_{\xi=i+1}^{j} \hat{\beta}_\xi, \text{ if } 1 \le i < j \le m, \\ 0 \text{ for all } i \text{ from } 1 \le i < j, \text{ if} \mathrm{G}_j = 0, \\ 0 \text{ fora all } j \text{ from } i < j \le m, \text{ if } \Lambda_i = 0. \end{cases} \tag{2.29}$$

Representation 2.12(of multiplicative type)

$$B_{ij} = \begin{cases} \prod\limits_{\xi=j+1}^{i} \hat{c}_\xi \omega_j, \text{ if } 1 \le j < i \le m, \\ 0 \text{ for all } j \text{ from } 1 \le j < i, \text{ if } \mathrm{G}_i = 0, \\ 0 \text{ for all } i \text{ from } j < i \le m, \text{ if } \Lambda_j = 0, \\ \prod\limits_{\xi=i+1}^{j} c_\xi \omega_j, \text{ when } 1 \le i < j \le m, \\ 0, \text{ for all } j \text{ from } i < j \le m, \text{ when } \Lambda_i = 0, \\ 0, \text{ for all } i \text{ from } 1 \le i < j, \text{ when } \mathrm{G}_j = 0. \end{cases} \tag{2.30}$$

Diagonal elements B_{ii} of the matrices B and the quantities ω_i in (2.29) $\div$ (2.30) can be represented in the form:

$$\begin{cases} B_{ii} = (\Lambda_{i+1} + \mathrm{G}_{i-1} - q_i)^{-1}, \text{ when } \Lambda_i \ne 0 \ne \mathrm{G}_i \\ \text{and } \omega_i = B_{ii}; \\ B_{ii} = 0, B_{i-1i-1} = \mathrm{G}_{i-1}\omega_i; B_{i+1i+1} = \mathrm{G}_i^{-1}, \text{ when } \Lambda_i = 0 \\ \text{and } \omega_i = (-p_i r_i)^{-1}; \\ B_{ii} = 0, B_{i-1i-1} = \Lambda_i^{-1}, B_{i+1i+1} = \Lambda_{i+1}\omega_i, \text{ when } \mathrm{G}_i = 0 \\ \text{and } \omega_i = (-r_{i+1}p_{i+1})^{-1}, \\ i = 1, 2, ..., m. \end{cases} \tag{2.31}$$

In (2.29) $\div$ (2.31), the elements Λ and G are given by (2.17) $\div$ (2.18), with Remark 2 included into consideration. Structure elements β, $\hat{\beta}$, $c, \hat{c}$ and products $\prod \beta_\xi$, $\prod \hat{\beta}_\xi$, $\prod c_\xi$, $\prod \hat{c}_\xi$ in (2.29) $\div$ (2.30) are of the form:

$$\beta_i = \begin{cases} -p_i \Lambda_i^{-1}, \text{ when } \Lambda_i \ne 0, \\ -p_i, \text{ when } \Lambda_i = 0, \\ \text{and } \beta_{i+1} = -p_{i+1}\omega_i; \end{cases} \quad c_i = \begin{cases} -\Lambda_i^{-1} r_i, \text{ when } \Lambda_i \ne 0, \\ -r_i, \text{ when } \Lambda_i = 0, \\ \text{and } c_{i+1} = -\omega_i r_{i+1}; \end{cases} \tag{2.32}$$

$$\hat{c}_{i+1} = \begin{cases} -\mathrm{G}_i^{-1} p_{i+1}, \text{ if } \mathrm{G}_i \ne 0, \\ -p_{i+1}, \text{ if } \mathrm{G}_i = 0, \\ \text{and } \hat{c}_i = -\omega_i p_i; \end{cases} \quad \hat{\beta}_{i+1} = \begin{cases} -r_{i+1}\mathrm{G}_i^{-1}, \text{ when } \mathrm{G}_i \ne 0, \\ -r_{i+1}, \text{ when } \mathrm{G}_i = 0, \\ \text{and } \hat{\beta}_i = -r_i\omega_i; \end{cases} \tag{2.33}$$

$$\prod_{\xi=j+1}^{i} \beta_\xi = \begin{cases} \beta_i \cdots \beta_{j+1}, \text{ when } j < i, \\ 1, \quad \text{when} j \ge i; \end{cases} \quad \prod_{\xi=i+1}^{j} c_\xi = \begin{cases} c_{i+1} \cdots c_j, \text{ when } i < j, \\ 1, \quad \text{when} i \ge j; \end{cases} \tag{2.34}$$

$$\prod_{\xi=i+1}^{j} \hat{\beta}_\xi = \begin{cases} \hat{\beta}_{i+1} \cdots \hat{\beta}_j, \text{ when } i < j, \\ 1, \quad \text{when} i \ge j; \end{cases} \quad \prod_{\xi=j+1}^{i} \hat{c}_\xi = \begin{cases} \hat{c}_i \cdots \hat{c}_{j+1}, \text{ when} j < i, \\ 1, \quad \text{when} j \ge i. \end{cases} \tag{2.35}$$

Proof. Representations (2.11) and (2.12) were derived with the use of results (specifically, on the structure of the inverse matrix B) of theorems теорем 7 and 8 from [1,2] and the combination [8] of representations 2.9 and 2.10 for C. B [1,2], the following sequences constructed on C–matrix elements

$$\begin{cases} \alpha_{i+1} = q_i\alpha_i - p_i r_i \alpha_{i-1}, \ i = 1, 2, ..., m, \ \alpha_0 = \alpha_1 = 1, \\ \beta_{i-1} = q_i\beta_i - r_{i+1}p_{i+1}\beta_{i+1}, \ i = m, ..., 2, 1, \ \beta_{m+1} = \beta_m = 1, \end{cases} \tag{2.36}$$

were considered. In those refs., it was shown that the elements B_{ij} of the matrix $B = C_-$ obeyed the following relations:

- if $\alpha_i = 0$,, then $B_{ii-1} = r_i^{-1}$ and $B_{ij} \overset{j \ge i}{=} 0, B_{ki} \overset{k \ge i}{=} 0$;
- if $\beta_i = 0$,,then $B_{ii+1} = p_{i+1}^{-1}$ and $B_{ij} \overset{j \le i}{=} 0, B_{ki} \overset{k \le i}{=} 0$;
- when $\alpha_i = 0$, то $B_{ii} = 0$ but here B_{i+1i+1} and B_{i-1i-1} do not vanish simultaneously;
- when $\beta_i = 0$,, then $B_{ii} = 0$ but B_{i-1i-1} and B_{i+1i+1} do not vanish simultaneously;
- when $\alpha_i = 0 = \beta_i$,, then $\det(C) = 0$.

From (2.36) it follows*) that

$$\begin{cases} \left[\frac{\alpha_{i+1}}{\alpha_{i-1}} = (-p_i r_i)\right] \equiv \omega_i^{-1}, \text{ when } \alpha_i = 0; \\ \left[\frac{\beta_{i-1}}{\beta_{i+1}} = (-r_{i+1}p_{i+1})\right] \equiv \omega_i^{-1}, \text{ when } \beta_i = 0. \end{cases} \tag{2.37}$$

Here we introduced the notation ω_k^{-1} for the corresponding products $(-p_k r_k)$ and $(-r_{k+1}p_{k+1})$.

*)For nonsingular matrices C(2.1), if $\alpha_i = 0$, then $\alpha_{i-1} \ne 0$ and if $\beta_i = 0$, then $\beta_{i+1} \ne 0$(see Lemma 10 from [2,7]).

If now we go over from the sequences $\{\alpha\}(2.36)_{1)}$ and $\{\beta\}(2.36)_{2)}$ to sequences $\{\Lambda\}(2.17)$ and $\{G\}$ (2.18), then having eqs. (2.37) at hand, we obtain the structure elements β, $\hat{\beta}$, $c, \hat{c}$ and products $\prod \beta_\xi$, $\prod \hat{\beta}_\xi$, $\prod c_\xi$, $\prod \hat{c}_\xi$ of the form (2.32) $\div$ (2.35). The validity of expressions (2.29) $\div$ (2.35) is established via verifying*) basic equalities основных равенств $BC = E = CB$. The theorem is proved.

Thus, matrices $B = C^{-1}$ can be obtained with the use of either multiplicative representations 2.11 and 2.12, or additive–multiplicative representation 2.1 provided that $\tilde{B}^{\nu}_{\rho}$ is obtained with the help of representation 2.11 or 2.12. In this case, the matrix Z in (2.3) $\div$ (2.4) has an analytic representation to be derived in a subsequent paper [10].

Now, let us apply to the solution of the system of linear algebraic equations $CX = Y$. Representation 2.1 combined with representation 2.11(or 2.12) gives [4] the following

Representation 2.13(of the type $X = \overset{\circ}{X} + B_1 \Delta X$)

$$\underbrace{\begin{bmatrix} \left[X_1^{l_1-2}\right] \\ \begin{bmatrix} x_{l_1-1} \\ x_{l_1} \end{bmatrix} \\ \left[X_{l_1+1}^{l_2-2}\right] \\ \begin{bmatrix} x_{l_2-1} \\ x_{l_2} \end{bmatrix} \\ \left[X_{l_2+1}^{l_3-2}\right] \\ \cdots\cdots \\ \begin{bmatrix} x_{l_n-1} \\ x_{l_n} \end{bmatrix} \\ \left[X_{l_n+1}^{m}\right] \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} \left[\tilde{B}_1^{l_1-2}\right] \times \begin{bmatrix} y_1 \\ \vdots \\ y_{l_1-2} \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left[\tilde{B}_{l_1+1}^{l_2-2}\right] \times \begin{bmatrix} y_{l_1+1} \\ \vdots \\ y_{l_2-2} \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left[\tilde{B}_{l_2+1}^{l_3-2}\right] \times \begin{bmatrix} y_{l_2+1} \\ \vdots \\ y_{l_3-2} \end{bmatrix} \\ \cdots\cdots\cdots\cdots \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \left[\tilde{B}_{l_n+1}^{m}\right] \times \begin{bmatrix} y_{l_n+1} \\ \vdots \\ y_m \end{bmatrix} \end{bmatrix}}_{\overset{\circ}{X}} +$$

*)Hereafter we do not dwell upon separate details of the calculational technique and verification of the given representations.

$$+ \underbrace{\begin{bmatrix} \begin{bmatrix} (\tilde{B}_{1l_1-2}r_{l_1-1}) & 0 \\ \vdots & \vdots \\ (\tilde{B}_{l_1-2l_1-2}r_{l_1-1}) & 0 \end{bmatrix} & & & \\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & & & \\ \begin{bmatrix} 0 & (\tilde{B}_{l_1+1l_1+1}p_{l_1+1}) \\ \vdots & \vdots \\ 0 & (\tilde{B}_{l_2-2l_1+1}p_{l_1+1}) \end{bmatrix} & \begin{bmatrix} (\tilde{B}_{l_1+1l_2-2}r_{l_2-1}) & 0 \\ \vdots & \vdots \\ (\tilde{B}_{l_2-2l_2-2}r_{l_2-1}) & 0 \end{bmatrix} & & \\ & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & & \\ & \begin{bmatrix} 0 & (\tilde{B}_{l_2+1l_2+1}p_{l_2+1}) \\ \vdots & \vdots \\ 0 & (\tilde{B}_{l_3-2l_2+1}p_{l_2+1}) \end{bmatrix} & \cdots & \begin{bmatrix} (\tilde{B}_{l_{n-1}+1l_n-2}r_{l_n-1}) & 0 \\ \vdots & \vdots \\ (\tilde{B}_{l_n-2l_n-2}r_{l_n-1}) & 0 \end{bmatrix} \\ & & & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ & & & \begin{bmatrix} 0 & (\tilde{B}_{l_n+1l_n+1}p_{l_n+1}) \\ \vdots & \vdots \\ 0 & (\tilde{B}_{ml_n+1}p_{l_n+1}) \end{bmatrix} \end{bmatrix}}_{B_1} \underbrace{\begin{bmatrix} \begin{bmatrix} x_{l_1-1} \\ x_{l_1} \end{bmatrix} \\ \begin{bmatrix} x_{l_2-1} \\ x_{l_2} \end{bmatrix} \\ \cdots \\ \begin{bmatrix} x_{l_n-1} \\ x_{l_n} \end{bmatrix} \end{bmatrix}}_{\Delta X}. \tag{2.38}$$

