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NUMERICAL ANALYSIS OF THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF A BOUNDARY PROBLEM FOR A NONLINEAR PARABOLIC EQUATION

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1 Introduction

One of the simplest models of evolution of nonlinear reaction-diffusion systems but with great variety of solutions and many interesting features is the problem

$$u_t = \Delta u^{\sigma+1} + u^{\beta}, \quad t > 0,\ x \in \Omega, \tag{1}$$
$$u(0,x) = u_0(x) \geq 0, \quad x \in \Omega, \tag{2}$$
$$u(t,x) = 0, \quad t \geq 0,\ x \in \partial\Omega, \tag{3}$$

where Ω is a bounded domain in $I\!R^N$ with smooth boundary, $\sigma > 0$, $\beta > 1$, $u_0 \in C(\bar{\Omega})$, $u_0^{\sigma+1} \in H_0^1(\Omega)$.

The solution of the Cauchy problem (1), (2), $\Omega \equiv I\!R^N$ is investigated in detail from theoretical and numerical point of view (see for example [1–8]). When $\beta < \sigma+1+2/N$ and $u_0 \not\equiv 0$ this problem has blow-up solution. The asymptotic behaviour of an wide class of solutions with various initial data is described by the self-similar solution of (1). But the presence of finite boundaries changes essentially the properties of the solution. When $\beta < \sigma + 1$ the problem (1)–(3) is known to have a global time solution [9, 10, 1]. There is a unique positive steady state solution of (1), (3) to which all solutions of (1)–(3) tend as $t \to \infty$ [1, 11].

For $\beta = \sigma + 1$ the behaviour of the solution of (1)–(3) is determined from the first eigenvalue λ_1 of the Dirichlet problem

$$-\Delta w = \lambda w,\ x \in \Omega, \quad w = 0,\ x \in \partial\Omega. \tag{4}$$

Let us denote by w_1 the corresponding eigenfunction. Then if $\lambda_1 > 1$ the problem (1)–(3) has a global time solution which tends to zero as $t \to \infty$ [1, 11]. For $\lambda_1 = 1$ the problem has a global time solution which tends to $\alpha_* w_1^{1/(\sigma+1)}$, $\alpha_* = (u_0, w_1)(\int_\Omega w_1^{(\sigma+2)/(\sigma+1)} dx)^{-1}$ as $t \to \infty$ [11]. When $\lambda_1 < 1$ the results from [9, 10] show that a local time solution exists but (1)– (3) has no nontrivial positive solution which exists for all time. It is proved in [11] that there exists $T_0 > 0$, $T_0 < \infty$ such that (1)–(3) has solution in $[0, T_0)$ and

$$\lim_{t \to T_0^-} \| u(t) \|_{L^\infty(\Omega)} = \infty.$$

For $\beta > \sigma + 1$ (and $\beta < (\sigma + 1)(N + 2)/(N - 2)$ for $N \geq 3$) a local time solution of (1)–(3) exists but it may or may not exist for all time — the existence of a global time solution depends on Ω and u_0 [1, 11, 12]. In particular $u \equiv 0$ is an asymptotically stable equilibrium solution of (1)–(3) while any positive equilibrium solution is unstable.

From now on we shall assume that Ω is a ball of radius X around the origin and we are only interested in the radially-symmetric solutions of (1)-(3), i.e

$$u_t = \frac{1}{r^{N-1}}(r^{N-1}u^\sigma u_r)_r + u^\beta, \quad t>0,\ 0<r<R, \tag{5}$$
$$u(0,r) = u_0(r) \geq 0, \quad 0 \leq r \leq R, \tag{6}$$
$$u_r(t,0) = 0, \quad t \geq 0, \tag{7}$$
$$u(t,R) = 0, \quad t \geq 0, \tag{8}$$

where $r = (\frac{1}{\sigma+1}\sum_{i=1}^N x_i^2)^{1/2}$, $R = (\frac{1}{\sigma+1})^{1/2}X$.

Some numerical illustrations of the cases of explosive growth, decreasing evolution and of the limiting case of approach the equilibrium solution for this problem may be found in [13].

In the presented work more detailed numerical analysis of the asymptotic behaviour of the solutions of (5)–(8) is done. In the case $\beta = \sigma + 1$ blow-up and global time self-similar solutions are sought. Some of them are obtained numerically. It is shown that they describe the asymptotic behaviour of an wide class of initial perturbations.

For $\beta > \sigma + 1$ the behaviour of the nontrivial equilibrium solution is investigated. It seems to be the limit between the blow-up and the global solutions. The results from the numerical experiments done here and in some other works [1, 4, 5, 6, 8] show that on the asymptotic stage an wide class of blow-up solutions evolve in consistency with the self-similar law for the Cauchy problem. For the global time solutions an hypothesis is done that approximate self-similar solutions describe their behaviour for large times.

2 Self-similar solutions for $\beta = \sigma + 1$

When Ω is a ball of radius X around the origin, the first eigenvalue and the first eigenfunction of the Dirichlet problem (4) are

$$\lambda_1 = \left(\frac{z^{(1)}_{(2-N)/2}}{X}\right)^2, \quad w_1 = z^{(2-N)/2} J_{(2-N)/2}(\sqrt{\lambda_1}\, z)$$

where $J_{(2-N)/2}(z)$ is the Bessel function of the first kind of order $(2-N)/2$, $z^{(1)}_{(2-N)/2}$ is the least positive root of $J_{(2-N)/2}(z)$, $z = \left(\sum_{i=1}^N x_i^2\right)^{1/2}$. Then the problem (5)–(8) will have steady state solution if

