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дубна

E11-93-163

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COST-EFFECTIVE COMPUTATIONS WITH BOUNDARY INTERFACE OPERATORS IN ELLIPTIC PROBLEMS

1993

1 Introduction

The domain decomposition techniques result in the efficient elliptic problem solvers with high serial and parallel performance. Efficiency of the iterative substructuring algorithms strongly depends on the complexity of the residual computations for a chosen interface operator. In fact, the typical case expense for the residual computations is of the order of $O(N^{d/(d-1)}logN)$ for d-dimensional problems, while the costs of the preconditioners treatment are usually of the order of $O(Nlog^qN)$ with $q \leq 2$, see e.g. [4], [8], [9], [11], [12]. Here N is the dimension of the discrete interface problem. But in the special case of elliptic boundary value problems (BVPs) with piecewise constant coefficients the computation expenditures taken over the interface operators can be essentially reduced. Such a decrease is essential to enable fast computations with the classical boundary integral operators via domain decomposition [10].

In this paper the numerical algorithm for fast computations with the Poincaré – Steklov (PS) interface operators associated with the elliptic BVPs defined on step-type domains, is presented. It is based on the asymptotically almost optimal techniques developed for treatment of the discrete PS operators on the rectangle boundary associated with the finite-difference (FD) Laplacian on the uniform grid with a "displacement by $h/2$". The approach can be regarded as an extension of the method for the partial solution of the FD Laplace equation proposed in [3] to the cases of displaced grids and mixed boundary conditions. It is shown that the action of the PS operator for the Dirichlet or mixed BVPs can be computed with expense of the order of $O(Nlog^2N)$ both for arithmetical operations and computer memory needs, where N is the number of unknowns on the rectangle boundary. Note that the corresponding computing complexity for the panel clustering techniques in the boundary element method developed in [6] is estimated by the quantity $O(Nlog^{d+2}N)$. The single domain algorithm is applied to solving the multidomain interface problems with piecewise constant coefficients. The algorithm proposed can be extended to the case of 3D elliptic BVPs, including the case of stretched geometries [11], [7], as well as to the finite element (FE) approximation of PS operator [1]. Note that the boundary element Galerkin discretization of the PS operators have been developed in [14].

The remainder of the paper is organized as follows. In Section 2 we describe the discrete PS operator defined for the finite-difference BVP for the Laplacian on shifted grids in a rectangular domain. In Section 3 the method for the fast matrix-vector multiplications of the discrete PS operator is developed. The extensions to the case of mixed boundary conditions are considered in Section 4. In Section 5 we present the numerical results characterizing the efficiency of the algorithm for a single domain. Finally, in Section 6 we outline the results of numerical experiments when solving the interface problem arising in the iterative substructuring techniques for elliptic BVP. The numerical experiments confirm the almost linear growth of computational costs with respect to the dimension of the discrete interface problem.

Acknowledgments. The authors express their sincere gratitude to Prof. Dr. W.L. Wendland and Dr. G. Schmidt for stimulating and helpful discussions of the problem under consideration.

2 The discretization of the inverse Poincaré-Steklov operator

Let $\overline{\Omega} = [0, A] \times [0, B] \in I\!R^2$ be a rectangular domain with a Lipschitz boundary $\partial\Omega$ partitioned into two nonoverlapping parts $\tilde{\Gamma}_D$ and $\tilde{\Gamma}_N$, $\partial\Omega = \tilde{\Gamma}_D \cup \tilde{\Gamma}_N$, where $\tilde{\Gamma}_D$ and $\tilde{\Gamma}_N$ are supposed to be a union of any edges $\tilde{\Gamma}_i$, $i = 1 \div 4$, of the rectangle $\overline{\Omega}$, see Figure 1. Consider the following mixed boundary value problem:
given $\tilde{\varphi} \in H^{\frac{1}{2}}(\tilde{\Gamma}_N)$ and $\tilde{\psi} \in H^{-\frac{1}{2}}(\tilde{\Gamma}_D)$, find $\tilde{u} \in H^1(\Omega) \cap C^2(\Omega)$ such that

$$-\Delta\tilde{u} = 0, \quad x \in \Omega, \tag{1}$$

$$\begin{aligned} \tilde{\gamma}_0\tilde{u} &:= \tilde{u}\,|_{\tilde{\Gamma}_D} = \tilde{\varphi}, \\ \tilde{\gamma}_1\tilde{u} &:= \partial\tilde{u}/\partial n\,|_{\tilde{\Gamma}_N} = \tilde{\psi} \end{aligned} \tag{2}$$

holds. In (2) n is the external vector with respect to the boundary $\partial\Omega$ normal.

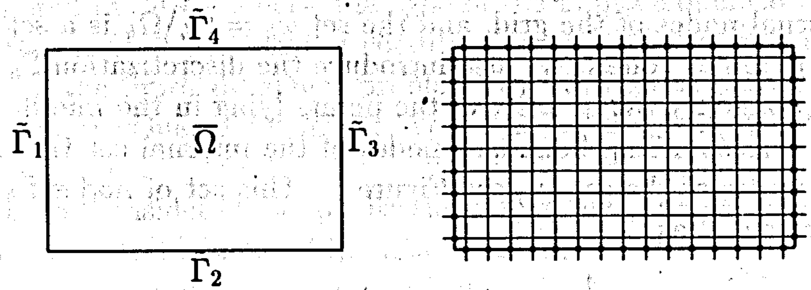

Figure 1: Displaced rectangular grid

For the boundary value problem (1), (2) with $\tilde{\varphi} = 0$ we introduce the Poincaré – Steklov operator $\tilde{S}$

$$\tilde{S} : \tilde{H}^{-\frac{1}{2}}(\tilde{\Gamma}_N) \to \tilde{H}^{\frac{1}{2}}(\tilde{\Gamma}_N).$$

It is known, see e. g. [1],[2], that the operator $\tilde{S}$ is a bounded, symmetric in $L_2(\tilde{\Gamma}_N)$ and positive-definite operator. There also exists the inverse operator $\tilde{S}^{-1}$

$$\tilde{S}^{-1} : \tilde{H}^{\frac{1}{2}}(\tilde{\Gamma}_N) \to \tilde{H}^{-\frac{1}{2}}(\tilde{\Gamma}_N).$$

with the same properties.

