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E11-93-468

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ASYS2: A NEW VERSION OF COMPUTER ALGEBRA PACKAGE ASYS FOR ANALYSIS AND SIMPLIFICATION OF POLYNOMIAL SYSTEMS

Submitted to «The Rhine Workshop on Computer Algebra», Karlsruhe, Germany, March 22-24, 1994

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1993

1 Introduction

The most universal and developed algorithmic method for analysis and solution of nonlinear algebraic equation systems is that based on the Gröbner basis construction [1]. This method allows in a completely algorithmic way to obtain the following information on the algebraic system:

- To verify its compatibility, i.e. the existence of common roots.
- To find the dimension of the solution space (algebraic variety) or, in other words, the dimension of the polynomial ideal generated by the algebraic system.
- To detect whether the system has finitely or infinitely many solutions.
- In the case of the finitely many solutions (a zero-dimensional ideal) to transform the system into an equivalent "triangular" form and therefore to reduce the initial multivariate problem to successive solving univariate equations.
- In the case of the infinitely many solutions (a positive dimensional ideal) to find all the maximal sets of (algebraically) independent variables modulo polynomial ideal. These variables can be considered as free parameters providing the parametrization of the solution space.

Earlier the first version (ASYS1) of a program package ASYS (Algebraic SYstem Simplifier) for analysis of nonlinear algebraic equations, based on Gröbner basis technique, has been developed [2]. The package was especially designed to investigate and solve polynomial algebraic equations with infinitely many solutions. It was written in the symbolic mode language Rlisp of computer algebra system Reduce[1]. The ASYS package provides a user with all the facilities listed above.

The ASYS1 package has been used in integrability analysis of polynomial-nonlinear evolution equations with arbitrary parameters [5] and in isomorphism verification of finite-dimensional Lie algebras [4].

In this paper we describe the general structure of ASYS2, its data structure and some implementation details. We also present the results of computation of a complete set of solutions for a large system of polynomial equation which comes from the bifurcation analysis of dynamical system. A number of examples are considered, and results of comparison for these examples between ASYS1, ASYS2, standard Reduce package GROEBNER and a new package CALI for commutative algebra computation, are given.

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[1]Actually ASYS1 was written in and used with Reduce 3.4. In this paper the latest version, Reduce 3.5 [3], is used

2 Basic notations

Throughout this paper we use the following notations:

K – integral domain;
a,b,c,d – elements of K;
i,j,k,l,m,n - natural numbers;
$K[x_1,\ldots,x_n]$ – polynomial ring over K;
f,g,h,p,q – polynomials from $K[x_1,\ldots,x_n]$;
F,G – finite subsets in $K[x_1,\ldots,x_n]$;
s,t,u,w - power products in the form $x_1^{i_1},\ldots,x_n^{i_n}$;
$lt(f)$ - leading power product of f (w.r.t. chosen admissible ordering);
$lc(f)$ - leading coefficient of f;
$cf(f,u)$ – coefficient of u in f;
$lcm(u,w)$ – least common multiple of u and w;
$SP(f_i,f_j)$ - Buchberger S– polynomial of f_i,f_j
$SP(f_i,f_j) = lc(f_j)(lcm(f_i,f_j)/lt(f_i))f_i - lc(f_i)(lcm(f_i,f_j)/lt(f_j))f_j$;
$Ideal(F)$ – ideal generated by F, i.e. the set $\{\sum_i h_i f_i \mid h_i \in K[x_1,\ldots,x_n], f_i \in F\}$.

3 Data structure

3.1 Representation of polynomials

As the basic recursive polynomial representation used in Reduce does not provide reasonable efficiency of constructing the Gröbner basis, the ASYS package as well as the Reduce standard package GROEBNER [3],[9] uses the distributive representation.

Let a polynomial be given in the form $f = \sum_{i=1}^{m} c_i u_i = \sum T_i$, where u_i are power products $x_1^{i_1},\ldots,x_n^{i_n}$, and c_i are their coefficients. Then, in the distributive representation used in the package, that polynomial is presented in the form $((T_1)(T_2)\ldots(T_m))$, where $T_i = (D_i.c_i)$ is a dot pair: $D_i = (i_1 i_2 \ldots i_n)$ is a list of exponents of power product u_i, $c_i = < s.q. >$ is the coefficient at power product u_i presented in the form of the Reduce standard quotient.

We illustrate basic facilities of ASYS2 at a following simple example

$$\begin{cases} -2w^2 + y^2 = 0, \\ yw + 10(x-1)z^2 = 0. \end{cases}$$

The distributive form of polynomials for this example is

((((0 2 0 0) 1.1) ((0 0 0 2) –2.1)) (((1 0 2 0) 10.1) ((0 1 0 1) 1.1)((0 0 2 0) –10.1)))

3.2 Coefficient domain mode

By contrast to the previous version of ASYS, all manipulations are done, by default, over the coefficient ring $D[x_1,\ldots,x_n]$ rather than over the coefficient field, where D is considered to be an integral domain. ASYS2 has two internal modes: a field mode and a ring one. One should note, that computations over the ring are often faster than over the field.

4 Description of the ASYS package

4.1 General structure

The ASYS package is written in the symbolic mode language Rlisp of the computer algebra system Reduce. It consists of a number of modules providing a user with the following facilities:

- Gröbner basis construction by Buchberger's algorithm [1, 10];
- determination of the dimension of a variety for a given polynomial system, computation of all sets of independent variables [11] and reduction by these sets [2];
- verification of homogeneity properties and homogeneity reduction [2];
- polynomial decomposition [12].

4.2 Special switches

The package contains various switches **lexord, setord, setdim, setgb, scale, scaletest, sugar, ringz** for control over the reduction process and the Gröbner basis construction:

lexord Selects a term ordering (pure lexicographical one if the switch is on and degree reverse lexicographical ordering, otherwise).

setord Generates an heuristically optimal ordering [13].

setdim Computes the dimension of a polynomial ideal and the maximal independent sets of variables.

setgb Performs reduction by maximal independent sets with the Gröbner basis construction for each subsystem has been computed over the reduction process.

scale Performs the homogeneity reduction.

scaletest Verifies of homogeneity properties without doing the homogeneity reduction.

sugar Determines a selection strategy, namely, the "sugar" one [8] if the switch is on and the "normal" selection strategy [1], otherwise.

ringz Computes a Gröbner basis over the ring of integers **Z**. Use of this switch for that particular case of a coefficient ring speeds up the computation.

By default, switches **lexord** and **sugar** are on and the others are off. In this case the call to the main procedure of ASYS, which has the same syntax as standard Reduce package GROEBNER, provides nothing more than computation of the lexicographical Gröbner basis for the system under consideration.

4.3 The structure of the package

In this subsection we consider general structure of the ASYS2 package. The package consists of a number of modules. The most important ones are described below.