$$R = R_{st} = z^{(1)}_{(2-N)/2}(\sigma+1)^{-1/2}.$$

2.1 Blow-up self-similar solutions

If $R > R_{st}$ then $\lambda_1 < 1$ and the problem (5)–(8) has a solution which exists finite time. If as for the Cauchy problem [1, 2, 3, 4] blow-up self-similar solutions of the kind

$$u_{s,R}(t,r) = g(t)\theta_{s,R}(r), \quad g(0) = 1, \quad \theta'_{s,R}(0) = 0,\ \theta_{s,R}(R) = 0$$

are sought, then

$$g(t) = (1 - t/T_0)^{-1/\sigma}, \quad T_0 > 0,$$

and $\theta_{s,R}(r)$ satisfies

$$\begin{aligned} &\frac{1}{r^{N-1}}(r^{N-1}\theta^\sigma_{s,R}\theta'_{s,R})' - \frac{\theta_{s,R}}{T_0\sigma} + \theta^{\sigma+1}_{s,R} = 0, \quad 0<r<R, \\ &\theta'_{s,R}(0) = 0, \\ &\theta_{s,R}(R) = 0. \end{aligned} \tag{9}$$

It is known from [1] that the problem

$$\begin{aligned} &\frac{1}{r^{N-1}}(r^{N-1}|\theta|^\sigma\theta')' - \frac{1}{\sigma T_0}\theta + |\theta|^\sigma\theta = 0, \\ &\theta(0) = \mu, \\ &\theta'(0) = 0, \end{aligned} \tag{10}$$

has unique solution $\theta_\mu(r)$ for any $\mu > 0$ which is positive in $I\!R^+$ for small μ. For

$$\mu_s = \sup\{\mu^0 |\ \theta_\mu(r) > 0 \text{ in } I\!R^+ \text{ for } 0 < \mu < \mu^0\} < \infty$$

the solution $\theta_{\mu_s}(r)$ is the *eigenfunction* $\theta_s(r) \equiv \theta_{s,\infty}(r)$ for the Cauchy problem (5)–(7), $R = \infty$. It vanishes at the point r_s, $(|\theta_s|^\sigma\theta'_s)(r_s) = 0$ and therefore it can be extended as $\theta_s(r) = 0$ for $r > r_s$ [1]. But then $\theta_s(r)$ will be solution of (9) for any $R \geq r_s$ — i.e. $\theta_{s,R}(r) \equiv \theta_s(r)$ when $R \geq r_s$. Let us note that for $N = 1$ the explicit form of θ_s has been obtained in [2, 3, 1]. For $N > 1$ numerical methods for finding θ_s are developed in [4, 14, 15].

To investigate the solutions of the problem (10) it was transformed to a system of two ordinary differential equations (ODE) of first order for $v = \theta$ and $w = |\theta|^\sigma\theta'$

$$\begin{aligned} v' &= \frac{w}{|v|^\sigma}, & v(0) &= \mu, \\ w' &= -\frac{N-1}{r}w + \frac{1}{\sigma T_0}v - |v|^\sigma v, & w(0) &= 0. \end{aligned} \tag{11}$$

This system was solved numerically by a modification of an explicit Runge-Kutta method [16] which has second order of accuracy and an extended region of stability. The step Δr is chosen automatically so as to guarantee relative stability and a desired accuracy ε at the end of the interval. In our calculations $\varepsilon = 10^{-7}$ and the solution was calculated until $v < 10^{-7}$ or $\Delta r < 10^{-16}$. The numerical experiments done for $T_0 = 1/\sigma$ and $\mu > \mu_s$ show that $\theta_\mu(r)$ vanishes at $r_\mu < r_s$, $(|\theta_\mu|^\sigma\theta'_\mu)(r_\mu) \neq 0$. When we increase μ then r_μ decreases and tends to R_{st} when $\mu \to \infty$. The obtained $\theta_\mu(r)$ are solutions of (9) for $R = r_\mu > R_{st}$ and $T_0 = 1/\sigma$. But if (9) has a solution for a given $T_{0,1} > 0$ then it has solution for arbitrary $T_{0,2} > 0$. If by $\theta_{s,R}(r;T_{0,i})$, $i = 1,2$ the solutions, corresponding to the blow-up times $T_{0,i}$ are denoted, then [4, 6]

$$\theta_{s,R}(r;T_{0,2}) = \left(\frac{T_{0,1}}{T_{0,2}}\right)^{1/\sigma}\theta_{s,R}(r;T_{0,1}). \tag{12}$$

2.2 Global self-similar solutions

If $R < R_{st}$ then $\lambda_1 > 1$ and the problem (5)–(8) has a global time solution. If the possible self-similar solutions are of the kind

$$u_{s,R}(t,r) = g(t)f_{s,R}(r),\quad g(0) = 1,\quad f'_{s,R}(0) = 0,\ f_{s,R}(R) = 0$$

then

$$g(t) = (1 + t/T)^{-1/\sigma},\quad T > 0,$$

and

$$\begin{aligned} &\frac{1}{r^{N-1}}(r^{N-1}f^\sigma_{s,R}f'_{s,R})' + \frac{f_{s,R}}{T\sigma} + f^{\sigma+1}_{s,R} = 0, \quad 0 < r < R,\\ &f'_{s,R}(0) = 0,\\ &f_{s,R}(R) = 0. \end{aligned} \tag{13}$$

The problem

$$\begin{aligned} &\frac{1}{r^{N-1}}(r^{N-1}|f|^\sigma f')' + \frac{1}{\sigma T}f + |f|^\sigma f = 0,\\ &f(0) = \mu,\\ &f'(0) = 0, \end{aligned} \tag{14}$$

has unique solution $f_\mu(r)$ for any $\mu > 0$ which vanishes at some point $r_\mu < \infty$ and $(|f_\mu|^\sigma f'_\mu)(r_\mu) \neq 0$ [1]. The problem (14) was solved numerically by the same method as (10). The experiments show that when μ increases then r_μ increases also and tends to R_{st} when $\mu \to \infty$. In this way we find solutions of (13) when $R < R_{st}$, $T = 1/\sigma$. The solution $f_{s,R}$ for arbitrary T may be found replacing $\theta_{s,R}$ in (12) by $f_{s,R}$ and T_0 by T.