The discretization of the inverse Poincaré – Steklov operator $\tilde{S}^{-1}$ is done in the framework of the finite–difference approach for an approximation of the boundary value problem (1), (2). Here and below we suppose that there exists a solution of the problem (1), (2) which is sufficiently smooth to use finite–difference schemes for discretization of the problem, see [5]. Introduce a uniform rectangular grid $\overline{\Omega}_h$ on $\overline{\Omega}$ with a "displacement by $h/2$", see e.g. [5], [13], [8]:

$$\overline{\Omega}_h = \{(x_i, y_j) : \ x_i = (i - 0.5)h_1, \ y_j = (j - 0.5)h_2, \\ i = 0 \div M + 1, \ j = 0 \div N + 1\},$$

where $h_1 = A/M$ and $h_2 = B/N$ are the grid steps. The set of nodes $\Omega_h = \{(x_i, y_j), \ i = 1 \div M, \ j = 1 \div N\}$ is a set of the internal nodes of the grid, and the set $\omega_h = \overline{\Omega}_h \backslash \Omega_h$ is a set of the external nodes. We also introduce the discretization Γ_h of the boundary $\partial\Omega$ as a set of the points lying in the middle of the corresponding boundary nodes of the internal set Ω_h and the nodes of the set ω_h, see Figure 1. This set of nodes Γ_h is represented as

$$\Gamma_h = \bigcup_{i=1}^{4} \Gamma_i = \Gamma_D \cup \Gamma_N, \quad \Gamma_i = \Gamma_h \cap \tilde{\Gamma}_i,$$

where the sets Γ_D and Γ_N, $\Gamma_D \cap \Gamma_N = \emptyset$, are subsets of Γ_h where the Dirichlet or Neumann conditions are given.

Next, define discrete analogues γ_0, γ_1 of the boundary operators $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ as

$$\gamma_0 u := \frac{(u_{\Gamma_h + h/2} + u_{\Gamma_h - h/2})}{2}, \tag{3}$$

$$\gamma_1 u := \frac{\Delta u}{\Delta n}\Big|_{\Gamma_h} = \frac{(u_{\Gamma_h + h/2} - u_{\Gamma_h - h/2})}{h}, \tag{4}$$

where $h = h_1$ or $h = h_2$ depending on the point of the boundary set Γ_h where the values of $\gamma_0 u$ and $\gamma_1 u$ are calculated.

Set grid functions φ and ψ as the projections on Γ_D and Γ_N correspondingly of the given data $\tilde{\varphi}$ and $\tilde{\psi}$ from (2), and consider the finite–difference boundary value problem approximating the problem (1), (2): find a discrete harmonic function $u \in X$ such that

$$-\Delta_h u = 0, \ in \ \Omega_h, \tag{5}$$

$$\begin{aligned} \gamma_0 u &= \varphi, \\ \gamma_1 u &= \psi \end{aligned} \tag{6}$$

holds. Here X is a space of discrete harmonic functions on Ω_h. Any element of X belongs to the direct product of the Euclidean spaces E_M and E_N of vectors of the dimensions M and N correspondingly, $X \subset E_M \otimes E_N$. Δ_h is a standard five–point centered finite–difference analogue of the Laplace operator.

Remark 1. Under the condition $\tilde{\varphi}$, $\tilde{\psi} \in C(\partial\Omega)$ and $\tilde{u} \in C^4(\Omega)$ the finite–difference problem (5), (6) approximates the boundary value problem (1), (2) to the accuracy $O(h_1^2 + h_2^2)$, see e.g. [5].

First, consider for simplicity the case of the Dirichlet boundary value problem (5), (6), with $\Gamma_h \equiv \Gamma_D$, $\Gamma_N = \emptyset$. Now the trace $\gamma_0 u = \varphi$ can be represented as

$$\varphi = [\varphi_1, \varphi_2, \varphi_3, \varphi_4] \in Y, \ \varphi_i = \varphi\,|_{\Gamma_i} \ \ \varphi_1, \varphi_3 \in E_N, \ \varphi_2, \varphi_4 \in E_M,$$

where we introduce according to [8] the boundary trace space Y of discrete harmonic functions from X as the direct sum of the spaces E_M and E_N: $Y = E_M \oplus E_N \oplus E_M \oplus E_N$. In this case the matrix S^{-1} approximating the inverse Poincaré–

Steklov operator $\tilde{S}^{-1}$ is defined as follows, see e.g. [8]:

$$\gamma_1 u \equiv \begin{pmatrix} \Delta u/\Delta n_1 \\ \Delta u/\Delta n_2 \\ \Delta u/\Delta n_3 \\ \Delta u/\Delta n_4 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \equiv S^{-1}\varphi. \tag{7}$$