The algorithms implemented in the current version of the package allow one to compute a Gröbner basis in different term ordering, namely, in lexicographical and degree reverse lexicographical (DegRevLex) ones. Besides, using properly different switches one can find the dimension of the polynomial ideal generated by the algebraic system, to find all the maximal sets of independent variables and perform the reduction by these sets, verify whether the original system possesses non-trivial homogeneity properties and, if yes, then perform the homogeneity reduction. One can also combine different switches in order to control over the form of the final result. Eventually, such reductions generate a finite number of "triangular" subsystems. Each of them has, typically, the form

$$\begin{cases} g_1(x_1,x_2,\ldots,x_n)=0 \\ g_2(x_2,\ldots,x_n)=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ g_{n-1}(x_{n-1},x_n)=0 \\ g_n(x_n)=0 \end{cases}$$

For our illustrative example calling to the main procedure

Groebner$(\{-2w^2+y^2, yw+10(x-1)z^2\}, \{x,y,z,w\})$;

with the switches **scale, setgb** are on, produces the output

variables = (x y z) % list of variables
parameters = (w) % list of non-zero homogeneous variables
zeros = NIL % list of zero-assigned homogeneous variables

$\{10x^2z+yw-10z^2, y^2-2w^2\}$

D i m e n s i o n % [11]

M = ((x) (z)) % set of maximal independent sets
S = (x) % the first among the independent sets of maximal length
D = 1 % dimension of the polynomial ideal

S u b s y s t e m f o r s e t (x) % here x is a free parameter

$\{yw+10z^2x-10z^2, 50z^4x^2-100z^4x+50z^4-w^4\}$

% "triangular" system in (y,z)

S u b s y s t e m f o r s e t (z) % here z is a free parameter

$\{10xz^2+yw-10z^2, y^2-2w^2\}$

% "triangular" system in (x,y)

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variables = (x) % list of variables
parameters = (z) % list of non-zero homogeneous variables
zeros = (w y) % list of zero-assigned homogeneous variables

$\{x-1\}$

D i m e n s i o n % [11]

M = NIL % set of maximal independent sets
S = NIL % the first among the independent sets of maximal length
D = 0 % dimension of the polynomial ideal

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variables = (x) % list of variables
parameters = NIL % list of non-zero homogeneous variables
zeros = (w y z) % list of zero-assigned homogeneous variables

$\{0\}$

D i m e n s i o n % [11]

M = ((x)) % set of maximal independent sets
S = (x) % the first among the independent sets of maximal length
D = 1 % dimension of the polynomial ideal

Two different versions of Buchberger's algorithm have been implemented in the ASYS2 package

1. the "classical" version [1],

2. that based on the ideas of papers [10],[9].

The previous version (ASYS1) of the package has the only selection strategy. Namely, the normal one [1], when the pair (f_i, f_j) with the minimal least common multiple of the leading terms is selected for computation of S-polynomial. In the present version we have also implemented a sugar selection strategy [8]. In accordance with this strategy a "sugar" weight $s_i = deg(f_i)$ is assigned to each polynomial f_i. Here degree $deg(f)$ means a degree of polynomial f. Let a critical pair be presented as $((f_i, f_j), s_{ij}, lcm(f_i, f_j))$, where $lcm(f_i, f_j$ is the least common multiple of f_i, f_j, s_{ij} is the sugar of the pair

$$s_{ij} = max(s_i - deg(lt(f_i)), s_j - deg(lt(f_j))) + deg(lcm(f_i, f_j))) .$$

In actual practice, the sugar strategy, i.e. a strategy when a pair with the minimal sugar is selected, is often the best one, especially for the lexicographical ordering.

So, a user of the ASYS2 package has an opportunity to chose different strategies for Gröbner basis computation. In addition, ASYS2 applies the new form [10] of criteria for detecting superfluous reductions rather than Buchberger's form of criteria [1] used in the previous version.

The basic algorithmic structure of ASYS2 can be presented as follows

ASYS2:

Input: List of polynomials $polys = \{f_1, \ldots, f_m\}$ and list of variables $vars = \{x_1, \ldots, x_n\}$ ordered according to their arrangement in the list. By contrast to the standard package GROEBNER, some of f_i may be zeros.

Output: Depends on the switch combination; a Gröbner basis by default.

F := **pftodf(polys,vars)**
  if **setord** is on *then* $vars$:= **orderv(F,vars)**
  else if **scale** is on *then* **gb1scale(F,vars)**
    else **gb21dist(F,vars)**

The program modules of ASYS2 have the following functional destination:

pftodf Converts a list of polynomials in the prefix form into the list in the distributive form.

orderv Returns a heuristically optimal order of variables [13].

gb1scale Provides the homogeneity reduction. This module has been improved to avoid generating the superfluous subsystems. As a result, one gets a number of subsystems which contains the complete set of solutions. If one applies switches **scale** and **setgb** simultaneously, then both procedures for computation of independent sets and for reduction by each of them will be performed.

gb21dist Selects either module **gb1dist** or **gb2dist** dependending on the switch **sugar** is on or off, respectively.

gb1dist Performs a Gröbner basis computation by Buchberger's algorithm with the normal selection strategy.

gb2dist Computes a Gröbner basis by Buchberger's algorithm with the sugar selection strategy.

5 Examples

In this section a number of polynomial systems are considered and the complete set of solutions for one of them is given.

5.1 List of examples

Example 1 [5]
Ordering - $\lambda_7 > \lambda_6 > \lambda_5 > \lambda_4 > \lambda_3 > \lambda_2 > \lambda_1$.
Dimension of polynomial ideal - 3.
Homogeneity degree - 1.

$$\lambda_1(\lambda_4 - \lambda_5/2 + \lambda_6) = (2/7\lambda_1^2 - \lambda_4)(-10\lambda_1 + 5\lambda_2 - \lambda_3) = 0,$$
$$(2/7\lambda_1^2 - \lambda_4)(3\lambda_4 - \lambda_5 + \lambda_6) = 0,$$
$$a_1(-3\lambda_1 + 2\lambda_2) + 21a_2 = a_1(2\lambda_4 - 2\lambda_5) + a_2(-45\lambda_1 + 15\lambda_2 - 3\lambda_3) = 0,$$
$$2a_1\lambda_7 + a_2(12\lambda_4 - 3\lambda_5 + 2\lambda_6) = b_1(2\lambda_2 - \lambda_1) + 7b_2 = b_1\lambda_3 + 7b_2 = 0,$$
$$b_1(-2\lambda_4 - 2\lambda_5) + b_2(2\lambda_2 - 8\lambda_1) + 84b_3 = 0,$$
$$b_1(8/3\lambda_5 + 6\lambda_6) + b_2(11\lambda_1 - 17/3\lambda_2 + 5/3\lambda_3) - 168b_3 = 0,$$
$$15b_1\lambda_7 + b_2(5\lambda_4 - 2\lambda_5) + b_3(-120\lambda_1 + 30\lambda_2 - 6\lambda_3) = 0,$$
$$-3b_1\lambda_7 + b_2(-\lambda_4/2 + \lambda_5/4 - \lambda_6/2) + b_3(24\lambda_1 - 6\lambda_2) = 0,$$
$$3b_2\lambda_7 + b_3(40\lambda_4 - 8\lambda_5 + 4\lambda_6) = 0,$$

where

$$a_1 = -2\lambda_1^2 + \lambda_1\lambda_2 + 2\lambda_1\lambda_3 - \lambda_2^2 - 7\lambda_5 + 21\lambda_6, \; a_2 = 7\lambda_7 - 2\lambda_1\lambda_4 + 3/7\lambda_1^3,$$
$$b_1 = \lambda_1(5\lambda_1 - 3\lambda_2 + \lambda_3), \; b_2 = \lambda_1(2\lambda_6 - 4\lambda_4), \; b_3 = \lambda_1\lambda_7/2,$$