3 Asymptotic and structural stability of the self-similar solutions

In [1] *self-similar representation* $\theta(t,r)$ of the solution $u(t,r)$ with blow-up time $T_0 > 0$, $T_0 < \infty$ for the Cauchy problem is introduced

$$\theta(t,r) = (1 - t/T_0)^{1/\sigma}u(t,r),\quad t \in (0,T_0). \tag{15}$$

It is proved for $N = 1$ that if $u_0(r)$ has a finite support and nonincreases for $r > 0$ then

$$\theta(t,r) \to \theta_s(r),\quad t \to T_0^-$$

where $\theta_s(r)$ corresponds to the same blow-up time T_0, i.e. the self-similar solution $u_s(t,r) \equiv u_{s,\infty}(t,r)$ is asymptotically stable.

Usually the exact blow-up time is not known. In the numerical experiments the blow-up time can be found approximately, but a small change of T_0 produces a great change of $\theta(t,r)$ for t near T_0. That is why in [4, 6] the self-similar representation that does not use explicitly T_0 is introduced for the Cauchy problem

$$\theta(t,r) = \frac{\max_r \theta_s(r)}{\max_r u(t,r)}u(t,r), \tag{16}$$

and $u_s(t,r)$ is called *structurally stable* if there exists a class of initial perturbations $u_0(r) \neq \theta_s(r)$ whose self-similar representations $\theta(t,r)$ defined by (16) tend to $\theta_s(r)$ when $t \to T_0^-$. These definitions may be extended for the boundary problem replacing θ_s by $\theta_{s,R}$.

The numerical experiments (see for example [2, 3, 6, 8]) show structural stability of $u_s(t,r)$ for $N = 1,2,3$ and arbitrary initial data. In [7] some kind of self-similar representation is not used but it is also asserted that the numerical solutions with various initial data approach the self-similar solution asymptotically in the case $N = 1$.

For $N = 1$ the localization of the solution of (1), (2), $\Omega = \mathbb{R}$ with an arbitrary initial condition with a finite connected support is proved in [1]. Moreover if by L_s the so called *fundamental length* (L_s depends only on the medium parameters – $L_s = 2\pi(\sigma+1)\sigma^{-1}$ for $N = 1$, $L_s = mes\,supp\,\theta_s(x)$ for $N > 1$), by $u_s(t,x;x_0;T_0)$ the self-similar solution symmetric about x_0 with blow-up time T_0, and by $T_0(u_0)$ the blow-up time of the solution with initial data $u_0(x)$ are denoted, then

1. if the initial condition $u_0(x)$, $u_0(-x) = u_0(x)$, is nonincreasing for $x > 0$; if $mes\,supp\,u_0 < L_s$; if there exists T_0 such that $u_s(0,x;0;T_0) \geq u_0(x)$ in $\mathbb{R}$ and $u_0(x)$ intersects $u_s(0,x;0;T)$ for all $T > T_0$ in two points then

   $$mes\,supp\,u(t,x) \leq L_s \quad \text{for } t \in (0, T_0(u_0)).$$

   In this case the Cauchy problem (1), (2) with $\Omega = \mathbb{R}$ and the boundary problem (1)–(3) with Ω – symmetric about the origin, $mes\,\Omega \geq L_s$ are equivalent.

2. if $supp\, u(t,x) = (h_-(t), h_+(t))$, $x_0 = h_+(0) - L_s/2$; if $mes\, supp\, u_0 > L_s$; if there exists T_0 such that $u_s(0,x;x_0;T_0) \le u_0(x)$ in $I\!R$ and $u_0(x)$ intersects $u_s(0,x;x_0;T)$ for all $0 < T < T_0$ in one point then

$$h_+(t) \equiv h_+(0) \quad \text{for } t \in (0, T_0(u_0)).$$

The conditions for inmobility of the left front point are analogous and hence in this case the Cauchy problem (1), (2) with $\Omega = I\!R$ and the boundary problem (1)–(3) with $\Omega \supset supp\, u_0(x)$ are equivalent.

The numerical experiments done with various initial data confirm these assertions in the radially-symmetric case for $N = 2,\ 3$.

Therefore for the boundary problem with $mes\,\Omega \ge L_s$ the self-similar solution $u_{s,R}(t,r) \equiv u_s(t,r)$ will be asymptotically and structurally stable for wide class of initial perturbations. That is why in this work the case when $mes\,\Omega < L_s$ is considered — then we may expect a different behaviour of the solution on the asymptotic stage. Below we show numerically the structural stability of the self-similar solutions when $mes\,\Omega < L_s$. Also if the initial perturbation $u_0(r)$ is the solution $\theta_{s,R}(r;T_0)$ of (9), corresponding to the blow-up time T_0, the numerical solution blows-up for $t \approx T_0$, i.e. the blow-up time is restored in the calculations.