The matrices P_{sr}, $s = 1 \div 4$, $r = 1 \div 4$, where P_{11}, $P_{33} \in L(E_N, E_N)$, P_{22}, $P_{44} \in L(E_M, E_M)$, have the following form:

$$\begin{array}{ll} P_{11} = P_{33} = F_N^{-1} D_1 F_N, & P_{22} = P_{44} = F_M^{-1} D_2 F_M, \\ P_{13} = P_{31} = F_N^{-1} D_3 F_N, & P_{24} = P_{24} = F_M^{-1} D_4 F_M, \\ P_{12} = Z_\nu T_M D_5 F_M, & P_{14} = Z_\nu D_5 F_M, \\ P_{21} = Z_\mu T_N D_6 F_N, & P_{23} = Z_\mu D_6 F_N, \\ P_{32} = Z_\nu T_M D_7 F_M, & P_{34} = Z_\nu D_7 F_M, \\ P_{41} = Z_\mu T_N D_8 F_N, & P_{43} = Z_\mu D_8 F_N. \end{array} \tag{8}$$

The matrices F_N and F_M are matrices of the discrete sine Fourier transform

$$F_p = \{f_{kl} = \rho_k \sqrt{\frac{2}{p}} \sin \frac{\pi k(2l-1)}{2p};\ k, l = 1 \div p\}, \tag{9}$$
$$\rho_k = 1,\ k = 1 \div p-1, \quad \rho_p = 0.5, \quad p = M, N;$$

while the matrices $T_N \in L(E_N, E_N)$, $T_M \in L(E_M, E_M)$ are permutation matrices of the form

$$T_p = \begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & 1 & 0 \\ \vdots & 1 & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}, \ p = M, N.$$

The matrices D_i, $i = 1 \div 8$ are diagonal ones with the following

elements

$$\begin{aligned} D_1 &= \{d_k = h_1^{-1} \frac{\mu_k - 1}{\mu_k + 1} \frac{\mu_k^M + \mu_k^{-M}}{\mu_k^M - \mu_k^{-M}},\ k = 1 \div N\}, \\ D_2 &= \{d_k = h_2^{-1} \frac{\nu_k - 1}{\nu_k + 1} \frac{\nu_k^N + \nu_k^{-N}}{\nu_k^N - \nu_k^{-N}},\ k = 1 \div M\}, \\ D_3 &= \{d_k = 2h_1^{-1} \frac{\mu_k - 1}{\mu_k + 1} \frac{1}{\mu_k^M - \mu_k^{-M}},\ k = 1 \div N\}, \\ D_4 &= \{d_k = 2h_2^{-1} \frac{\nu_k - 1}{\nu_k + 1} \frac{1}{\nu_k^N - \nu_k^{-N}},\ k = 1 \div M\}, \\ D_5 &= \{d_k = -\rho_k h_1^{-1} \sin \frac{\pi k}{2M},\ k = 1 \div M\}, \\ D_6 &= \{d_k = -\rho_k h_2^{-1} \sin \frac{\pi k}{2N},\ k = 1 \div N\}, \\ D_7 &= \{d_k = -\rho_k h_1^{-1} \sin \frac{\pi k(2M-1)}{2M},\ k = 1 \div M\}, \\ D_8 &= \{d_k = -\rho_k h_2^{-1} \sin \frac{\pi k(2N-1)}{2N},\ k = 1 \div N\}, \end{aligned} \tag{10}$$

where ρ_k are defined in (9). The matrices Z_μ, Z_ν are dense rectangular matrices with the entries

$$\begin{aligned} Z_\mu &= \{z_{ik} = 2\frac{\mu_k^i - \mu_k^{1-i}}{(1+\mu_k)(\mu_k^M - \mu_k^{-M})},\ i = 1 \div M,\ k = 1 \div N\}, \\ Z_\nu &= \{z_{ik} = 2\frac{\nu_k^i - \nu_k^{1-i}}{(1+\nu_k)(\nu_k^N - \nu_k^{-N})},\ i = 1 \div N,\ k = 1 \div M\}. \end{aligned} \tag{11}$$

In the representations (10) and (11) the quantities μ_k and ν_k are given by the following formulae

$$\begin{aligned} \mu_k &= 1 + 2\alpha_k^2 + 2\alpha_k\sqrt{1+\alpha_k^2},\ \alpha_k = \frac{h_1}{h_2} \sin \frac{\pi k}{2N},\ k = 1 \div N, \\ \nu_k &= 1 + 2\beta_k^2 + 2\beta_k\sqrt{1+\beta_k^2},\ \beta_k = \frac{h_2}{h_1} \sin \frac{\pi k}{2M},\ k = 1 \div M. \end{aligned} \tag{12}$$

Remark 2. The discrete inverse Poincaré–Steklov operator S^{-1} defined by formulae (7) – (12) with the domain of definition

$Y \backslash Ker S^{-1}$, where $Ker S^{-1} = \{\varphi \in Y : \varphi = const \ on \ \Gamma_h\}$, is a symmetric and positive-definite one. This follows from the discrete Green formula

$$(S^{-1}\gamma_0 u, \gamma_0 w)_{L_2(\Gamma_h)} = D(u, w) + \frac{1}{2}(\gamma_1 u, \gamma_1 w)_{L_2(\Gamma_h)},$$

which holds for all $u \in X$ and for any w, see e.g. [13], [8]. The form $D(\cdot, \cdot)$ generates the discrete Dirichlet sum corresponding to the internal nodes Ω_h.

3 The fast method to compute the product $S^{-1}\varphi$

First, we consider a method of fast multiplication by a vector for the matrices of the form

$$P = ZDF, \tag{13}$$

constituting the block representation (7) of the discrete counterpart of the inverse Poincaré – Steklov operator $\tilde{S}^{-1}$. We assume the matrix Z in the expression (13) has one of the forms indicated in the representation (11), F is the matrix of the discrete sine Fourier transform (9), and the matrix D is a diagonal one.