Example 2 [5]
Ordering - $l_6 > l_5 > l_4 > l_3 > l_2 > l_1$.
Dimension of polynomial ideal - 0.

$$a_1 = -12/7l_2 + 3/7l_3 - 6/49l_1^2,\ a_2 = 3/7l_5 - 6/49l_2^2,\ b_1 = 5/7l_2,$$
$$b_2 = 30/49l_1l_2 - 10/49l_1l_3 - 5/7l_4 + 15/343l_1^3,$$
$$b_3 = -30/7l_5 - 30/49l_1l_4 - 10/49l_2l_3 + 45/343l_1^2l_2 + 80/49l_2^2,$$
$$b_4 = -10/49l_2l_5 + 5/7l_6 + 15/343l_2^3,$$
$$12/7l_2 + 3/7l_3 - 6/49l_1^2 = 3/7l_5 - 6/49l_2^2 = 5/7l_2 = 0,$$
$$30/49l_1l_2 - 10/49l_1l_3 - 5/7l_4 + 15/343l_1^3 = 0,$$
$$-30/7l_5 - 30/49l_1l_4 - 10/49l_2l3 + 45/343l_1^2l_2 + 80/49l_2^2 = 0,$$
$$-10/49l_2l_5 + 5/7l_6 + 15/343l_2^3 = a_1l_1 = a_1l_2 + 14a_2 = a_1l_4 = 0,$$
$$a_1(6l_2 + 2l_3 + 3l_4) + 168a_2 = a_1l_5 + 5a_2l_2 = 5b_1l_1 + 21b_2 = 0,$$
$$10b_1l_2 + 14b_3 = 105b_4 - 5b_1l_5 - b_3l_2 = 5b_1l_4 + 2b_2l_2 = 0.$$

Example 3 [5]
Ordering - $t > x > y > z$.
Dimension of polynomial ideal - 2.
Homogeneity degree - 1.

$$-2z^3t + (3z^2t - 2z^2 - 6zyt + 6zy + 6y^2t - 6y^2)x - ztx^2 = 0,$$
$$18z^3t^2 - 9z^3t - 18z^2yt^2 + 18z^2yt + 18zy^2t^2 - 18zy^2t +$$
$$(-27z^2t^2 + 24z^2t - 5z^2 + 63zyt^2 - 78zyt + 15zy - 63y^2t^2 +$$
$$78y^2t - 15y^2)x + 9zt^2x^2 = 0,$$
$$-8z^4t + (6z^3t - 6z^3 - 12z^2yt + 12z^2y + 12zy^2t - 12zy^2)x +$$
$$(5z^2t - 4z^2 - 18zyt + 18zy + 18y^2t - 18y^2)x^2 - 3ztx^3 = 0,$$
$$(3t-5)z^2y - 15(t-1)zy^2 + 10(t-1)y^3 +$$
$$(zy + 3y^2t - 3y^2)x - ytx^2 = 0.$$

Example 4 [7]
Ordering - $a_2 > b_2 > a_4 > b_4 > a_1 > b_1 > a_3 > b_3 > a_0 > b_0$.
Dimension of polynomial ideal - 6.
Homogeneity degree - 3.

$$e_k = \hat{e}_k = 0, \quad (k = 1 \div 6),$$

where $\hat{e}_k = e_k \,|_{a_i \Leftrightarrow b_i}$ and

$$e_1 = a_1\ (a_3 - a_4) - a_4\ (b_3 - b_4)\ ,$$

$$e_2 = (2a_3 - a_4)\ y_1 - b_2y_2\ , \qquad y_1 = 6a_0a_3b_2 + (a_0 - b_0)\ (a_1^2 + a_4b_2)\ ,$$
$$e_3 = a_2y_1 - (2b_3 - b_4)\ y_2\ , \qquad y_2 = 6a_0a_2b_3 + (a_0 - b_0)\ (a_1a_2 + a_4b_1)\ ,$$
$$e_4 = 3a_0\ (a_2b_2 + a_3b_3) + (a_0 - b_0)\ (a_1 + b_3)\ a_4\ ,$$
$$e_5 = 2\ (2a_0^2 + 8a_0b_0 - b_0^2)\ a_3b_3 + 2\ (a_0 - b_0)\ (4a_0 - b_0)\ a_3b_4 -$$
$$6a_0\ (a_0 + 2b_0)\ a_2b_2 + (a_0 - b_0)^2\ (5a_1a_3 - 5a_1a_4 + a_4b_4) -$$
$$(a_0 - b_0)\ (7a_0 - b_0)\ a_4b_3\ ,$$
$$e_6 = 3a_0\ [\ (a_0 - b_0)^3 - 3a_0\ (a_0 + 2b_0)^2\ ]\ (a_2b_2 + a_3b_3) +$$
$$(a_0 - b_0)^3\ [\ 3a_0a_1a_3 - 2\ (2a_0 + b_0)a_1a_4\ ] + 9a_0^2\ (a_0 - b_0)$$
$$[\ (a_0 - b_0)\ a_4 - (a_0 + 2b_0)\ a_3\ ]\ b_4 - (a_0 - b_0)\ (2a_0^3 - 30a_0^2b_0 + b_0^3)\ a_4b_3.$$

Example 5 [6, 8]
Ordering - $x_1 > x_2 > x_3 > x_4 > x_5$.
Dimension of polynomial ideal - 0 (number of solutions - 70).

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0,$$
$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 0,$$
$$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 = 0,$$
$$x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 = 0,$$
$$x_1x_2x_3x_4x_5 - 1 = 0.$$

Example 6 [6]
Ordering - $x_4 > x_1 > x_2 > x_5 > x_3$.
Dimension of polynomial ideal - 0 (number of solutions - 64).

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0,$$
$$x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 0,$$
$$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 = 0,$$
$$x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 = 0,$$
$$x_1x_2x_3x_4x_5 - 1 = 0.$$

Example 7 [6]
Ordering - $a > b > c$.
Dimension of polynomial ideal - 0.

$$a^2bc + ab^2c + abc^2 + abc + ab + ac + bc = 0,$$
$$a^2b^2c + ab^2c^2 + a^2bc + abc + bc + a + c = 0,$$
$$a^2b^2c^2 + a^2b^2c + ab^2c + abc + ac + c + 1 = 0.$$

Example 8 [13]

Ordering - $u_0 > u_1 > u_2 > u_3$.
Dimension of polynomial ideal - 0.

$$u_0^2 - u_0 + 2u_1^2 + 2u_2^2 + 2u_3^2 = 2u_0u1 + 2u_1u_2 + 2u_2u_3 - u_1 = 0,$$
$$2u_0u_2 + u_1^2 + 2u_1u_3 - u_2 = u_0 + 2u_1 + 2u_2 + 2u_3 - 1 = 0.$$

Example 9 [8, 13]

Ordering - $u_4 > u_0 > u_3 > u_2 > u_1$.
Dimension of polynomial ideal - 0.

$$u_0^2 - u_0 + 2u_1^2 + 2u_2^2 + 2u_3^2 + 2u_4^2 = 0,$$
$$2u_0u_1 + 2u_1u_2 + 2u_2u_3 + 2u_3u_4 - u_1 = 0,$$
$$2u_0u_2 + u_1^2 + 2u_1u_3 + 2u_2u_4 - u_2 = 0,$$
$$2u_0u_3 + 2u_1u_2 + 2u_1u_4 - u_3 = 0,$$
$$u_0 + 2u_1 + 2u_2 + 2u_3 + 2u_4 - 1 = 0.$$