For global time solutions the self-similar representation may be defined as

$$f(t,r) = (1 + t/T)^{1/\sigma} u(t,r), \quad t > 0 \tag{17}$$

and then the self-similar solution corresponding to $f_{s,R}(r)$ will be called asymptotically stable if for some class of initial perturbations $u_0(r) \not\equiv f_{s,R}(r)$

$$\lim_{t\to\infty} \| f(t,r) - f_{s,R}(r) \| = 0,$$

In this case the choice of T is not essential — if

$$f(t,r;T_1) \equiv (1 + t/T_1)^{1/\sigma} u(t,r) \to f_{s,R}(r;T_1), \quad t \to \infty$$

then

$$\begin{aligned} f(t,r;T_2) \equiv\ & (1 + t/T_2)^{1/\sigma} u(t,r) = \\ & \left(\frac{1+T_2/t}{1+T_1/t}\right)^{1/\sigma} \left(\frac{T_1}{T_2}\right)^{1/\sigma} (1 + t/T_1)^{1/\sigma} u(t,r) \to \\ & \left(\frac{T_1}{T_2}\right)^{1/\sigma} f_{s,R}(r;T_1) \equiv f_{s,r}(r;T_2), \quad t \to \infty. \end{aligned}$$

The numerical experiments show the asymptotical stability of the global time self-similar solutions. If T is chosen in an appropriate way suggested in [4]

$$T = T_n = \frac{t_n \max\limits_r u(t_n,r)^\sigma - t_{n+1} \max\limits_r u(t_{n+1},r)^\sigma}{\max\limits_r u(t_{n+1},r)^\sigma - \max\limits_r u(t_n,r)^\sigma} \tag{18}$$

where $0 < t_1 < t_2 < \ldots < t_n < \ldots < t_{n+k}$, n is chosen so that $T_n \approx T_{n+i}$, $i = 1,2,\ldots,k$, then $f(t,r)$ is close to $f_{s,R}(r)$ for $t \ll \infty$.

The problem (5)–(8) was solved numerically by the method described in [8]. For discretization in space we use the lumped mass finite element method [17] with interpolation [18, 19] of the nonlinear terms $G(u) = \int_0^u w^\sigma dw = u^{\sigma+1}/(\sigma+1)$ and $f(u) = u^\beta$ on the same basis, as for the solution. Linear finite elements (in the calculations their number was between 30 and 60) on uniform and nonuniform grids are used. The obtained system of ODE was solved numerically in time by the modification of Runge-Kutta method [16] described in Section 2. The prescribed accuracy for solving the system was $\varepsilon = 10^{-3}$ and the stop criterion was $\tau < 10^{-16}$, τ beeing the time step.

Example 1. The evolution in time of $\theta_{s,R} \equiv \theta_\mu(r)$ for $\sigma = 1.5$, $N = 3$, $\mu = 3$, $R = r_\mu = 2.359628$ is illustrated on Fig.1.a. The blow-up time obtained in calculations was $\tilde{T}_0 = 0.664230$, the blow-up time of the exact self-similar solution is $T_0 = 1/\sigma = 0.666667$. Taking into account the approximations made when solving the problems (11) and (5)–(8) we think that this is a good restoration of the blow-up time. As it is seen on Fig.1.b the all of the self-similar representations (16) coincide within the plotting resolution. This allows us to think that the self-similar solution is structurally stable.

Example 2. Fig.2.a shows the evolution of the non-self-similar initial data

$$u_0(r) = \begin{cases} 1 - r, & r \le 1 \\ 0, & r > 1 \end{cases}$$

for $\sigma = 2$, $N = 2$, $R = 1.713262 = r_\mu$, $\mu = 2$. The corresponding self-similar representations $\theta(t,r)$ are shown on Fig.2.b. As it is seen they tend to $\theta_{s,R}(r) \equiv \theta_\mu(r)$ which is also drawn and signed with $\times$ on Fig.2.a, i.e. the self-similar solution $u_{s,R}$ is structurally stable.

Example 3. The evolution of the global time self-similar initial data $u_0(r) = f_{s,R}(r) = f_\mu(r)$ for $\sigma = 3$, $N = 3$, $\mu = 1$, $R = r_\mu = 0.988611$ is shown on Fig.3.a. The corresponding self-similar representations $f(t,r;T)$, $T = 1/\sigma = 0.(3)$ are shown on Fig.3.b and as it is seen they coincide for all time, i.e. the self-similar solution is asymptotically stable.

Example 4. Fig.4.a shows the evolution of the non-self-similar initial data

$$u_0(r) = \begin{cases} 2r, & r \le 1/2 \\ 2 - 2r, & 1/2 < r \le 1 \\ 0 & r > 1 \end{cases}$$

for $\sigma = 2$, $N = 3$, $R = 1.547856 = r_\mu$, $\mu = 2$. Fig.4.b illustrates the asymptotical stability of the self-similar solution corresponding to $f_{s,R}(r;T)$, $T = T_n = 17.6712$. The finction $f_{s,R}(r;T)$ is signed with $\times$.

Example 5. An illustration of the case $\lambda_1 = 1$ for $\sigma = 2$, $N = 2$, is given on