Here we follow the idea of special truncation of the matrix Z presented in [3] which is based on the exponential decay of the matrix entries.

Let for definiteness $Z \equiv Z_\mu$, $F \equiv F_N$ and

$$D = \{d_k = \sin\frac{\pi k(2j_0 - 1)}{2N}, \ k = 1 \div N\}. \tag{14}$$

Note that by virtue of the fact that index j_0 in (14) is chosen arbitrarily, all the below arguments and conclusions are valid for any matrix of the form (13) in the block representation (7), (8).

Multiplication of the matrix P with the chosen matrices Z, D, F, by a vector $\varphi = \{\varphi_j\}_{j=1}^{N}$ gives the exact solution $u = \{u_{ij}\}$ on the set of the grid points $\{(x_i, y_j), \ i = 1 \div M, \ j = j_0\}$ for the discrete Laplace equation

$$-\Delta_h u = 0, \ in \ \Omega_h, \tag{15}$$

with the boundary conditions

$$\begin{aligned} &\frac{u_{i0} + u_{i1}}{2} = 0, && \frac{u_{iN} + u_{iN+1}}{2} = 0, \quad i = 1 \div M, \\ &\frac{u_{0j} + u_{1j}}{2} = 0, && \frac{u_{Mj} + u_{M+1j}}{2} = \varphi_j, j = 1 \div N. \end{aligned} \tag{16}$$

We denote this partial solution as $v = \{u_{ij}, \ i = 1 \div M, \ j = j_0\}$, and so

$$v = ZDF\varphi. \tag{17}$$

Now introduce a vector $l = \{l(i)\}_{i=1}^{M}$ with integer components $l(i) : 1 \leq l(i) \leq N$, $i = 1 \div M$, and denote as Z_l the rectangular matrix

$$Z_l = \{z_{ik}^l\}, \quad z_{ik}^l = \begin{cases} z_{ik} \ , & k = 1 \div l(i) \\ 0 \ , & k = l(i) + 1 \div N \end{cases} ; i = 1 \div M; \tag{18}$$

with the elements z_{ik} coinciding with the entries of the matrix Z_μ. Consider an approximate solution $v^l = \{u_{ij}^l, \ i = 1 \div M, \ j = j_0\}$ of the problem (15), (16) on the grid line $\{(x_i, y_j), \ i = 1 \div M, \ j = j_0\}$, which is defined similarly to (17) by the expression

$$v^l = Z_l DF\varphi. \tag{19}$$

Note that if we set $l(i) = N$ in (18) for all $i = 1 \div M$, then $Z_l \equiv Z$ and $v^l \equiv v$.

The estimate for the approximation error $\|v - v^l\|_\infty$ is established in the following theorem

THEOREM 1. 1. *For any $\varepsilon > 0$ and for any index j_0, $j_0 = 1 \div N$, there exists the vector l with the components $l(i)$ satisfying the estimate*

$$l(i) \geq \xi log \frac{8\|\varphi\|_\infty}{\pi\varepsilon}\xi, \quad \xi = \frac{N}{a(M+1-i)},$$

where $a = ln(1 + 2\eta^2 + 2\eta\sqrt{1+\eta^2})$, $\eta = h_1/h_2$, such that

$$\|v - v^l\|_\infty \leq \varepsilon$$

holds.

2. *For any $\varepsilon > 0$ and for any index j_0, $j_0 = 1 \div N$, there exists the vector l with the components $l(i)$ satisfying the inequality*

$$\|\varphi\|_\alpha^2 \sum_{k=l(i)+1}^{N} k^{-\alpha} exp(-2k/\xi) \leq \varepsilon^2,$$

where $\|\varphi\|_\alpha^2 = \sum k^\alpha c_k^2$, $\alpha > 0$, $\{c_k\}_{k=1}^N = F_N\varphi$, such that

$$\|v - v^l\|_\infty \leq \varepsilon$$

holds.

To prove this theorem we first establish some auxiliary statements.

LEMMA 1. For any $\eta > 0$ and any $x \in [0;\eta]$ the inequality

$$1 + 2x^2 + 2x\sqrt{1+x^2} \geq exp(ax/\eta)$$

holds, where $a = ln(1 + 2\eta^2 + 2\eta\sqrt{1+\eta^2})$.

Proof. After the substitution $y = x/\eta$, Lemma 1 follows from the inequality

$$\frac{ln(1 + 2\eta^2 y^2 + 2\eta y\sqrt{1+\eta^2 y^2})}{y} \geq a,$$

taking into account the monotonicity of the left-hand side for all $y \in [0;1]$ ∎

LEMMA 2. For all $k = 1 \div N$ and $\eta = h_1/h_2$ the inequality

$$\mu_k \geq exp(ak/N)$$

holds, where μ_k are given by (12). The value of a is given in Lemma 1.

Proof. From the assertion of Lemma 1 and the expressions (12) it follows that

$$\mu_k \geq exp(a\alpha_k/\eta) = exp(a \sin\frac{\pi k}{2N})$$

holds. Taking into account that

$$\sin\frac{\pi k}{2N} \geq \frac{k}{N}, \quad for\ all\ k = 1 \div N,$$

we obtain the assertion of Lemma 2 ∎

Proof of Theorem 1. According to (17), (19) we have

$$| u_{ij_0} - u_{ij_0}^l | = 2 \left| \sum_{k=l(i)+1}^{N} \frac{c_k \mu_k^{i-M}(\mu_k^{-1} - \mu_k^{-2i})}{(1-\mu_k^{-2M})(1+\mu_k^{-1})} \sin\frac{\pi k(2j_0 - 1)}{2N} \right|,$$

where $\{c_k\}_{k=1}^N = F_N\varphi$. It is obvious that:

a) $| \sin\frac{\pi k(2j_0-1)}{2N} | \leq 1$ for all $k = 1 \div N$;

b) $\frac{\mu_k^{i-M}(\mu_k^{-1}-\mu_k^{-2i})}{(1-\mu_k^{-2M})(1+\mu_k^{-1})} \leq \mu_k^{i-M-1}$ for all $k = 1 \div N$.