Here ΔX is the solution of the system [4] of linear equations

$$\Omega \Delta X = \Delta Y, \tag{2.39}$$

where Ω is a nonsingular tridiagonal matrix of the general form (2.5), and

$$\Delta Y = \begin{bmatrix} \begin{bmatrix} y_{l_1-1} - p_{l_1-1}\overset{\circ}{x}_{l_1-2} \\ y_{l_1} - r_{l_1+1}\overset{\circ}{x}_{l_1+1} \end{bmatrix} \\ \begin{bmatrix} y_{l_2-1} - p_{l_2-1}\overset{\circ}{x}_{l_2-2} \\ y_{l_2} - r_{l_2+1}\overset{\circ}{x}_{l_2+1} \end{bmatrix} \\ \cdots\cdots\cdots\cdots \\ \begin{bmatrix} y_{l_n-1} - p_{l_n-1}\overset{\circ}{x}_{l_n-2} \\ y_{l_n} - r_{l_n+1}\overset{\circ}{x}_{l_n+1} \end{bmatrix} \end{bmatrix}, \tag{2.40}$$

where

$$\begin{aligned} \overset{\circ}{x}_{l_k-1} &= \sum_{j=l_{k-1}+1}^{l_k-2} \tilde{B}_{l_k-2j}y_j, \\ \overset{\circ}{x}_{l_k} &= \sum_{j=l_k+1}^{l_{k+1}-2} \tilde{B}_{l_k+1j}y_j, \quad k = 1, 2, \dots, n. \end{aligned} \tag{2.41}$$

However, it is to be noted [3,4] that if $C(2.1)$ ill–posed, matrix $\Omega(2.39)$ becomes also ill–posed. Therefore, a direct use of representation 2.13 as a generator of the methods of numerical solution of the systems of equations $CX = Y$ is not always justified. The problem for Ω being ill–posed in inverting matrices $C(2.1)$ is significantly relaxed by the possibility of explicit computation of $Z(2.3) \div (2.4)$, as was said above.

3. Critical–component method for solving a system of linear algebraic equations with tridiagonal matrices of the general form

As follows from the above consideration, if Ω is ill–posed, then components of the vector $\Delta X(2.38) \div (239)$ are ill–posed; but not all of them, obviously. In this section, we carry out modifications of representation 2.13 that allow us to separate well– and ill–posed components from ΔX. A method of that sort of deriving a solution X will be called the critical–component method.

In what follows, we will employ theorem 2.1 and results of ref. [9].

Let matrices $C(2.1)$ have leading upper $[\Delta_1^{l_i} \neq 0)$, where $i = 1, 2, ..., n]$ or leading lower $[(\Delta_{l_j+1}^{m} \neq 0)$, where $j = 1, 2, ..., n]$ angular minors that differ from zero and let the matrices $\{\tilde{C}_{l_i+1}^{l_{i+1}}\}_{i=0}^{n}$ be well–defined, where

$$\left\{ \begin{array}{l} \tilde{C}_{l_i+1}^{l_{i+1}} = C_{l_i+1}^{l_{i+1}} - [0, ..., 0, p_{l_i+1}]\left[\tilde{C}_{l_{i-1}+1}^{l_i}\right]^{-1}[0, ..., 0, r_{l_i+1}]^T, \\ \tilde{C}_1^{l_1} = C_1^{l_1}, \ i = 1, 2, ..., n \\ \text{or} \\ \tilde{C}_{l_{i-1}+1}^{l_i} = C_{l_{i-1}+1}^{l_i} - [r_{l_i+1}, 0, ..., 0]\left[\tilde{C}_{l_i+1}^{l_{i+1}}\right]^{-1}[p_{l_i+1}, 0, ..., 0]^T, \\ \tilde{C}_{l_n+1}^{m} = C_{l_n+1}^{m}, \ i = n, ..., 2, 1 \end{array} \right. \tag{3.1}$$

and

$$C_{l_k+1}^{l_{k+1}} = \begin{bmatrix} q_{l_k+1} & r_{l_k+2} & & & \\ p_{l_k+2} & q_{l_k+2} & r_{l_k+3} & & \\ & \ddots & \ddots & \ddots & \\ & & p_{l_{k+1}-1} & q_{l_{k+1}-1} & r_{l_{k+1}} \\ & & & p_{l_{k+1}} & q_{l_{k+1}} \end{bmatrix}, \ k = 0, 1, .., n, \ l_0 = 0, \ l_{n+1} = m, \tag{3.2}$$

i.e. $C_{l_k+1}^{l_{k+1}} = tridiag\{q_{l_k+1}, q_i, p_i, r_i\}_{i=l_k+2}^{l_{k+1}}$ are submatrices of matrices $C(2.1)$, respectively. If we represent the matrix $C(2.1)$ in the following block form:

$$C = \begin{bmatrix} q_1 & r_2 & & & \\ p_2 & q_2 & r_3 & & \\ & \ddots & \ddots & \ddots & \\ & & p_{m-1} & q_{m-1} & r_m \\ & & & p_m & q_m \end{bmatrix} \equiv \begin{bmatrix} \left[C_1^{l_1}\right] & r_{l_1+1} & & \\ p_{l_1+1} & \left[C_{l_1+1}^{l_2}\right] & r_{l_2+1} & \\ & & \cdots\cdots & \\ & & p_{l_n+1} & \left[C_{l_n+1}^{m}\right] \end{bmatrix}, \tag{3.3}$$

we arrive at the following lemma 1. Let C be a nonsingular tridiagonal matrix of the general form (2.1) satisfying the conditions $(3.1) \div (3.3)$. Then $C(2.1)$ possess the following representations:

Representation 3.1

$$C = \begin{bmatrix} \left[\tilde{E}_1^{l_1}\right] & & \\ \left[\bar{\beta}_1^{l_1}\right]\left[\tilde{E}_{l_1+1}^{l_2}\right] & & \\ \cdots\cdots & & \\ & \left[\bar{\beta}_{l_n+1}^{m}\right]\left[\tilde{E}_{l_n+1}^{m}\right] \end{bmatrix} \begin{bmatrix} \left[\tilde{C}_1^{l_1}\right] & & \\ & \left[\tilde{C}_{l_1+1}^{l_2}\right] & \\ & \cdots\cdots & \\ & & \left[\tilde{C}_{l_n+1}^{m}\right] \end{bmatrix} \begin{bmatrix} \left[\tilde{E}_1^{l_1}\right]\left[\bar{c}_1^{l_1}\right] & & \\ & \left[\tilde{E}_{l_1+1}^{l_2}\right]\left[\bar{c}_{l_1+1}^{l_2}\right] & \\ & \cdots\cdots & \\ & & \left[\tilde{E}_{l_n+1}^{m}\right] \end{bmatrix}, \tag{3.4}$$

where

$$\tilde{C}_{l_k+1}^{l_{k+1}} = \begin{bmatrix} \tilde{q}_{l_k+1} & r_{l_k+2} & & & \\ p_{l_k+2} & q_{l_k+2} & r_{l_k+3} & & \\ & \ddots & \ddots & \ddots & \\ & & p_{l_{k+1}-1} & q_{l_{k+1}-1} & r_{l_{k+1}} \\ & & & p_{l_{k+1}} & q_{l_{k+1}} \end{bmatrix};$$

$$\bar{\beta}_{l_k+1}^{l_{k+1}} = \begin{bmatrix} \bar{\beta}_{l_k+1} \cdots \bar{\beta}_{l_{k+1}} \\ \\ 0 \end{bmatrix}, \bar{c}_{l_k+1}^{l_{k+1}} = \begin{bmatrix} \bar{c}_{l_k+1} & \\ \vdots & 0 \\ \bar{c}_{l_{k+1}} & \end{bmatrix}, \tag{3.5}$$

$$\tilde{q}_{l_k+1} = q_{l_k+1} - p_{l_k+1}\tilde{B}_{l_k l_k} r_{l_k+1}, \tilde{q}_1 = q_1,$$
$$\bar{\beta}_i = -p_{l_k+1}\tilde{B}_{l_k i}, \ \bar{c}_i = -\tilde{B}_{i l_k} r_{l_k+1}, \ i = l_k + 1, l_k + 2, ..., l_{k+1},$$
$$k = 0, 1, ..., n, \ l_0 = 0, \ l_{n+1} = m;$$

Representation 3.2

$$C = \begin{bmatrix} \left[\tilde{E}_1^{l_1}\right]\left[\tilde{\beta}_{l_1+1}^{l_2}\right] & & \\ & \left[\tilde{E}_{l_1+1}^{l_2}\right]\left[\tilde{\beta}_{l_1+1}^{l_2}\right] & \\ & \cdots\cdots & \\ & & \left[\tilde{E}_{l_n+1}^{m}\right] \end{bmatrix} \begin{bmatrix} \left[\tilde{C}_1^{l_1}\right] & & \\ & \left[\tilde{C}_{l_1+1}^{l_2}\right] & \\ & \cdots\cdots & \\ & & \left[\tilde{C}_{l_n+1}^{m}\right] \end{bmatrix} \begin{bmatrix} \left[\tilde{E}_1^{l_1}\right] & & \\ \left[\tilde{c}_{l_1+1}^{l_2}\right]\left[\tilde{E}_{l_1+1}^{l_2}\right] & & \\ \cdots\cdots & & \\ & \left[\tilde{c}_{l_n+1}^{m}\right]\left[\tilde{E}_{l_n+1}^{m}\right] \end{bmatrix}, \tag{3.6}$$

where

$$
\tilde{C}_{l_k+1}^{l_{k+1}} = \begin{bmatrix} q_{l_k+1} & r_{l_k+2} & & & \\ p_{l_k+2} & q_{l_k+2} & r_{l_k+3} & & \\ & \ddots & \ddots & \ddots & \\ & & p_{l_{k+1}-1} & q_{l_{k+1}-1} & r_{l_{k+1}} \\ & & & p_{l_{k+1}} & \tilde{q}_{l_{k+1}} \end{bmatrix},
$$

$$
\tilde{\beta}_{l_k+1}^{l_{k+1}} = \begin{bmatrix} & 0 & \\ \tilde{\beta}_{l_k+1} & \cdots & \tilde{\beta}_{l_{k+1}} \end{bmatrix}, \tilde{c}_{l_k+1}^{l_{k+1}} = \begin{bmatrix} & \tilde{c}_{l_k+1} \\ 0 & \vdots \\ & \tilde{c}_{l_{k+1}} \end{bmatrix}, \qquad (3.7)
$$