Example 10 [13]

Ordering - $u_5 > u_3 > u_4 > u_2 > u_1 > u_0$.
Dimension of polynomial ideal - 0.

$$u_0^2 - u_0 + 2u_1^2 + 2u_2^2 + 2u_3^2 + 2u_4^2 + 2u_5^2 = 0,$$
$$2u_0u_1 + 2u_1u_2 + 2u_2u_3 + 2u_3u_4 + 2u_4u_5 - u_1 = 0,$$
$$2u_0u_2 + u_1^2 + 2u_1u_3 + 2u_2u_4 + 2u_3u_5 - u_2 = 0,$$
$$2u_0u_3 + 2u_1u_2 + 2u_1u_4 + 2u_2u_5 - u_3 = 0,$$
$$2u_0u_4 + 2u_1u_3 + 2u_1u_5 + u_2^2 - u_4 = 0,$$
$$u_0 + 2u_1 + 2u_2 + 2u_3 + 2u_4 + 2u_5 - 1 = 0.$$

Example 11 [13]

Ordering - $w > p > z > t > s > b$.
Dimension of polynomial ideal - 0.

$$45p + 35s - 165b - 36 = 35p + 40z + 25t - 27s = 0,$$
$$15w + 25ps + 30z - 18t - 165b^2 = -9w + 15pt + 20zs = 0,$$
$$wp + 2zt - 11b^3 = 99w - 11sb + 3b^2 = 0.$$

Example 12 Communicated by H.-G.Gräbe

Ordering - $x > y > z$.
Dimension of polynomial ideal - 0.

$$x^3 + y^2 + z - 3 = y^3 + z^2 + x - 3 = z^3 + x^2 + y - 3 = 0.$$

Example 13 Communicated by J.Apel and U.Klaus

Ordering - $l > s > z > y > x$.
Dimension of polynomial ideal - 0.

$$x - sl^2 - 1 = y - l^2 - 4sl + l = z - 6ls^2 + s = 0.$$

Example 14 [6]

Ordering - $x > y > z > t$.
Dimension of polynomial ideal - 0.

$$y^2z + 2xyt - 2x - z = 0,$$
$$-x^3z + 4xy^2z + 4x^2yt + 2y^3t + 4x^2 - 10y^2 + 4xz - 10yt + 2 = 0,$$
$$2yzt + xt^2 - x - 2z = 0,$$
$$-xz^3 + 4yz^2t + 4xzt^2 + 2yt^3 + 4xz + 4z^2 - 10yt - 10t^2 + 2 = 0.$$

Example 15 [13]

Ordering - $b_1 > a_{21} > a_{31} > a_{32} > b_2 > b_3 > c_3 > c_2$.
Dimension of polynomial ideal - 0.

$$c_2 - a_{21} = c_3 - a_{31} - a_{32} = b_1 + b_2 + b3 - 1 = 0,$$
$$b_2c_2 + b_3c_3 - 1/2 = b_2c_2^2 + b_3c_3^2 - 1/3 = b_3a_{32}c_2 - 1/6 = 0.$$

Example 16 [8]

Ordering - $x > y > z > t$.
Dimension of polynomial ideal - 0.

$$x^{31} - x^6 - x - y = x^8 - z = x^{10} - t = 0.$$

Example 17 [14]

Ordering - $x > y > z$.
Dimension of polynomial ideal - 0.
Homogeneity degree - 1.

$$x^2 + 4x + 2y^2 + 11y + 17z^2 + 10z + 2 = 0,$$
$$xy + 17x + 5y^2 + yz + 4y + 10z^2 + 6z + 4 = 0,$$
$$xz + 18x + 18y^2 + yz + 2y + 18z^2 + 18z + 14 = 0.$$

Example 18 Communicated by N.G.Lloyd, J.M.Pearson

Ordering - $a_4 > a_6 > a_5 > a_1 > a_2 > a_3$.
Dimension of polynomial ideal - 3.

$\eta_4 = a_5 + a_3a_2 + a_2a_1 = 0,$

$\eta_6 = 15a_6a_5+23a_6a_3a_2+19a_6a_2a_1-3a_5a_4-112a_5a_3^2-46a_5a_3a_1+a_5a_2^2-10a_5a_1^2+13a_4a_3a_2+17a_4a_2a_1-124a_3^3a_2-190a_3^2a_2a_1+a_3a_2^3-76a_3a_2a_1^2+a_2^3a_1-10a_2a_1^3 = 0,$

$\eta_8 = 621a_6^2a_5 + 1365a_6^2a_3a_2 + 993a_6^2a_2a_1 - 738a_6a_5a_4 - 28372a_6a_5a_3^2 - 17284a_6a_5a_3a_1 - 54a_6a_5a_2^2 - 4024a_6a_5a_1^2 + 1062a_6a_4a_3a_2 + 1134a_6a_4a_2a_1 - 36144a_6a_3^3a_2 - 56408a_6a_3^2a_2a_1 - 254a_6a_3a_2^3 - 26360a_6a_3a_2a_1^2 - 154a_6a_2^3a_1 - 4544a_6a_2a_1^3 + 81a_5^3 - 4597a_5^2a_3a_2 - 1529a_5^2a_2a_1 - 675a_5a_4^2 - 5844a_5a_4a_3^2 - 11388a_5a_4a_3a_1 - 610a_5a_4a_2^2 - 4800a_5a_4a_1^2 + 77258a_5a_3^4 + 64304a_5a_3^3a_1 - 9949a_5a_3^2a_2^2 + 20168a_5a_3^2a_1^2 - 10526a_5a_3a_2^2a_1 - 1960a_5a_3a_1^3 + 11a_5a_2^4 - 2881a_5a_2^2a_1^2 - 1850a_5a_1^4 + 909a_4^2a_3a_2 + 1305a_4^2a_2a_1 - 19720a_4a_3^3a_2 - 43440a_4a_3^2a_2a_1 - 1010a_4a_3a_2^3 - 27888a_4a_3a_2a_1^2 - 1110a_4a_2^3a_1 - 5720a_4a_2a_1^3 + 86378a_3^5a_2 + 171186a_3^4a_2a_1 - 4971a_3^3a_2^3 + 114176a_3^3a_2a_1^2 - 8709a_3^2a_2^3a_1 + 28328a_3^2a_2a_1^3 + 11a_3a_2^5 - 5009a_3a_2^3a_1^2 - 2890a_3a_2a_1^4 + 11a_2^5a_1 - 1271a_2^3a_1^3 - 1850a_2a_1^5 = 0,$