From inequalities a), b) and Lemma 2 we obtain:

$$| u_{ij_0} - u_{ij_0}^l | \leq 2 \sum_{k=l(i)+1}^{N} | c_k exp(-k/\xi) | . \qquad (20)$$

1. Taking into account that $| c_k |\leq \frac{4}{\pi}\|\varphi\|_\infty$, $k = 1 \div N$ holds, from inequality (20) and the sum of the geometrical progression

$$\sum_{k=l(i)+1}^{N} exp[-k/\xi] = \frac{exp[-l(i)/\xi](1-exp[(l(i)-N)/\xi])}{1-exp[-1/\xi]}$$

it follows that

$$| u_{ij_0} - u^l_{ij_0} |\leq \frac{8\|\varphi\|_\infty}{\pi}\xi exp[-l(i)/\xi] \leq \varepsilon,$$

which completes the proof of item 1.

2. Assertion 2 follows from the estimate (20) and Hölder inequality

$$| u_{ij_0} - u^l_{ij_0} |\leq 2\left(\sum_{k=l(i)+1}^{N} k^\alpha c_k^2\right)^{\frac{1}{2}}\left(\sum_{k=l(i)+1}^{N} k^{-\alpha} exp[-2k/\xi]\right)^{\frac{1}{2}} \leq$$

$$\leq 2\|\varphi\|_\alpha\left(\sum_{k=l(i)+1}^{N} k^{-\alpha} exp[-2k/\xi]\right)^{\frac{1}{2}} \leq \varepsilon \quad \blacksquare$$

Remark 3. Note that the strengthened assertion 2 of the Theorem enables one to use a priori information about the smoothness of the incoming data to reduce the number of nonzero elements of the truncated matrix Z^l.

The following lemma is a consequence of the Theorem

LEMMA 3. To compute the product (17) with a given accuracy $\varepsilon > 0$ using the approximate formulae (19), (18), one needs

$$O(NlogM(logN + logM + log\varepsilon^{-1}))$$

both arithmetical operations and computer memory.

Proof. The calculation of the product $\phi = DF_N\varphi$ needs $O(NlogN)$ operations. According to [3], the calculation of each

element z_{ik} of the matrix Z_μ needs $O(logM)$ operations. The number of nonzero elements z^l_{ik} and the number of arithmetical operations to calculate the product $v^l = Z_l\phi$ with a given accuracy ε, due to Theorem 1, is estimated by the quantity

$$O\left(\sum_{i=1}^{M} l(i)\right) = O\left(\sum_{i=1}^{M}\frac{N}{M+1-i} log\frac{N}{(M+1-i)\varepsilon}\right).$$

Lemma is proved $\blacksquare$

Taking into account that the multiplication by a vector of the diagonal blocks P_{ii}, $i = 1 \div 4$ and of the blocks P_{13}, P_{31}, P_{24}, P_{42} in the block representation (7), (8) is done by the fast Fourier transform and all the other blocks of the form (14) are multiplied using the approximate formulae (19), (18) we obtain

THEOREM 2. *To compute the product $\psi = S^{-1}\varphi$ with a given accuracy $\varepsilon > 0$, where the matrix S^{-1} is given by formulae (7) – (12), one needs*

$$O(NlogN+MlogM+(NlogM+MlogN)(logM+logN+log\varepsilon^{-1}))$$

both arithmetical operations and computer memory.

4 The mixed problem

In the previous section we have considered the fast method to compute conormal derivatives of the discrete harmonic function when its trace on the boundary is given. Consider a more general problem: how to compute effectively the complementary Cauchy data provided that some mixed Dirichlet – Neumann boundary conditions are given on the edges Γ_i, $i = 1 \div 4$ of the rectangle Ω. For example, if we have a mixed boundary value problem for the discrete Laplace equation (15) with Neumann boundary conditions ψ_1, ψ_2 on the edges Γ_1, Γ_2 and with

Dirichlet boundary conditions φ_3, φ_4 on the edges Γ_3, Γ_4 then, there exists a matrix S_{mix} which defines the mapping

$$S_{mix}: \begin{pmatrix} \psi_1 \\ \psi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \rightarrow \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix},$$

where φ_1, φ_2 are unknown traces on the edges Γ_1, Γ_2, and ψ_3, ψ_4 – conormal derivatives on the edges Γ_3, Γ_4. This matrix S_{mix} can be represented in the same block form (7) – (11) as the matrix S^{-1}, but the representation of the elements of the factor matrices Z, D, F will depend on the type of boundary conditions on the edges Γ_i, $i = 1 \div 4$. Considering that fact, in order not to clutter up the presentation with new designations, we keep the previous ones (P, Z, D, F) for the blocks forming the matrix S_{mix}.