$$
\tilde{q}_{l_{k+\nu}} = q_{l_{k+1}} - r_{l_{k+1}+1}\tilde{B}_{l_{k+1}+1 l_{k+1}+1}p_{l_{k+1}+1}, \tilde{q}_m = q_m,
$$

$$
\tilde{\beta}_i = -r_{l_{k+1}}\tilde{B}_{l_{k+1}i}, \ \tilde{c}_i = -\tilde{B}_{il_{k+1}}p_{l_{k+1}}, i = l_k+1, l_k+2, ..., l_{k+1},
$$

$$
k = n, n-1, ..., 0, \ l_0 = 0, \ l_{n+1} = m.
$$

Here $\tilde{B}_{ij}$ are elements of the last (first) rows and columns of submatrices $\tilde{B}_{\rho}^{\nu} = [\tilde{C}_{\rho}^{\nu}]^{-1}$ that can be derived by using either (2.8) $\div$ (2.13) or (2.29) $\div$ (2.35).

Proof. Decompositions of the type (3.4) and (3.7) immediately follow from the results of refs. [9], if matrix $C(2.1)$ is decomposed in the block form (3.3).

The validity of these decompositions is verified by multiplying the corresponding factorizing matrices. The Lemma is proved.

Further, making use of the results of Theorem 2.1 and Lemma 3.1, we shall prove the validity of the following

Theorem 3.1. Let $C(2.1)$ be a nonsingular tridiagonal matrix obeying the conditions of Lemma 3.1. Then for $B = C^{-1}$ there exist the following representations:

Representation 3.3(of the type $B = [\mathring{B} \equiv \tilde{B}\bar{\beta}] + [\Delta B = \bar{c}\bar{b}\bar{\beta}]$)

$$
B = \underbrace{\begin{bmatrix} \begin{bmatrix} \tilde{B}_1^{l_1} \end{bmatrix} & & & \\ & \begin{bmatrix} \tilde{B}_{l_1+1}^{l_2} \end{bmatrix} & & \\ & & \begin{bmatrix} \tilde{B}_{l_2+1}^{l_3} \end{bmatrix} & \\ & & & \ddots \\ & & & & \begin{bmatrix} \tilde{B}_{l_n+1}^{m} \end{bmatrix} \end{bmatrix}}_{\tilde{B}} \underbrace{\begin{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} & & & \\ \begin{matrix} \bar{\beta}_1 \cdots \bar{\beta}_{l_1} \\ 0 \end{matrix} & \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} & & \\ \begin{matrix} [(\bar{\beta}_{l_1+1})\bar{\beta}_1^{l_1}] \\ 0 \end{matrix} & \begin{matrix} \bar{\beta}_{l_1+1}\cdots\bar{\beta}_{l_2} \\ 0 \end{matrix} & \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} & \\ \cdots & \cdots & \cdots & \cdots \\ \begin{matrix} [(\prod_{\xi=1}^{n-1}\bar{\beta}_{l_\xi+1})\bar{\beta}_1^{l_1}] \\ 0 \end{matrix} & \cdots & \begin{matrix} \bar{\beta}_{l_{n-1}+1}\cdots\bar{\beta}_{l_n} \\ 0 \end{matrix} & \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \end{bmatrix}}_{\bar{\beta}} +
$$

$$
+ \underbrace{\begin{bmatrix} \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} \begin{matrix} \bar{c}_1 \\ \vdots \\ \bar{c}_{l_1} \end{matrix} & & & \\ & \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} \begin{matrix} \bar{c}_{l_1+1} \\ \vdots \\ \bar{c}_{l_2} \end{matrix} & & \\ & & \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} \begin{matrix} \bar{c}_{l_2+1} \\ \vdots \\ \bar{c}_{l_3} \end{matrix} & \\ & & \ddots & \\ & & & \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} \end{bmatrix}}_{\bar{c}} \underbrace{\begin{bmatrix} \begin{bmatrix} 0 \end{bmatrix} & & & \\ & \begin{bmatrix} \bar{b}_{l_1+1}^{l_2} \\ 0 \end{bmatrix} & [(\bar{c}_{l_1+1})\bar{b}_{l_2+1}^{l_3}] \cdots & [(\prod_{\xi=1}^{n-1}\bar{c}_{l_\xi+1})\bar{b}_{l_n+1}^{m}] \\ & 0 & \begin{bmatrix} \bar{b}_{l_2+1}^{l_3} \\ 0 \end{bmatrix} \cdots & [(\prod_{\xi=2}^{n-1}\bar{c}_{l_\xi+1})\bar{b}_{l_n+1}^{m}] \\ & & \ddots & 0 \\ & & & \begin{bmatrix} \bar{b}_{l_n+1}^{m} \\ 0 \end{bmatrix} \end{bmatrix}}_{\bar{b}} \times
$$

$$\times\underbrace{\begin{bmatrix}
\begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix} & & & \\
\begin{matrix}\bar\beta_1 \cdots \bar\beta_{l_1}\\ 0\end{matrix} & \begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix} & & \\
\begin{matrix}[(\bar\beta_{l_1+1})\bar\beta_1^{l_1}]\ \bar\beta_{l_1+1}\cdots\bar\beta_{l_2}\\ 0\end{matrix} & & \begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix} & \\
\cdots & \cdots & \cdots & \cdots \\
\begin{matrix}[(\prod_{\xi=1}^{n-1}\bar\beta_{l_\xi+1})\bar\beta_1^{l_1}]\ \cdots\ \bar\beta_{l_{n-1}+1}\cdots\bar\beta_{l_n}\\ 0\end{matrix} & & & \begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix}
\end{bmatrix}}_{\bar\beta}, \qquad (3.8)$$