$\eta_{10} = 8505a_6^3a_5 + 32481a_6^3a_3a_2 + 20493a_6^3a_2a_1 - 35685a_6^2a_5a_4 - 2491143a_6^2a_5a_3^2 - 1846200a_6^2a_5a_3a_1 + 5202a_6^2a_5a_2^2 - 392013a_6^2a_5a_1^2 + 20331a_6^2a_4a_3a_2 + 19215a_6^2a_4a_2a_1 - 3651435a_6^2a_3^3a_2 - 5823699a_6^2a_3^2a_2a_1 - 13158a_6^2a_3a_2^3 - 2905209a_6^2a_3a_2a_1^2 - 3978a_6^2a_2^3a_1 - 490737a_6^2a_2a_1^3 + 22635a_6a_5^3 - 923251a_6a_5^2a_3a_2 - 109487a_6a_5^2a_2a_1 - 88245a_6a_5a_4^2 - 1148694a_6a_5a_4a_3^2 - 2084784a_6a_5a_4a_3a_1 + 35712a_6a_5a_4a_2^2 - 856050a_6a_5a_4a_1^2 + 27375497a_6a_5a_3^4 + 29749976a_6a_5a_3^3a_1 - 1675251a_6a_5a_3^2a_2^2 + 12941638a_6a_5a_3^2a_1^2 - 1898806a_6a_5a_3a_2^2a_1 + 2013176a_6a_5a_3a_1^3 + 5118a_6a_5a_2^4 - 139343a_6a_5a_2^2a_1^2 - 77647a_6a_5a_1^4 - 5661a_6a_4^2a_3a_2 + 567a_6a_4^2a_2a_1 - 4039134a_6a_4a_3^3a_2 - 8624022a_6a_4a_3^2a_2a_1 - 28296a_6a_4a_3a_2^3 - 5779482a_6a_4a_3a_2a_1^2 - 17928a_6a_4a_2^3a_1 - 1337922a_6a_4a_2a_1^3 + 32956461a_6a_3^5a_2 + 69451805a_6a_3^4a_2a_1 - 355785a_6a_3^3a_2^3 + 53340790a_6a_3^3a_2a_1^2 - 959559a_6a_3^2a_2^3a_1 + 18868838a_6a_3^2a_2a_1^3 + 13654a_6a_3a_2^5 - 528319a_6a_3a_2^3a_1^2 + 2423661a_6a_3a_2a_1^4 + 9386a_6a_2^5a_1 + 35119a_6a_2^3a_1^3 - 95987a_6a_2a_1^5 - 6345a_5^3a_4 + 171030a_5^3a_3^2 + 7606a_5^3a_3a_1 + 34363a_5^3a_2^2 - 24592a_5^3a_1^2 - 263691a_5^2a_4a_3a_2 - 160567a_5^2a_4a_2a_1 + 8140468a_5^2a_3^3a_2 + 5952428a_5^2a_3^2a_2a_1 + 10649a_5^2a_3a_2^3 + 1622916a_5^2a_3a_2a_1^2 + 150019a_5^2a_2^3a_1 + 98116a_5^2a_2a_1^3 - 46575a_5a_4^3 - 233451a_5a_4^2a_3^2 - 1304640a_5a_4^2a_3a_1 - 59238a_5a_4^2a_2^2 - 735345a_5a_4^2a_1^2 + 7599099a_5a_4a_3^4 + 15008640a_5a_4a_3^3a_1 - 735755a_5a_4a_3^2a_2^2 + 8949810a_5a_4a_3^2a_1^2 - 769598a_5a_4a_3a_2^2a_1 + 781536a_5a_4a_3a_1^3 + 47372a_5a_4a_2^4 - 306087a_5a_4a_2^2a_1^2 - 511005a_5a_4a_1^4 - 51490240a_5a_3^6 - 65210418a_5a_3^5a_1 + 15852909a_5a_3^4a_2^2 - 33688314a_5a_3^4a_1^2 + 24816406a_5a_3^3a_2^2a_1 - 4238684a_5a_3^3a_1^3 - 31316a_5a_3^2a_2^4 + 12960556a_5a_3^2a_2^2a_1^2 + 2667284a_5a_3^2a_1^4 - 4188a_5a_3a_2^4a_1 + 3265586a_5a_3a_2^2a_1^3 + 738990a_5a_3a_1^5 - 409a_5a_2^6 + 178444a_5a_2^4a_1^2 + 273183a_5a_2^2a_1^4 - 33530a_5a_1^6 + 38529a_4^3a_3a_2 + 59805a_4^3a_2a_1 - 2189439a_4^2a_3^3a_2 - 5788647a_4^2a_3^2a_2a_1 - 153174a_4^2a_3a_2^3 - 4644549a_4^2a_3a_2a_1^2 - 176658a_4^2a_2^3a_1 - 1177245a_4^2a_2a_1^3 + 16481807a_4a_3^5a_2 + 45298967a_4a_3^4a_2a_1 + 258539a_4a_3^3a_2^3 + 43349882a_4a_3^3a_2a_1^2 +$

$802485a_4a_3^2a_2^3a_1 + 16307434a_4a_3^2a_2a_1^3 + 64444a_4a_3a_2^5 + 494565a_4a_3a_2^3a_1^2 + 621991a_4a_3a_2a_1^4 + 68712a_4a_2^5a_1 - 9045a_4a_2^3a_1^3 - 656465a_4a_2a_1^5 - 56836444a_3^7a_2 - 137004890a_3^6a_2a_1 + 7399835a_3^5a_2^3 - 125995776a_3^5a_2a_1^2 + 17206823a_3^4a_2^3a_1 - 53659078a_3^4a_2a_1^3 - 20406a_3^3a_2^5 + 14509150a_3^3a_2^3a_1^2 - 5092364a_3^3a_2a_1^4 - 137876a_3^2a_2^5a_1 + 6062318a_3^2a_2^3a_1^3 + 3623834a_3^2a_2a_1^5 - 409a_3a_2^7 - 54682a_3a_2^5a_1^2 + 1510631a_3a_2^3a_1^4 + 850920a_3a_2a_1^6 - 409a_2^7a_1 + 62788a_2^5a_1^3 + 150475a_2^3a_1^5 - 33530a_2a_1^7 = 0,$