We consider the method for fast multiplication of the matrix S_{mix} by a vector assuming that homogeneous Dirichlet or Neumann boundary conditions are given on the edges Γ_i, $i = 2, 3, 4$, and on the edge Γ_1 nonhomogeneous ones are given. This case describes the multiplication of the first block column (P_{i1}, $i = 1 \div 4$) in the representation (7) by a given on Γ_1 vector. We indicate the cases of Dirichlet or Neumann boundary conditions on the edge Γ_i as $\mathcal{D}_i$ or $\mathcal{N}_i$, correspondingly. Depending on the type of these conditions on the edges Γ_2 and Γ_4, the elements f_{kl}, $k, l = 1 \div N$, the Fourier transform matrix F are

defined by the following expressions

$$\begin{aligned} (\mathcal{D}_2\mathcal{D}_4): \quad & f_{kl} = \sin\frac{\pi k(2l-1)}{2N}, \\ (\mathcal{D}_2\mathcal{N}_4): \quad & f_{kl} = \sin\frac{\pi(2k-1)(2l-1)}{4N}, \\ (\mathcal{N}_2\mathcal{D}_4): \quad & f_{kl} = \sin\frac{\pi(2k-1)(2(N-l)-1)}{4N}, \\ (\mathcal{N}_2\mathcal{N}_4): \quad & f_{kl} = \cos\frac{\pi k(2l-1)}{2N}. \end{aligned} \qquad (21)$$

The representation of the elements of the matrix Z depends on the type of boundary conditions on the edges Γ_1, Γ_3:

$$\begin{aligned} (\mathcal{D}_1\mathcal{D}_3): \quad & z_{ik} = \frac{2}{\mu_k^M - \mu_k^{-M}}\left(\frac{\mu_k^i}{1+\mu_k} - \frac{\mu_k^{-i}}{1+\mu_k^{-1}}\right), \\ (\mathcal{D}_1\mathcal{N}_3): \quad & z_{ik} = \frac{h_1}{\mu_k^M - \mu_k^{-M}}\frac{(1+\mu_k^{-1})\mu_k^i - (1+\mu_k)\mu_k^{-i}}{\mu_k - \mu_k^{-1}}, \\ (\mathcal{N}_1\mathcal{D}_3): \quad & z_{ik} = \frac{2}{\mu_k^M + \mu_k^{-M}}\frac{(1-\mu_k^{-1})\mu_k^i - (1-\mu_k)\mu_k^{-i}}{\mu_k - \mu_k^{-1}}, \\ (\mathcal{N}_1\mathcal{N}_3): \quad & z_{ik} = \frac{h_1}{\mu_k^M - \mu_k^{-M}}\left(\frac{\mu_k^i}{\mu_k - 1} - \frac{\mu_k^{-i}}{\mu_k^{-1} - 1}\right). \end{aligned} \qquad (22)$$

Here μ_k are the roots of the equation

$$\mu_k^2 + \left(-2 - 2\frac{h_1^2}{h_2^2} + \frac{h_1^2}{h_2^2}\frac{f_{kl+1} + f_{kl-1}}{f_{kl}}\right)\mu_k + 1 = 0.$$

Multiplication of the blocks P_{11}, P_{31} by a vector can be done by the fast Fourier transform. For multiplication of the blocks P_{21}, P_{41} of the form (13) we use the approximate formulae (19), (18). Comparing the expression (11) and (22) for the matrix elements one can easily obtain the assertion of Theorem 1. So, the partial solution of the mixed problem with a given accuracy ε is done with the same asymptotical expenditures as the solution of the Dirichlet problem (Theorem 2).

5 Numerical experiments (single domain)

The numerical experiments presented in this section demonstrate the computational characteristics of the algorithm to compute the matrix – vector product of the discrete Poincaré – Steklov operator (7) using the Bakhvalov – Orekhov (BO) method from [3] adapted for displaced grids. All the below numerical experiments are done with the Poincaré – Steklov operator defined for the discrete Laplacian in the unit square on the uniform mesh with steps $h_1 = h_2 = h$ and the number of unknowns $N = M = 2^m$ along each edge, provided that the approximate multiplication $S^{-1}\varphi$ is done to an accuracy $\varepsilon = \varepsilon_0 \times h^2$, $0 < \varepsilon_0$ (see Theorem 1).

Tables 1, 2 show expenditures and accuracy of the BO algorithm to compute the product $S^{-1}\varphi$ compared with the exact one without optimization. As a test function φ we take the projection of the trace of the harmonic function

$$\tilde{u}(x, y) = x^2 - y^2, \quad 0 \le x, y \le 1$$

on the mesh boundary Γ_h, and denote as ψ, ψ^* the computed and exact values of the conormal derivatives on Γ_h. Indices "BO" and "P" in Tables 1, 2 denote the quantities characterizing the approximate fast method and the exact one without optimization correspondingly. Columns N_{BO}, N_P/N_{BO} of Table 1 show the number of nonzero elements of the truncated matrix Z^l from (18) and its ratio to the total number $N_P = 2^m$ of Z^l matrix elements. Columns T^M_{BO}, T^M_P present the total time in seconds to compute the elements of all the matrices constituting the representation (8). The last three columns show the time characteristics to compute the product $S^{-1}\varphi$, provided that all the needed matrix elements have already been found. These computations were performed on IBM PC – 486/25. Table 2 presents the accuracy of the fast approximate computation

Table 1: Expenditures for computation of the product $\psi = S^{-1}\varphi$

$\varepsilon_0 \times h^2$	m	N_{BO}	N_P/N_{BO}	T^M_{BO}	T^M_P	T^{Op}_{BO}	T^{Op}_P	T^{Op}_P/T^{Op}_{BO}
1.6e–02	3	35	1.8	9.0e–03	9.3e–03	7.1e–03	7.7e–03	1.08
3.9e–03	4	126	2.03	2.4e–02	5.0e–02	1.4e–02	1.6e–02	1.14
9.8e–04	5	401	2.55	5.5e–02	0.23	2.8e–02	4.2e–02	1.5
2.4e–04	6	1161	3.53	0.13	1.17	6.1e–02	0.13	2.13
6.1e–05	7	3192	5.27	0.32	3.9	0.14	0.41	2.93
1.5e–05	8	8419	7.78	0.76	16.2	0.32	1.47	4.59
3.8e–06	9	21522	12.18	2.02	69.8	0.71	5.6	7.89
9.5e–07	10	53679	19.53	5.8	300.2	1.59	22.2	13.96