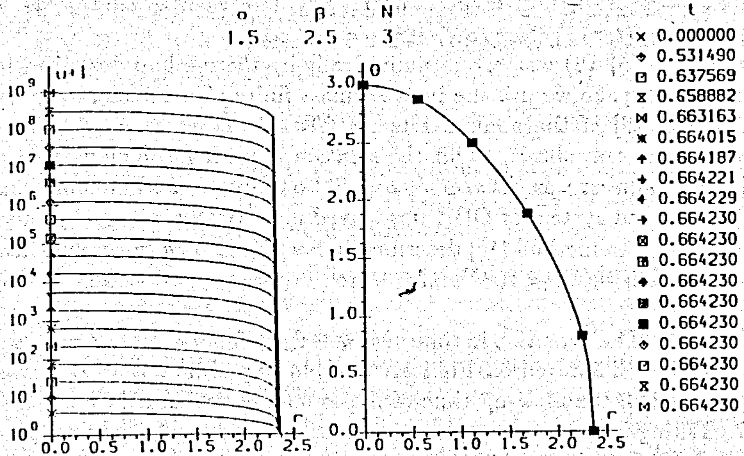

Fig.1.a,b

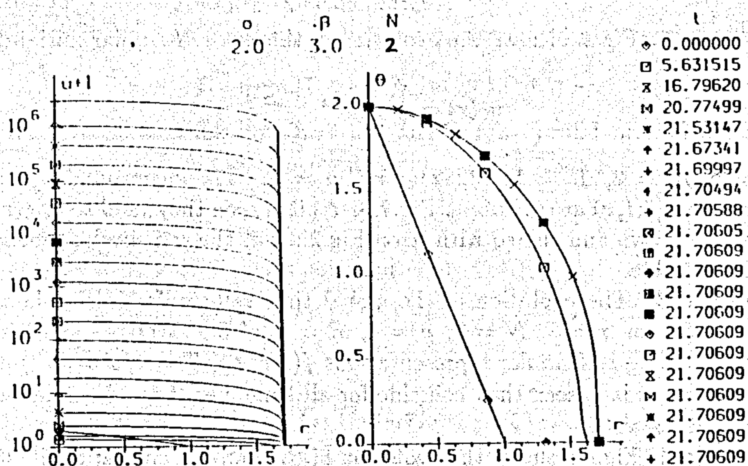

Fig.2.a,b

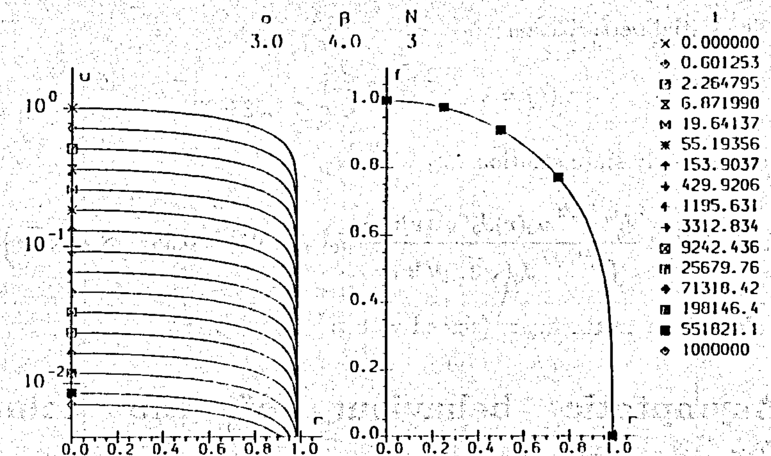

Fig.3.a,b

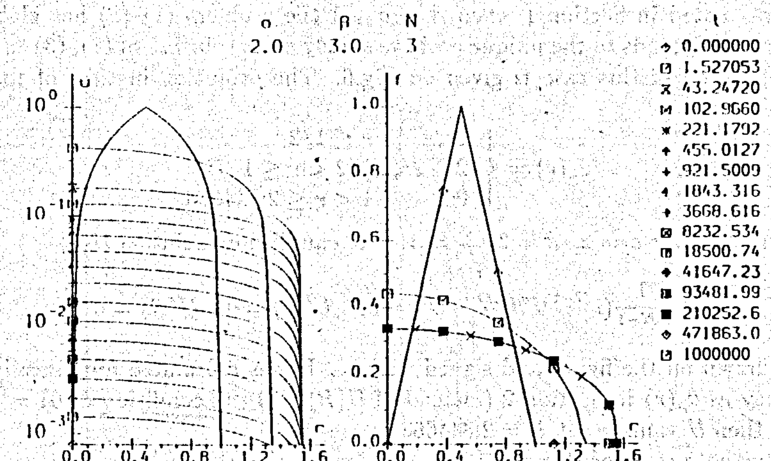

Fig.4.a,b

Fig.5. The initial perturbation

$$u_0(r) = \begin{cases} 1-r, & r \le 1 \\ 0, & 1 < r \le 1.388427 \end{cases}$$

tends to the steady state solution

$$\alpha_* w_1^{1/(\sigma+1)} = \frac{\int_0^{z_0^{(1)}/\sqrt{3}} u_0(r) J_0(\sqrt{3}\,r) r\, dr}{\int_0^{z_0^{(1)}/\sqrt{3}} J_0(\sqrt{3}\,r)^{4/3} r\, dr} J_0(\sqrt{3}\,r)^{1/3} = 0.381228\, J_0(\sqrt{3}\,r)^{1/3}$$

that is also drawn on the figure (signed with $\times$).

4 Asymptotic behaviour of the solutions for $\beta \neq \sigma + 1$

4.1 The case $\beta < \sigma + 1$

As it was noted in Section 1 when $\beta < \sigma + 1$ the problem (1)–(3) has global time solution which tends to the unique positive steady state solution of (1), (3) as $t \to \infty$. An illustration of this case is given on Fig.6. The evolution in time of the initial perturbation

$$u_0(r) = \begin{cases} 2r, & r \le 1/2 \\ 2-2r, & 1/2 < r \le 1 \\ 0 & 1 < r \le 2.344663 \end{cases}$$

is shown there for $\sigma = 2$, $\beta = 2$, $N = 3$. The equilibrium solution $U(r)$

$$\frac{1}{r^{N-1}} (r^{N-1} U^\sigma U')' + U^\beta = 0, \quad U'(0) = 0, \quad U(R) = 0$$

is also drawn on the figure and signed with $\times$. It was calculated numerically in the same way as $\theta_\mu(r)$ in Section 2 (instead of $U(R) = 0$ the condition $U(0) = 1$ is put on and then U vanishes at $R = 2.344663$).