$\eta_{12} = -4359825a_6^4a_5 + 11288295a_6^4a_3a_2 + 3464235a_6^4a_2a_1 - 50940900a_6^3a_5a_4 - 6945562548a_6^3a_5a_3^2 - 5752062648a_6^3a_5a_3a_1 + 53186760a_6^3a_5a_2^2 - 1198505412a_6^3a_5a_1^2 - 51965820a_6^3a_4a_3a_2 - 40312620a_6^3a_4a_2a_1 - 11936254248a_6^3a_3^3a_2 - 19272843756a_6^3a_3^2a_2a_1 + 56171520a_6^3a_3a_2^3 - 9928850400a_6^3a_3a_2a_1^2 + 54679140a_6^3a_2^3a_1 - 1663915212a_6^3a_2a_1^3 + 42421725a_6^2a_5^3 - 3476776041a_6^2a_5^2a_3a_2 - 307568637a_6^2a_5^2a_2a_1 - 211648950a_6^2a_5a_4^2 - 5050453356a_6^2a_5a_4a_3^2 - 8971009848a_6^2a_5a_4a_3a_1 + 175475664a_6^2a_5a_4a_2^2 - 3426207660a_6^2a_5a_4a_1^2 + 232769672430a_6^2a_5a_3^4 + 300261411984a_6^2a_5a_3^3a_1 - 6444303033a_6^2a_5a_3^2a_2^2 + 150900741252a_6^2a_5a_3^2a_1^2 - 6070706994a_6^2a_5a_3a_2^2a_1 + 33378826800a_6^2a_5a_3a_1^3 - 30021660a_6^2a_5a_2^4 + 39194943a_6^2a_5a_2^2a_1^2 + 2290962030a_6^2a_5a_1^4 - 269868510a_6^2a_4^2a_3a_2 - 248954310a_6^2a_4^2a_2a_1 - 20274509376a_6^2a_4a_3^3a_2 - 42737231844a_6^2a_4a_3^2a_2a_1 + 216460944a_6^2a_4a_3a_2^3 - 28812545928a_6^2a_4a_3a_2a_1^2 + 217574244a_6^2a_4a_2^3a_1 - 6523875540a_6^2a_4a_2a_1^3 + 305733906990a_6^2a_3^5a_2 + 680273558454a_6^2a_3^4a_2a_1 + 951543993a_6^2a_3^3a_2^3 + 571817691516a_6^2a_3^3a_2a_1^2 + 385382427a_6^2a_3^2a_2^3a_1 + 231205732812a_6^2a_3^2a_2a_1^3 + 3316620a_6^2a_3a_2^5 + 942010539a_6^2a_3a_2^3a_1^2 + 434471415\,90a_6^2a_3a_2a_1^4 - 13352520a_6^2a_2^5a_1 + 878524905a_6^2a_2^3a_1^3 + 2557552830a_6^2a_2a_1^5 - 17564850a_6a_5^3a_4 + 4968312318a_6a_5^3a_3^2 + 2538582192a_6a_5^3a_3a_1 + 170584740a_6a_5^3a_2^2 + 286506882a_6a_5^3a_1^2 - 113241555\,0a_6a_5^2a_4a_3a_2 - 21065526a_6a_5^2a_4a_2a_1 + 157615216462a_6a_5^2a_3^3a_2 + 126771659130a_6a_5^2a_3^2a_2a_1 + 1654454668a_6a_5^2a_3a_2^3 + 39470327658a_6a_5^2a_3a_2a_1^2 + 684935024a_6a_5^2a_2^3a_1 + 4374858830a_6a_5^2a_2a_1^3 - 202054500a_6a_5a_4^3 + 1032864372a_6a_5a_4^2a_3^2 - 7972386120a_6a_5a_4^2a_3a_1 - 170035560a_6a_5a_4^2a_2^2 - 4984887420a_6a_5a_4^2a_1^2 + 158504030916a_6a_5a_4a_3^4 + 315749523072a_6a_5a_4a_3^3a_1 - 10689551286a_6a_5a_4a_3^2a_2^2 + 220802905080a_6a_5a_4a_3^2a_1^2 - 3755220204a_6a_5a_4a_3a_2^2a_1 + 57606778560a_6a_5a_4a_3a_1^3 - 142789920a_6a_5a_4a_2^4 + 1056135258a_6a_5a_4a_2^2a_1^2 + 3012280260a_6a_5a_4a_1^4 - 1532611028400a_6a_5a_3^6 - 2377200210392a_6a_5a_3^5a_1 + 265487895254a_6a_5a_3^4a_2^2 - 1562329857704a_6a_5a_3^4a_1^2 + 459546704908a_6a_5a_3^3a_2^2a_1 - 484075509680a_6a_5a_3^3a_1^3 + 1766848280a_6a_5a_3^2a_2^4 + 244149470784a_6a_5a_3^2a_2^2a_1^2 - 38687873920a_6a_5a_3^2a_1^4 + 2597401120a_6a_5a_3a_2^4a_1 + 60734280628a_6a_5a_3a_2^2a_1^3 + 12165125000a_6a_5a_3a_1^5 - 15148140a_6a_5a_2^6 + 353791416a_6a_5a_2^4a_1^2 + 6176427434a_6a_5a_2^2a_1^4 + 1838576600a_6a_5a_1^6 - 196592940a_6a_4^3a_3a_2 - 212570460a_6a_4^3a_2a_1 - 18261988488a_6a_4^2a_3^3a_2 - 49933734228a_6a_4^2a_3^2a_2a_1 - 320371920a_6a_4^2a_3a_2^3 - 42477685200a_6a_4^2a_3a_2a_1^2 - 238117860a_6a_4^2a_2^3a_1 - 11647594260a_6a_4^2a_2a_1^3 + 336302396676a_6a_4a_3^5a_2 + 928808576148a_6a_4a_3^4a_2a_1 + 1332847134a_6a_4a_3^3a_2^3 + 952713443592a_6a_4a_3^3a_2a_1^2 + 9639030906a_6a_4a_3^2a_2^3a_1 + 449499505320a_6a_4a_3^2a_2a_1^3 + 28122600a_6a_4a_3a_2^5 + 11744615706a_6a_4a_3a_2^3a_1^2 +$