$\psi = S^{-1}\varphi$ in different norms, provided

$$\psi = [\psi^1, \psi^2, \psi^3, \psi^4], \quad \psi^i = \{\psi^i_k\}_{k=1}^N,$$
$$\|\psi\|_\infty = \max_{i,k} | \psi^i_k |,$$
$$\|\psi\|^2_{L_2} = (\psi, \psi)_{L_2} = h \sum_{i=1}^{4} \sum_{k=1}^{N} \psi^i_k \psi^i_k,$$
$$\|\psi\|_{S^{-1}} = (S^{-1}\varphi, v)_{L_2}, \quad v = \frac{\varphi}{\|\varphi\|_{L_2}}.$$

The structure of the matrices Z^l from (18) is shown on Figure 2. Dots and blank space in this Figure represent the relative ratio of nonzero and zero matrix elements.

In Tables 1, 2 and in Figure 2 we assume that $\varepsilon_0 = 1$. It is also interesting to know how the smoothness of the boundary function influences the computational characteristics of the method.

Let φ be the trace of the following test function

$$\tilde{u}(x, y) = \frac{e^{\pi k x} - e^{-\pi k x}}{e^{\pi k} - e^{-\pi k}} \sin \pi k y, \quad 0 \le x, y \le 1.$$

The data of Table 3 show the dependence of the accuracy $\delta = \|\psi - \psi_P\|_\infty$ and of the computational characteristics N_{BO}, T_{BO} on the chosen $\varepsilon = \varepsilon_0 \times h^2$ and on the smoothness of the trace

Table 2: Accuracy of the computations of the product $\psi = S^{-1}\varphi$

$\varepsilon_0 \times h^2$	m	$\lVert\psi - \psi^*\rVert_\infty$	$\lVert\psi - \psi^*\rVert_{L_2}$	$\lVert\psi - \psi^*\rVert_{S^{-1}}$	$\lVert\psi - \psi_P\rVert_\infty$
1.6e–02	3	6.3e–02	7.2e–02	1.4e–02	5.3e–03
3.9e–03	4	3.2e–02	2.6e–02	3.4e–03	1.4e–03
9.8e–04	5	1.6e–02	9.2e–03	9.1e–04	3.2e–04
2.4e–04	6	7.8e–03	3.3e–03	2.3e–04	1.0e–04
6.1e–05	7	3.9e–03	1.2e–03	6.0e–05	2.8e–05
1.5e–05	8	1.9e–03	4.1e–04	1.5e–05	7.4e–06
3.8e–06	9	9.7e–04	1.4e–04	3.8e–06	1.9e–06
9.5e–07	10	4.9e–04	5.1e–05	9.2e–07	5.5e–07

φ. Columns $k = 1, 5, 10, 20, 30$ present data for the accuracy δ. Columns N_{BO} and T_{BO} show the growth of the number of nonzero elements of the matrix Z^l from (18) and of the total time ($T_{BO} = T_{BO}^M + T_{BO}^{Op}$) to compute the product $S^{-1}\varphi$ to a given accuracy ε. This computational experiment was done for $N = 128$ on IBM PC – 386/25.

Table 3: Dependence of N_{BO}, T_{BO} and δ, on ε_0 and on k for N=128

ε_0	$k=1$	$k=5$	$k=10$	$k=20$	$k=30$	N_p	T_p
100	1.4e-13	4.8e-05	1.1e-04	1.7e-04	1.9e-04	2179	0.77
10	1.4e-13	1.9e-14	2.1e-06	3.7e-06	5.5e-06	2699	0.82
1	1.4e-13	1.9e-14	3.4e-08	7.9e-08	1.6e-07	3192	0.83
0.1	1.4e-13	1.9e-14	5.3e-10	1.7e-09	2.4e-09	3657	0.87
0.01	1.4e-13	1.9e-14	6.5e-12	2.2e-11	6.9e-11	4100	0.93

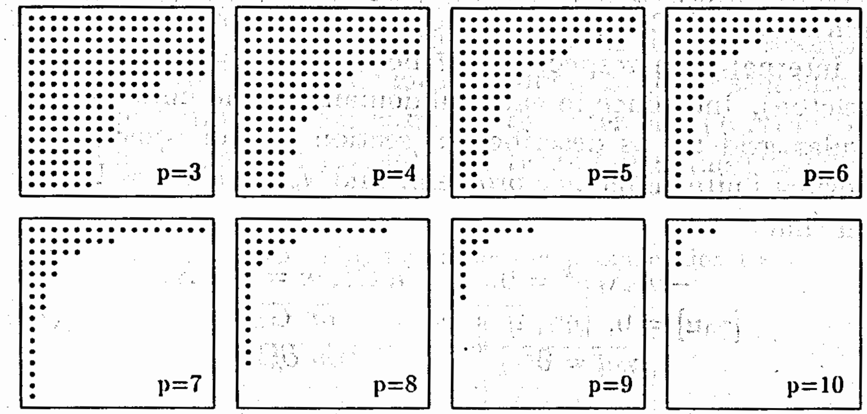

Figure 2: The structure of Z^l matrices for $N = 2^m$, $m = 3 \div 10$

6 Numerical experiments (multidomain case)

Here we present numerical experiments demonstrating the effectiveness of using the fast method to compute the product $S^{-1}\varphi$ for solving elliptic problems via domain decomposition. The interface problem arising in the substructuring methods is formulated via the local Poincaré – Steklov operators S_k^{-1} defined for each substructure. The computation of the residual in the iterative solution of the interface problem needs local multiplications $S_k^{-1}\varphi_k$ for all k. The application of the fast method to computation of this product finally leads to effectiveness of the whole domain decomposition algorithm.