Let us note that for $\beta \neq \sigma + 1$

$$U(r; R_2) = \alpha U(\alpha^m r; R_1)$$

where $U(r; R_i)$ is the equilibrium solution corresponding to boundary condition $U(R_i) = 0$, $i = 1, 2$, $m = (\beta - \sigma - 1)/2$, $\alpha = (R_1/R_2)^{1/m}$ [1].

4.2 The case $\beta > \sigma + 1$

For $\beta > \sigma + 1$ also a unique positive equilibrium solution of (5), (8) exists (see [1, 11, 12] and the references cited therein). It was calculated in the same way as for

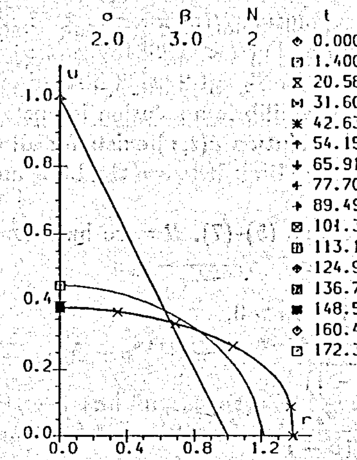

Fig.5

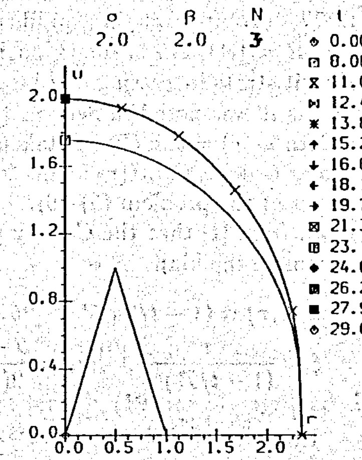

Fig.6

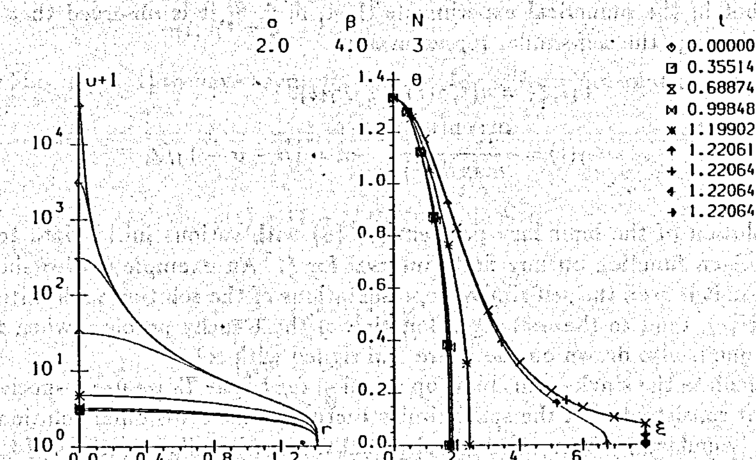

Fig.7.a,b

$\beta < \sigma + 1$. The evolution of the equilibrium solution $U(r)$ for $\sigma = 2$, $\beta = 4$, $N = 3$, $U(0) = 2$, $R = 1.415219$ is shown on Fig.7.a. For $t < 0.998489$ $u(t,r) \approx u(0,r)$ but after that it starts increasing and blows-up for $T_0 = 1.220638$ instead of existing for all time. As it was noted in Section 1 the equilibrium solution is unstable. But if as initial data $u_0(r) = 0.99U(r)$ is taken the solution $u(t,r)$ exists for all time and tends to zero as $t \to \infty$, i.e. $U(r)$ seems to be the limit between the blow-up and the global solutions of the problem (5)–(8).

It is known from [1] that the Cauchy problem (5)–(7), $R = \infty$ has blow-up self-similar solutions of the kind

$$\begin{aligned}
&u_s(t,r) = (1 - t/T_0)^{-1/(\beta-1)}\theta_s(\xi), \quad T_0 > 0,\\
&\xi = \frac{r}{(1-t/T_0)^{m_1}}, \quad m_1 = \frac{\beta-\sigma-1}{2(\beta-1)},\\
&\frac{1}{\xi^{N-1}}(\xi^{N-1}\theta_s^\sigma\theta_s')' - \frac{m_1}{T_0}\xi\theta_s' - \frac{\theta_s}{T_0(\beta-1)} + \theta_s^\beta = 0, \quad 0 < \xi < \infty,\\
&\theta_s'(0) = 0, \quad \theta_s(\infty) = 0.
\end{aligned}$$

They are not self-similar solutions for the boundary problem because $\theta_s(\xi) > 0$, $0 < \xi < \infty$ but in the numerical experiments [1, 4, 5, 6, 8] it is observed that on the asymptotic stage the self-similar representation

$$\begin{aligned}
&\theta(t,\xi) = u(t, \gamma(t)^{-m}\xi)/\gamma(t),\\
&\gamma(t) = \frac{\max_r u(t,r)}{\max_\xi \theta_s(\xi)}, \quad m = (\beta - \sigma - 1)/2,
\end{aligned}$$

of the solution of the boundary problem (5)–(8) with various initial data tends to the first eigen function on any finite interval for ξ. An example is also shown on Fig.7.b. As it is seen the self-similar representations of the solution with initial data $u_0(r) = U(r)$, tend to the first eigen function of the Cauchy problem when $t \to T_0$ (the last one is also drawn on the figure and signed with $\times$).

To calculate the single-point blow-up solution for t near T_0 we use a special mesh refinement consistent with the space-time structure of the self-similar solution [8].