where

$$\begin{cases}
\prod_{\xi=j}^{i}\bar\beta_{l_\xi+1} = \bar\beta_{l_i+1}\bar\beta_{l_{i-1}+1}\cdots\bar\beta_{l_j+1},\ \prod_{\xi=i}^{j}\bar c_{l_\xi+1} = \bar c_{l_i+1}\bar c_{l_{i+1}+1}\cdots\bar c_{l_j+1},\\
\bar\beta_j^i = [\bar\beta_j, \bar\beta_{j+1}, ..., \bar\beta_i],\ \bar b_i^j = [\tilde B_{ii}, \tilde B_{ii+1}, ..., \tilde B_{ij}],\\
\bar\beta_i = -p_{l_k+1}\tilde B_{l_k i},\ \bar c_i = -\tilde B_{il_k} r_{l_k+1},\\
k = 0, 1, ..., n;\ l_0 = 0,\ l_{n+1} = m;
\end{cases} \qquad (3.9)$$

Representation 3.4(of the type $B = [\mathring{B} \equiv \tilde B\tilde\beta] + [\Delta B = \tilde c\tilde b\tilde{\tilde\beta}]$).

$$B = \underbrace{\begin{bmatrix}
\begin{bmatrix}\tilde B_1^{l_1}\end{bmatrix} & & & & \\
& \cdots & & & \\
& & \begin{bmatrix}\tilde B_{l_1+1}^{l_2}\end{bmatrix} & & \\
& & & \begin{bmatrix}\tilde B_{l_2+1}^{l_3}\end{bmatrix} & \\
& & & & \begin{bmatrix}\tilde B_{l_n+1}^{m}\end{bmatrix}
\end{bmatrix}}_{\tilde B}
\underbrace{\begin{bmatrix}
\begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix} & \begin{matrix}0\\ \tilde\beta_{l_1+1}\cdots\tilde\beta_{l_2}\end{matrix} & \cdots & [(\prod_{\xi=2}^{n}\tilde\beta_{l_\xi})\tilde\beta_{l_n+1}^{m}] \\
\cdots & \cdots & \cdots & \cdots \\
& \begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix} & \begin{matrix}0\\ \tilde\beta_{l_{n-1}+1}\cdots\tilde\beta_{l_n}[(\tilde\beta_{l_n})\tilde\beta_{l_n+1}^{m}]\end{matrix} & \\
& & \begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix} & \begin{matrix}0\\ \tilde\beta_{l_n+1}\cdots\tilde\beta_m\end{matrix} \\
& & & \begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix}
\end{bmatrix}}_{\tilde\beta} +$$

$$+\underbrace{\begin{bmatrix}
\begin{bmatrix}0&&\\&\ddots&\\&&0\end{bmatrix} & & & \\
\cdots & & & \\
\begin{matrix}\tilde c_{l_{n-2}+1}\\ \vdots\\ \tilde c_{l_{n-1}}\end{matrix} & \begin{bmatrix}0&&\\&\ddots&\\&&0\end{bmatrix} & & \\
& \begin{matrix}\tilde c_{l_{n-1}+1}\\ \vdots\\ \tilde c_{l_n}\end{matrix} & \begin{bmatrix}0&&\\&\ddots&\\&&0\end{bmatrix} & \\
& & \begin{matrix}\tilde c_{l_n+1}\\ \vdots\\ \tilde c_m\end{matrix} & \begin{bmatrix}0&&\\&\ddots&\\&&0\end{bmatrix}
\end{bmatrix}}_{\tilde c}
\underbrace{\begin{bmatrix}
\begin{bmatrix}0\\ \tilde b_1^{l_1}\end{bmatrix} & & & \\
\cdots & & & \\
\begin{matrix}0\\ [(\prod_{\xi=2}^{n-1}\tilde c_{l_\xi})\tilde b_1^{l_1}]\ \cdots\end{matrix} & \begin{bmatrix}0\\ \tilde b_{l_{n-2}+1}^{l_{n-1}}\end{bmatrix} & & \\
\begin{matrix}0\\ [(\prod_{\xi=2}^{n}\tilde c_{l_\xi})\tilde b_1^{l_1}]\ \cdots\end{matrix} & [(\tilde c_{l_n})\tilde b_{l_{n-2}+1}^{l_{n-1}}] & \begin{bmatrix}0\\ \tilde b_{l_{n-1}+1}^{l_n}\end{bmatrix} & \\
& & & \begin{bmatrix}0\end{bmatrix}
\end{bmatrix}}_{\tilde b}\times$$

$$\times\underbrace{\begin{bmatrix}
\begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix} & \begin{matrix}0\\ \tilde\beta_{l_1+1}\cdots\tilde\beta_{l_2}\end{matrix} & \cdots & [(\prod_{\xi=2}^{n}\tilde\beta_{l_\xi})\tilde\beta_{l_n+1}^{m}] \\
\cdots & \cdots & \cdots & \cdots \\
& \begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix} & \begin{matrix}0\\ \tilde\beta_{l_{n-1}+1}\cdots\tilde\beta_{l_n}[(\tilde\beta_{l_n})\tilde\beta_{l_n+1}^{m}]\end{matrix} & \\
& & \begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix} & \begin{matrix}0\\ \tilde\beta_{l_n+1}\cdots\tilde\beta_m\end{matrix} \\
& & & \begin{bmatrix}1&&\\&\ddots&\\&&1\end{bmatrix}
\end{bmatrix}}_{\tilde{\tilde\beta}}, \qquad (3.10)$$

where

$$\begin{cases}
\prod_{\xi=i}^{j}\tilde\beta_{l_\xi} = \tilde\beta_{l_i}\tilde\beta_{l_{i+1}}\cdots\tilde\beta_{l_j},\ \prod_{\xi=j}^{i}\tilde c_{l_\xi} = \tilde c_{l_i}\tilde c_{l_{i-1}}\cdots\tilde c_{l_j},\\
\tilde\beta_i^j = [\tilde\beta_i, \tilde\beta_{i+1}, ..., \tilde\beta_j],\ \tilde b_j^i = [\tilde B_{ij}, \tilde B_{ij+1}, ..., \tilde B_{ii}],\\
\tilde\beta_i = -r_{l_k+1}\tilde B_{l_k+1 i},\ \tilde c_i = -\tilde B_{il_k+1} p_{l_k+1},\\
k = n, ..., 1, 0,\ l_0 = 0,\ l_{n+1} = m.
\end{cases} \qquad (3.11)$$