$8742365\,1060a_6a_4a_3a_2a_1^4 - 25763280a_6a_4a_2^5a_1 + 379668\,1854a_6a_4a_2^3a_1^3 +$
$2950523460a_6a_4a_2a_1^5 - 1779359020400a_6a_3^7a_2 - 462894349831\,2a_6a_3^6a_2a_1 +$
$82489756870a_6a_3^5a_2^3 - 4808931336496a_6a_3^5a_2a_1^2 + 225722736454a_6a_3^4a_2^3a_1 -$
$2571224760504a_6a_3^4a_2a_1^3 - 1195108748a_6a_3^3a_2^5 + 204398709564a_6a_3^3a_2^3a_1^2 -$
$677481143440a_6a_3^3a_2a_1^4 - 1489665700a_6a_3^2a_2^5a_1 + 76797599756a_6a_3^2a_2^3a_1^3 -$
$41204100920a_6a_3^2a_2a_1^5 - 41605700a_6a_3a_2^7 - 779036740a_6a_3a_2^5a_1^2 +$
$16230261662a_6a_3a_2^3a_1^4 + 15600225200a_6a_3a_2a_1^6 - 28376920a_6a_2^7a_1 -$
$351216668a_6a_2^5a_1^3 + 1860729486a_6a_2^3a_1^5 + 1972713400a_6a_2a_1^7 +$
$2903850a_5^5 + 754559226a_5^4a_3a_2 + 299936034a_5^4a_2a_1 - 24549075a_5^3a_4^2 +$
$3965660874a_5^3a_4a_3^2 + 3036111408a_5^3a_4a_3a_1 + 410561892a_5^3a_4a_2^2 +$
$367837110a_5^3a_4a_1^2 - 29130986708a_5^3a_3^4 - 18106696108a_5^3a_3^3a_1 +$
$8180222598a_5^3a_3^2a_2^2 - 90359328a_5^3a_3^2a_1^2 + 6166457100a_5^3a_3a_2^2a_1 +$
$2504507828a_5^3a_3a_1^3 - 142448821a_5^3a_2^4 + 1640958102a_5^3a_2^2a_1^2 +$
$423147148a_5^3a_1^4 - 173310705a_5^2a_4^2a_3a_2 - 87186213a_5^2a_4^2a_2a_1 +$
$55961159306a_5^2a_4a_3^3a_2 + 78410692974a_5^2a_4a_3^2a_2a_1 + 1895532836a_5^2a_4a_3a_2^3 +$
$36874292766a_5^2a_4a_3a_2a_1^2 + 2148151936a_5^2a_4a_2^3a_1 + 5334698794a_5^2a_4a_2a_1^3 -$
$666834432740a_5^2a_3^5a_2 - 792778940104a_5^2a_3^4a_2a_1 + 17668981886a_5^2a_3^3a_2^3 -$
$382093729740a_5^2a_3^3a_2a_1^2 + 17143847670a_5^2a_3^2a_2^3a_1 - 71261387484a_5^2a_3^2a_2a_1^3 -$
$118672559a_5^2a_3a_2^5 + 6305266362a_5^2a_3a_2^3a_1^2 + 3449452304a_5^2a_3a_2a_1^4 -$
$587596055a_5^2a_2^5a_1 + 1756826786a_5^2a_2^3a_1^3 + 1931197444a_5^2a_2a_1^5 -$
$121323825a_5a_4^4 - 288232020a_5a_4^3a_3^2 - 5764416840a_5a_4^3a_3a_1 -$
$274204656a_5a_4^3a_2^2 - 4056714900a_5a_4^3a_1^2 + 38504898126a_5a_4^2a_3^4 +$
$131107975920a_5a_4^2a_3^3a_1 + 1542506103a_5a_4^2a_3^2a_2^2 + 112371891300a_5a_4^2a_3^2a_1^2 +$
$3837349518a_5a_4^2a_3a_2^2a_1 + 25934141520a_5a_4^2a_3a_1^3 + 282911412a_5a_4^2a_2^4 +$
$1160977935a_5a_4^2a_2^2a_1^2 - 2340320850a_5a_4^2a_1^4 - 461642336592a_5a_4a_3^6 -$
$1063324269672a_5a_4a_3^5a_1 + 151157718498a_5a_4a_3^4a_2^2 - 903483925560a_5a_4a_3^4a_1^2 +$
$261810527876a_5a_4a_3^3a_2^2a_1 - 279336508560a_5a_4a_3^3a_1^3 - 16038616a_5a_4a_3^2a_2^4 +$
$189853992288a_5a_4a_3^2a_2^2a_1^2 + 22332382080a_5a_4a_3^2a_1^4 + 2649932272a_5a_4a_3a_2^4a_1 +$
$70310486844a_5a_4a_3a_2^2a_1^3 + 23688516600a_5a_4a_3a_1^5 - 179787196a_5a_4a_2^6 +$
$2566117560a_5a_4a_2^4a_1^2 + 10246748158a_5a_4a_2^2a_1^4 + 1302491400a_5a_4a_1^6 +$
$2376871863232a_5a_3^8 + 4041123600672a_5a_3^7a_1 - 1229061396268a_5a_3^6a_2^2 +$
$2984695617496a_5a_3^6a_1^2 - 2529842153540a_5a_3^5a_2^2a_1 + 982050494000a_5a_3^5a_1^3 +$
$10721745246a_5a_3^4a_2^4 - 1973424168172a_5a_3^4a_2^2a_1^2 - 26214495880a_5a_3^4a_1^4 +$
$24510969928a_5a_3^3a_2^4a_1 - 794735738616a_5a_3^3a_2^2a_1^3 - 109714908800a_5a_3^3a_1^5 -$
$26267783a_5a_3^2a_2^6 + 5050624836a_5a_3^2a_2^4a_1^2 - 155487802852a_5a_3^2a_2^2a_1^4 -$
$19402097400a_5a_3^2a_1^6 - 134836446a_5a_3a_2^6a_1 - 239861896\,8a_5a_3a_2^4a_1^3 -$
$5279995076a_5a_3a_2^2a_1^5 + 2125102000a_5a_3a_1^7 + 1302065a_5a_2^8 -$
$685129647a_5a_2^6a_1^2 + 1023502a_5a_2^4a_1^4 + 1893822844a_5a_2^2a_1^6 +$
$409431400a_5a_1^8 + 58437855a_4^4a_3a_2 + 103378275a_4^4a_2a_1 -$
$11651227920a_4^3a_3^3a_2 - 34206028380a_4^3a_3^2a_2a_1 - 779204736a_4^3a_3a_2^3 -$
$31191781320a_4^3a_3a_2a_1^2 - 905454756a_4^3a_2^3a_1 - 8988669900a_4^3a_2a_1^3 +$
$146268657486a_4^2a_3^5a_2 + 484451482806a_4^2a_3^4a_2a_1 + 12167472033a_4^2a_3^3a_2^3 +$
$566648845020a_4^2a_3^3a_2a_1^2 + 30395208819a_4^2a_3^2a_2^3a_1 + 274747102380a_4^2a_3^2a_2a_1^3 +$

$614296932a_4^2a_3a_2^5 + 22250068899a_4^2a_3a_2^3a_1^2 + 38934333270a_4^2a_3a_2a_1^4 +$
$697143312a_4^2a_2^5a_1 + 4429039473a_4^2a_2^3a_1^3 - 4569084450a_4^2a_2a_1^5 -$
$825535101712a_4a_3^7a_2 - 2682208744184a_4a_3^6a_2a_1 + 44826001186a_4a_3^5a_2^3 -$
$3368788658352a_4a_3^5a_2a_1^2 + 75409404898a_4a_3^4a_2^3a_1 - 1993616767720a_4a_3^4a_2a_1^3 -$
$45258142220a_4a_3^3a_2^5 + 51591640980a_4a_3^3a_2^3a_1^2 - 447320678960a_4a_3^3a_2a_1^4 -$
$9035897156a_4a_3^2a_2^5a_1 + 38874549380a_4a_3^2a_2^3a_1^3 + 55374298680a_4a_3^2a_2a_1^5 -$
$232702316a_4a_3a_2^7 - 4286482500a_4a_3a_2^5a_1^2 + 24261695786a_4a_3a_2^3a_1^4 +$
$33375532400a_4a_3a_2a_1^6 - 245931096a_4a_2^7a_1 + 90337316a_4a_2^5a_1^3 +$
$5133046074a_4a_2^3a_1^5 + 1410677800a_4a_2a_1^7 + 2585019162112a_3^9a_2 +$
$7321857833184a_3^8a_2a_1 - 557768982556a_3^7a_2^3 + 8521929383928a_3^7a_2a_1^2 -$
$1593379498376a_3^6a_2^3a_1 + 5184009078696a_3^6a_2a_1^3 + 2512827862a_3^5a_2^5 -$
$1778280508640a_3^5a_2^3a_1^2 + 1444847897160a_3^5a_2a_1^4 + 17683197138a_3^4a_2^5a_1 -$
$1044078708108a_3^4a_2^3a_1^3 - 80052078280a_3^4a_2a_1^5 - 10357705a_3^3a_2^7 +$
$19717403900a_3^3a_2^5a_1^2 - 374284078284a_3^3a_2^3a_1^4 - 151405331000a_3^3a_2a_1^6 +$
$342320721a_3^2a_2^7a_1 + 2630134084a_3^2a_2^5a_1^3 - 80038365296a_3^2a_2^3a_1^5 -$
$23406805800a_3^2a_2a_1^7 + 1302065a_3a_2^9 + 112696013a_3a_2^7a_1^2 -$
$2328777906a_3a_2^5a_1^4 - 6777229752a_3a_2^3a_1^6 + 2426347000a_3a_2a_1^8 +$
$1302065a_2^9a_1 - 239982413a_2^7a_1^3 - 411877366a_2^5a_1^5 + 385772548a_2^3a_1^7 +$
$409431400a_2a_1^9 = 0.$

5.2 The complete set of solutions of Example 18

This example comes from the bifurcation analysis of the following dynamical system with cubic non-linearity [15]-[16]

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 \end{cases}$$

The problem is to estimate a maximal number of the limit cycles which can be bifurcate out the critical point of a center type located in the origin. The interest to this particular problem derives from the investigation of Hilbert's sixteenth one, still unsolved. A part of the latter is estimation of the maximum possible number (Hilbert's number) H_n of limit cycles for polynomial-nonlinear dynamical systems of degree n

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases} \quad deg(P), deg(Q) \leq n.$$

General approach to the above problem [15]-[16] is based on the construction of Liapunov's function $V(x,y)$ such that $V > 0$ near the origin and

$$\dot{V} = \eta_2 r^2 + \eta_4 r^4 + \cdots \quad (r^2 = x^2 + y^2),$$

where the *focal values* η_{2k} are polynomials in the coefficients in p, q.