If the boundary problem (5)–(8) had global time self-similar solutions of the kind

$$u_{s,R}(t,r) = g(t)f_{s,R}(\xi), \quad g(0) = 1, \quad \xi = r\varphi(t), \quad \varphi(0) = 1, \quad u_{s,R}(R) = 0,$$

the functions $g(t)$ and $\varphi(t)$ should be

$$\begin{aligned}
&g(t) = (1 + t/T)^{-1/(\beta-1)}, \quad T > 0,\\
&\varphi(t) = (1 + t/T)^{-m_1}, \quad m_1 = \frac{\beta-\sigma-1}{2(\beta-1)}.
\end{aligned}$$

But if $f_{s,R}(\xi_0) > 0$ for some $\xi_0 > 0$, $\xi_0 < R$ then the value of $u_{s,R}(t_0,R) = g(t_0)f_s(\xi_0) > 0$ for $t_0 = T((R/\xi_0)^{1/m_1} - 1) > 0$ would not satisfy the boundary condition (8). Hence

the boundary problem has not self-similar solutions of this type. Because of the boundary condition (8) and the numerical experiments done it may be assumed that for large times the solutions of the boundary problem will evolve in consistency with some approximate self-similar solution of the kind

$$u_a(t,r) = g(t)f_a(r), \quad g(0) = 1, \quad f_a'(0) = 0, \quad f_a(R) = 0.$$

Assuming that

$$u_{a,t} - \frac{1}{r^{N-1}}(r^{N-1}u_a^\sigma u_{a,r})_r - u_a^\beta \to 0, \quad t \to \infty$$

we get

$$g'f_a - g^{\sigma+1}[\frac{1}{r^{N-1}}(r^{N-1}f_a^\sigma f_a')' + g^{\beta-\sigma-1}f_a^\beta] \to 0, \quad t \to \infty. \tag{19}$$

When $\beta > \sigma + 1$ and $\lim_{t\to\infty} g(t) = 0$ the last term in (19) may be ignored. But then $u_a(t,r)$ might be the self-similar solution of the equation without source

$$\begin{aligned}
&v_t = \frac{1}{r^{N-1}}(r^{N-1}v^\sigma v_r)_r, \quad t > 0,\ 0 < r < R,\\
&v_r(t,0) = 0, \quad t \geq 0,\\
&v(t,R) = 0, \quad t \geq 0.
\end{aligned}$$

which is known [20, 1] to have asymptotically stable self-similar solutions of the kind

$$u_a(t,r) = g(t)f_a(r),$$

where

$$g(t) = (1 + t/T)^{-1/\sigma}, \quad T > 0,$$

and

$$\begin{aligned}
&\frac{1}{r^{N-1}}(r^{N-1}f_a^\sigma f_a')' + \frac{1}{\sigma T}f_a = 0,\\
&f_a'(0) = 0,\\
&f_a(R) = 0.
\end{aligned} \tag{20}$$

Two examples, confirming that for large times the evolution of the global time solutions of the boundary problem (5)–(8), $\beta > \sigma+1$, is in consistency with the self-similar law for the corresponding problem without source, are given below. Let us note this assumption was prompted by the fact that for the Cauchy problem (5)–(7), $R = \infty$, $\beta > \sigma + 1 + 2/N$, the asymptotic behaviour of an wide class of global time solutions is described by the self-similar solution (in this case with a constant energy) of the corresponding problem without source. [1].

Example 1. The evolution in time of the initial data

$$u_0(r) = \begin{cases} (1-r)/2, & r \leq 1 \\ 0 & 1 < r \leq 1.226252 \end{cases}$$

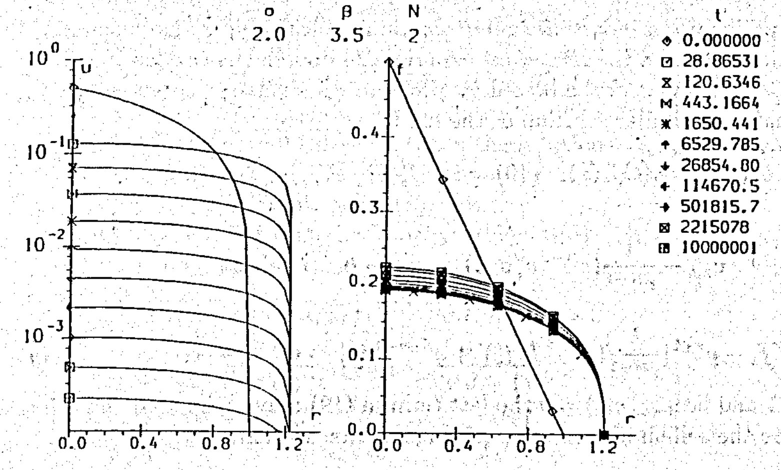

Fig.8.a,b

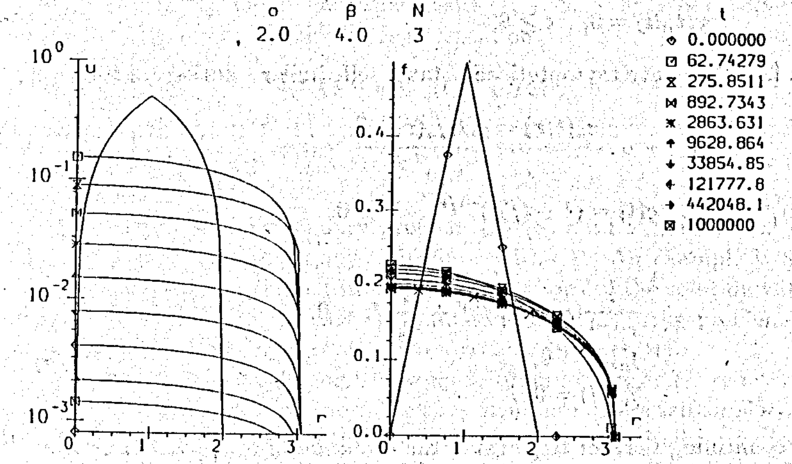

Fig.9.a,b

for $\sigma = 2$, $\beta = 3.5$, $N = 2$, $R = 1.226252$ is shown on Fig.8.a. On Fig.8.b the corresponding self-similar representations

$$f(t,r) = (1 + t/T)^{1/\sigma} u(t,r), \quad t > 0 \tag{21}$$

with $T = 13.6$ (determined from (18)) are drawn and as it is seen they tend to the solution $f_a(r;T)$ of (20) that is also drawn on the figure and signed with $\times$.