In these representations, $\tilde B_\rho^\nu = [\tilde C_\rho^\nu]^{-1}$. Elements $\tilde B_{ij}$ of submatrices $\tilde B_\rho^\nu$ can be represented either in the form (2.8) ÷ (2.13) or in the form (2.29) ÷ (2.35).

Proof. We have obtained representations 3.3 and 3.4 from decompositions (3.4) ÷ (3.5) and (3.6) ÷ (3.7) for $C(2.1)$, respectively. In this case, zeroth semirows (half-columns) of the matrix B are defined both as the corresponding zeroth semirows (half-columns) of matrices $\tilde{B}^{\nu}_{\rho}$ and the corresponding zeroth values of the elements $\{\bar{c}_i\}, \{\bar{\beta}_i\}, \{\tilde{c}_i\}$ and $\{\tilde{\beta}_i\}$. The validity of representations 3.3 and 3.4 is established by verifying the basic equalities $BC = E = CB$. The theorem is thus proved.

Finally, let us determine direct methods for solving the systems of linear equations $CX = Y$. There exists

Theorem 3.2. Let C(2.1) be a nonsingular tridiagonal matrix of the general form obeying the conditions of Lemma 3.1. Then, for x_i – components of the X – solution of the system of linear equations $CX = Y$ there hold the following representations:

Representation 3.5

$$\begin{cases} x_i = \overset{\circ}{x}_i + \bar{c}_i \gamma_k, \ k = n+1, n, ..., 1, \\ i = l_k, l_k - 1, ..., l_{k-1} + 1, \ l_{n+1} = m, \ l_0 = 0, \end{cases} \tag{3.12}$$

where

$$\begin{cases} \overset{\circ}{x}_i = \tilde{B}_{il_k+1}\tilde{y}_{l_k+1} + \sum\limits_{j=l_k+2}^{l_{k+1}} (\tilde{B}_{ij} y_j), i = l_{k+1}, ..., l_k + 1, \ k = n, ..., 1, 0; \\ [\gamma_k \equiv x_{l_k+1}] = \overset{\circ}{x}_{l_k+1} + \bar{c}_{l_k+1}[\gamma_{k+1} \equiv x_{l_{k+1}+1}], [\gamma_{n+1} \equiv x_{m+1}] = 0, \ k = n, ..., 2, 1; \\ \tilde{y}_{l_k+1} = y_{l_k+1} + \bar{\beta}_{l_{k-1}+1}\tilde{y}_{l_{k-1}+1} + \sum\limits_{j=l_{k-1}+2}^{l_k} (\bar{\beta}_j y_j), \tilde{y}_1 = y_1, \ k = 1, 2, ..., n; \\ \bar{\beta}_i = -p_{l_k+1}\tilde{B}_{l_k i}, \bar{c}_i = -\tilde{B}_{il_k} r_{l_k+1}, \ i = l_{k-1} + 1, ..., l_k, \ k = 1, 2, ..., n; \end{cases} \tag{3.13}$$

Representation 3.6

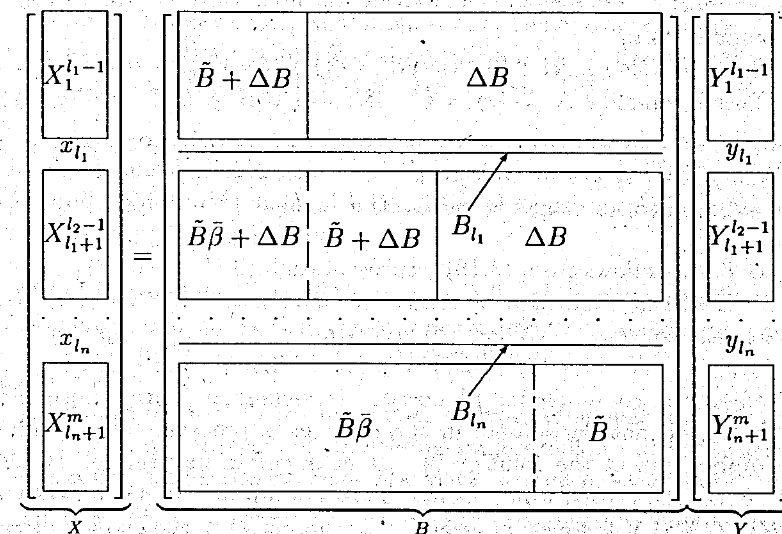
$$\begin{cases} x_i = \overset{\circ}{x}_i + \tilde{c}_i \gamma_k, \ k = 0, 1, ..., n, \\ i = l_k + 1, l_k + 2, ..., l_{k+1} - 1, \ l_0 = 0, \ l_{n+1} = m, \end{cases} \tag{3.14}$$