Let $\overline{\Omega}$ be a step–type domain with a boundary $\partial\Omega$ consisting of linear segments lying at right angles. We consider a decomposition of Ω into nonoverlapping subdomains Ω^k, $k = 1 \div K$, such that each substructure Ω^k is a rectangle and the common boundary of any two intersecting subdomains $\overline{\Omega}^k$, $\overline{\Omega}^l$ is their one common edge. The union of these common boundaries forms the internal with respect to $\partial\Omega$ boundary $G = \cup_{k=1}^{K}\partial\Omega^k\backslash\partial\Omega$ (skeleton). Introduce in each subdomain Ω^k the uniform rectangular grid Ω_h^k as described in section 1, and consider the following finite–difference problem: find $u_h = \{u^k,\ k = 1 \div K\}$ such that

$$\begin{aligned} -\mu_k\Delta_h u^k &= 0, && in\ \Omega_h^k,\ k = 1 \div K, \\ [\gamma_0 u] = 0,\ [\mu\gamma_1 u] &= \psi && on\ G_h, \\ \gamma_0 u &= 0 && on\ \partial\Omega_h, \end{aligned} \qquad (23)$$

holds. In (23) $[\,\cdot\,]$ defines a jump of the function or its conormal derivatives on the skeleton G; μ_k, $0 < \mu_k < \infty$, $k = 1 \div K$, are given constants.

The finite–difference problem (23) is reduced to the following interface problem, see e.g. [8], with respect to the trace $\varphi = \gamma_0 u$ of the unknown function on the skeleton G_h

$$(A\varphi, v)_{L_2(G_h)} \equiv \sum_{k=1}^{K} \mu_k(S_k^{-1}\varphi_k, v_k)_{L_2(\Gamma_h^k)} = (\psi, v)_{L_2(G_h)}, \qquad (24)$$

where the matrices S_k^{-1} are defined by (7) - (12).

Due to the properties of the local operators S^{-1}, see Remark 2, the interface matrix A is a symmetric and positive-definite one, so the solution of the interface problem (24) is done by the preconditioned conjugate gradient (PCG) method. The preconditioners B for the interface matrix A constructed for example in [8], need for their inversion $O(KNlogN)$ arithmetical operations, and the condition number κ of the operator $B^{-1}A$ is defined by the quantity $\kappa(B^{-1}A) = O(log^2N)$, where N is the maximal number of unknowns on the edge, $N = \max_{k=1\div K}(N_k, M_k)$.

The main laborious stage in the PCG method for solving the interface problem (24) is to compute the residual $A\varphi_n$ on each

iterative step n,which in general needs $O(KN^2)$arithmetical operations and computer memory. The use of the fast algorithm for the computation of the local products $S_k^{-1}\varphi_k$ essentially reduces the computational costs (to the oder of $O(KNlog^2N)$), see Table 4. Table 4 presents the results of numerical exper-

Table 4: Times and iterations for the decomposition 4 × 4

	1		2	
N	IT	T	IT	T
16	6	15.7	4	15.0
32	6	22.6	6	45.3
64	7	44.8	7	135.0
128	7	87.8	7	505.0
256	7	179.2	–	–

iments to solve the interface problem (24) arising from the Dirichlet boundary value problem for the Laplace equation in the unit square decomposed into 16 identical subdomains. Column 1 of the table demonstrates the application of the fast algorithm to computation of the residual, column 2 presents data for the algorithm without optimization. In this table N is the number of unknowns on each edge of the subdomains, IT is the number of iterations of the PCG method and T is time in seconds for IBM PC 386/25 to reduce the initial residual 10^{-5} times.

Table 5 presents the results of numerical experiments for the solution of the interface problem (24) arising from the Dirichlet boundary value problem for the Laplace equation in the step-type domains shown on Figure 3 , which are "cut" from the unit square decomposed into 100 identical subdomains. The columns marked B, E, M correspond to the geometries of the domains shown on Figure 3, the column marked O corresponds

Table 5: Times and iterations for the step–type domains

	O		B		E		M	
N	IT	T	IT	T	IT	T	IT	T
16	4	65	4	45	4	39	4	33
32	4	108	4	81	4	66	4	57
64	4	197	4	158	4	126	4	110

to the problem in the whole unit square. The other notations are as in Table 4.

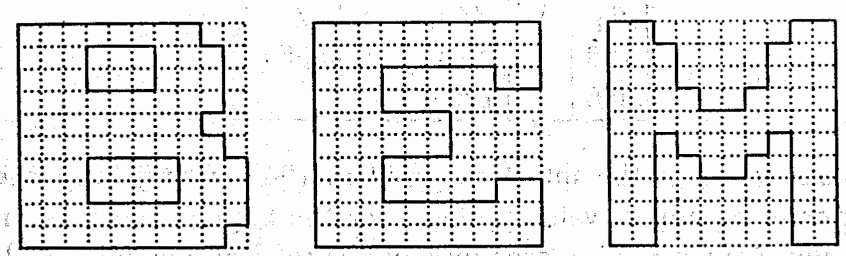

Figure 3: Step–type domains

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Received by Publishing Department on May 7, 1993.