Example 2. Fig.9.a illustrates the evolution in time of the initial data

$$u_0(r) = \begin{cases} r/2, & r \le 1 \\ (1-r)/2, & 1 < r \le 2 \\ 0, & 1 < r \le 3.052734 \end{cases}$$

for $\sigma = 2$, $\beta = 4$, $N = 3$, $R = 3.052734$. The corresponding self-similar representations (21), $T = 53.2$ drawn on Fig.9.b tend to $f_a(r;T)$ (signed with $\times$).

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References

[1] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov and A.P. Mikhailov, *Blow-up in Problems for Quasilinear Parabolic Equations*, Nauka, Moscow (1987). (English Trans. Walter de Gruyter, Berlin)

[2] N.V. Zmitrenko, S.P. Kurdyumov, A.P. Mikhailov and A.A. Samarskii, Generation of structures in nonlinear media and nonstationary thermodynamics of blow-up regimes, Preprint, the Keldysh Inst. of Appl. Math., Acad. Sci. USSR, Moscow, 74 (1976).

[3] A.A. Samarskii, N.V. Zmitrenko, S.P. Kurdyumov and A.P. Mikhailov, Thermal structures and fundamental length in a medium with non-linear heat conduction and bulk heat sources, *Dokl. Akad. Nauk SSSR*, **227** (1976) 321-324.

[4] G.G. Elenin and S.P. Kurdyumov, Conditions for the complexity of the organization of a nonlinear dissipative medium, Preprint, the Keldysh Inst. of Appl. Math., Acad. Sci. USSR, Moscow, 106 (1977).

[5] A.A. Samarskii, G.G. Elenin, N.V. Zmitrenko, S.P. Kurdyumov and A.P. Mikhailov, Combustion of a nonlinear medium in the form of complex structures, *Dokl. Akad. Nauk SSSR*, **237** (1977) 1330-1333.

[6] G.G. Elenin, S.P. Kurdyumov and A.A. Samarskii, Nonstationary dissipative structures in a nonlinear heat-conducting medium, *Zh. Vychisl. Mat. Mat. Fiz.*, **23** (1983) 380-390.

[7] H. Wilhelmsson, M. Benda, B. Etlicher, R. Jancel and T. Lehner, Nonlinear evolution of the densities in the presence of simultaneous diffusion and reaction processes, *Physica Scripta*, **38** (1988) 863-874.

[8] S.N. DIMOVA AND D.P. VASILEVA, Lumped mass finite element method with interpolation of the nonlinear coefficients for a quasilinear singular parabolic equation, submitted to *Num. Meth. PDE.*

[9] V.A. GALACTIONOV, A boundary problem for the nonlinear parabolic equation $u_t = \Delta u^{\sigma+1} + u^{\beta}$, *Differ. Equat.*, **17** (1981) 836-842.

[10] H.A. LEVINE, P.E. SACKS, Some existence and nonexistence theorems for solutions of degenerate parabolic equations, *J. Differ. Equat.*, **52** (1984) 135-161.

[11] P.E. SACKS, Global behavior for a class of nonlinear evolution equations, *SIAM J. Math. Anal.*, **16** (1985) 233-250.

[12] V.A. GALACTIONOV, Nonexistence and existence of global solutions of boundary value problems for quasilinear parabolic equations, *Zh. Vychisl. Mat. Mat. Fiz.*, **22** (1982) 1369-1385.

[13] M.N. LE ROUX AND H. WILHELMSSON, External boundary effects on simultaneous diffusion and reaction processes, *Physica Scripta*, **40** (1989) 674-681.

[14] S.P. KURDYUMOV, E.S. KURKINA, G.G. MALINETSKII AND A.A. SAMARSKII, Dissipative structures in nonhomogenious nonlinear burning medium, *Dokl. Akad. Nauk SSSR*, **251** (1980) 587-591.

[15] S.N. DIMOVA, M.S. KASCHIEV AND S.P. KURDYUMOV, Numerical analysis of eigenfunctions for combustion of a nonlinear medium in the radial-symmetric case, *USSR Comput. Maths. Math. Phys.*, **29** (1989) 61-73.

[16] V.A. NOVIKOV AND E.A. NOVIKOV, Stability control of explicit onestep methods for integration of ordinary differential equations, *Dokl. Akad. Nauk SSSR*, **272** (1984) 1058-1062.

[17] V. THOMEE, Galerkin finite elements methods for parabolic equations, *Lect. Notes Math.* **1054**, Springer Verlag, New York, etc. (1984).

[18] I. CHRISTIE, D.F. GRIFFITHS, A.R. MITCHEL AND J.M. SANZ-SERNA, Product approximation for nonlinear problems in the finite element method, *IMA J. Numer. Analys.*, **1** (1981) 153-266.

[19] C.M. CHEN, S. LARSSON AND N.-Y. ZHANG, Error estimates of optimal order for finite element methods with interpolated coefficients for the nonlinear heat equation, *IMA J. Numer. Analys.*, **9** (1989) 507-524.

[20] D.G. ARONSON AND L.A. PELETIER, Large time behaviour of solutions of the porous medium equation on bounded domains, *J. Differ. Equat.*, **39** (1981) 378-412.

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