The origin is

- *a fine focus* of order k if $\eta_2 = \eta_4 = \cdots = \eta_{2k} = 0$ but $\eta_{2k+2} \neq 0$
- *a nonlinear center* if $\eta_2 = \eta_4 = \cdots = 0$

The algorithmic procedure implemented in Reduce [15] consists of the following successive computational steps

1. The calculation of the focal values η_{2i}.
2. The reduction of η_{2k} modulo $\eta_2 = \eta_4 = \cdots = \eta_{2k-2} = 0$. Then the construction of *Liapunov's quantities* $L_0, L_1, \ldots$ which are sequential non-zero reduced η_{2k}.
3. Verification of the maximum possible order k_{max} of the fine focus.
4. Introduction of appropriate perturbations to generate the small amplitude of the limiting cycles, and in doing so at most k limit cycles can bifurcate out of a fine focus of order k.

For the above cubic dynamical system one obtains non-zero focal values η_{2k} given in Example 18. This example is too large to be solved by constructing the Gröbner basis. Fortunately, it reveals non-trivial homogeneity properties, and, hence, allows effective homogeneity reduction to a set of small subsystems. Each of them can be easily solved. The complete set of solutions is listed in Table 1. There are three different solution subsets coinciding with those found in [16] by another method. Each of them is parametrized by one of the maximal independent sets modulo ideal generated by the initial polynomial expressions for η_{2i}.

Table 1

Parameters	Solutions
a_1, a_3, a_4, a_6	$a_2 = a_5 = 0$
a_2, a_4, a_6	$a_1 = a_3 = a_5 = 0$
a_1, a_2, a_3	$a_4 = a_3(a_1 + a_3)$ $a_5 = -a_2(a_1 + a_3)$ $a_6 = -a_3^2(a_1 + a_3)/(a_1 + 2a_3)$

6 Comparison with other packages

In this section we present results of the comparison in efficiency between the current version ASYS2 of the ASYS package and two other packages GROEBNER and CALI, both based on the Gröbner basis technique and included in Reduce 3.5. In the case of a lexicographical order we give also the timings for the previous version ASYS1 of the ASYS package. All the computations have been performed on the SPARC station *IPX* with the 32 Mb memory. As a collection of the polynomial systems that one presented in Sect.5 was selected. Many from those examples have been used [2],[6],[8],[10],[13],[14] as benchmarks for the Gröbner basis software.

6.1 Lexicographical order

Table 2 contains the timings for the Gröbner bases construction in a lexicographical order, which is the most informative one for solving the polynomial systems.

Table 2

	ASYS1	ASYS2	GROEBNER	CALI
Example 1	43.3"	7.2"	2.6"	5.7"
Example 2	0.3"	0.19"	0.14"	0.1"
Example 5	>600"	5.0"	24.0"	>14550"
Example 8	3.3"	0.35"	3.1"	1.0"
Example 9	>600"	11.3"	1908.3"	>8615"
Example 12	>600"	100.3"	16.5"	>7336"
Example 13	>600"	1.5"	37.0"	1.0"
Example 15	0.7"	0.3"	0.2"	0.3"
Example 16	>600"	10.8"	21.3"	2.9"
Example 17	30.1"	0.7"	1.5"	1.3"

6.2 Degree-reverse-lexicographical order

The timings for results of comparison between ASYS2, GROEBNER and CALI for a degree-reverse-lexicographical term ordering, being the best one from the complexity point of view [6], are presented in Table 3.

Table 3

	ASYS2	GROEBNER	CALI
Example 5	5.1"	2.6"	3.8"
Example 6	141.7"	75.3"	157.7"
Example 7	5.3"	9.4"	6.3"
Example 8	0.3"	0.2"	0.3"
Example 9	2.1"	1.5"	2.9"
Example 10	57.3"	50.9"	27.6"
Example 11	2.7"	2.7"	3.5"
Example 14	3.0"	1.4"	2.7"
Example 15	0.2"	0.2"	0.3"
Example 16	0.3"	0.2"	0.2"

6.3 Examples with non-trivial homogeneity properties

In Table 4 all the examples of Sect.5 possessing non-trivial homogeneity properties are selected. In these examples all the packages were used as solvers rather than the Gröbner basis constructors. Doing the computations with GROEBNER we applied the **solve** instruction of Reduce 3.5 which exploits the internal facilities of the GROEBNER package. Using CALI we called its procedure **groebfactor** which splits the initial problem into a reduced list of smaller ones. As for ASYS2, the timings of Table 4 are given for the complete computational process which is accomplished by construction of "triangular" output subsystems when the switches **scale** and **setgb** are on.

Table 4

	ASYS2	GROEBNER	CALI
Example 1	6.9"	4.9"	3.2"
Example 3	0.2"	1.4"	0.54"
Example 4	36.7"	91.6"	170.3"
Example 18	20.3"	25.5"	25.7"

One should note the following.

(i) The systems of Table 4 have infinitely many solutions. Hence, their lexicographical Gröbner bases have no "triangular" form which could reduce the multivariate problem to a chain of univariate ones.

(ii) The Gröbner bases for examples 4 and 18 are rather large to be used for direct analysis of solutions. So, the lexicographical basis for Example 4 occupies about 3 Mb of the output space [2]. That is why the splitting into smaller subsystems is most useful for practical solving such systems.

Acknowledgements

The authors are grateful to W.Lassner for numerous productive discussions, J.Apel, H.Caprasse, U.Klaus and H.Melenk for helpful remarks; H.-G.Gräbe for consultations on CALI and some important comments, N.G.Lloyd and J.M.Pearson for providing us with Example 18 and related references. This research was supported in part by the Russian Foundation for Fundamental Research (Project 93-02-144281).

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Received by Publishing Department
on December 28, 1993.

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ASYS2: Новая версия пакета программ ASYS
для анализа и упрощения полиномиальных систем

В настоящей работе описана новая версия пакета программ ASYS, базирующаяся на технике базисов Гребнера и предназначенная для анализа полиномиальных систем. В дополнение к первой версии пакета ASYS последняя версия снабжена рядом новых возможностей, обеспечивающих ее высокую эффективность. Рассмотрены некоторые примеры и результаты сравнения пакета ASYS с пакетами GROEBNER и CALI в системе REDUCE 3.5.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1993

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ASYS2: a New Version of Computer Algebra Package ASYS
for Analysis and Simplification of Polynomial Systems

In this paper a new version of a package ASYS for analysis of nonlinear algebraic equations based on the Grobner basis technique is described. In addition to the first version ASYS1 of the package, the current one has a number of new facilities which provide its higher efficiency. Some examples and results of comparison between ASYS2, ASYS1 and two other Reduce packages GROEBNER and CALI included in REDUCE 3.5, are given.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1993