where

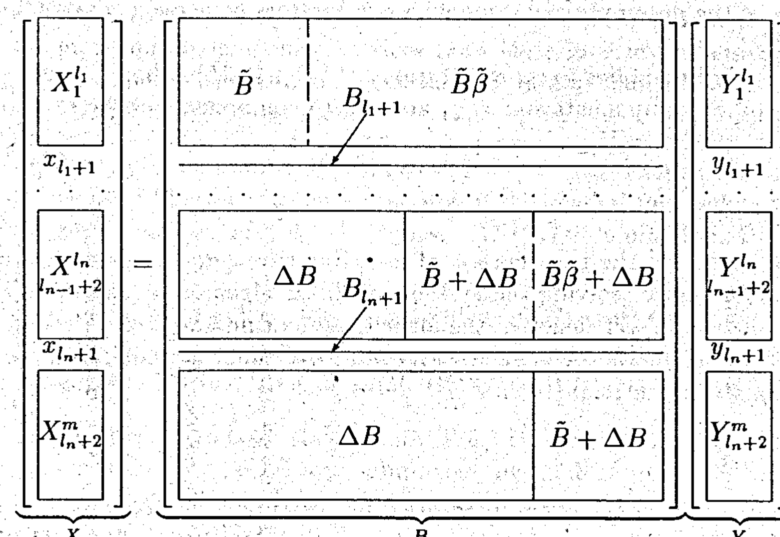
$$\begin{cases} \overset{\circ}{x}_i = \tilde{B}_{il_k}\tilde{y}_{l_k} + \sum\limits_{j=l_{k-1}+1}^{l_k - 1} (\tilde{B}_{ij} y_j), k = 1, 2, ..., n+1; \ i = l_{k-1} + 1, ..., l_k; \\ [\gamma_k \equiv x_{l_k}] = \overset{\circ}{x}_{l_k} + \tilde{c}_{l_k - 1}[x_{l_{k-1}} \equiv \gamma_{k-1}], \ [x_0 \equiv \gamma_0] = 0, \ k = 1, 2, ..., n; \\ \tilde{y}_{l_k} = y_{l_k} + \tilde{\beta}_{l_{k+1}}\tilde{y}_{l_{k+1}} + \sum\limits_{j=l_k+1}^{l_{k+1}-1} (\tilde{\beta}_j y_j), \tilde{y}_m = y_m, k = n, ..., 2, 1; \\ \tilde{\beta}_i = -r_{l_k+1}\tilde{B}_{l_k+1 i}, \tilde{c}_i = -\tilde{B}_{il_k+1} p_{l_k+1}, \ i = l_{k+1}, ..., l_k + 1, \ k = n, ..., 2, 1. \end{cases} \tag{3.15}$$

Here $\tilde{B}_{ij}$ are elements of submatrices $\tilde{B}^{\nu}_{\rho} = [\tilde{C}^{\nu}_{\rho}]^{-1}$ matrices $B(3.8) \div (3.11)$ respectively. They can be found with the help of any of the representations $(2.8) \div (2.13)$ and $(2.29) \div (2.35)$.

Proof. If for matrix C (2.1), divided into blocks of the form (3.3), the matrix $B = C^{-1}$ is in structure of the form $(3.8) \div (3.11)$, the vector X, a solution to the system $CX = Y$, can be represented as the following graphic scheme:



; (3.16)



(3.17)

From (3.16) and (3.17) it is seen that components of the solution X (3.12) and X (3.14) are sums of two constituents, the constituent $\overset{\circ}{X}$ and constituent

ΔX; also, note that in (3.12) $\overset{\circ}{X} = [\overset{\circ}{B} \equiv \tilde{B}\bar{\beta}]Y$ and $\Delta X = [\Delta B \equiv \bar{c}\bar{b}\bar{\beta}]Y$, whereas in (3.14) $\overset{\circ}{X} = [\overset{\circ}{B} \equiv \bar{B}\tilde{\beta}]Y$ and $\Delta X = [\Delta B \equiv \bar{c}\tilde{b}\tilde{\beta}]Y$. Here $\overset{\circ}{B}$ and ΔB are given by the representations 3.3 and 3.4. From the graphic representation it also follows that the vector X is in turn splitted into subvectors X_ρ^ν with $(x \equiv \gamma)$ – support components calculated by formulae $(3.13)_{2)}$, $(3.15)_{2)}$ and critical components x_{l_k}, x_{l_k+1}.

Since the solutions (3.12) ÷ (3.13) and (3.14) ÷ (3.15) have been derived from the formal equality $X = (B = C^{-1})Y$, where $B = C^{-1}$ (of the type $B = \overset{\circ}{B} + \Delta B$), its validity is verified by direct substitution of X (3.12)÷(3.13) or X (3.14)÷(3.15) into the equation $CX = Y$, respectively. Here, one should also take account of the basic equalities $BC = E = CB$. The theorem is proved.

Remark 5. As follows from (3.16), representation (3.12) ÷ (3.13) may be called the direct representation of critical components of the solution of a system of linear equations $CX = Y$. The reason is that any components x_i of the solution of the system $CX = Y$ are not recurrent functions of ($\gamma_k \equiv x_{l_k+1}$), well-posed components. Ill-posed components, x_{l_k}, are determined separately and do not participate in the recurrence processes of obtaining x_i, any components of the solution X. It is therefore natural to call the components x_{l_k} the critical components. This method of solving the system of equations $CX = Y$ belongs, in essence (see subsect.2) to the class of direct methods of the decomposition type whose generators have been constructed in refs.[3,4].

The said also applies to representations (3.14) ÷ (3.15), where ($\gamma_k \equiv x_{l_k}$) are well-posed components and x_{l_k+1} are critical components of the solution to the system $CX = Y$.

4. Conclusion

We have suggested the new method of inverting tridiagonal matrices of the general form and of solving the systems of linear algebraic equations with matrices like that. Efficiency of the algorithms and programs developed on the basis of this method and their advantages over known similar programs are demonstrated in refs. [10 and 11].

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