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ITERATION SCHEME FOR THE MULTIPARAMETER NONLINEAR BOUNDARY VALUE PROBLEM WITH THE ADDITIONAL CONDITIONS AND ITS APPLICATION TO SOME FIELD MODELS

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The simulation of some nonlinear effects in the polar medium and in the field models leads to the singular, nonlinear boundary value problems for the differential equations depending on the "external" physical parameters of a model. Taking in account their additional coupling allows to reduce the number of the parameters of the equations. It should be pointed out that in this case the additional conditions arises.As a rule, they connect the integral characteristics of the desired solutions with their asymptotic behavior. As far as now the group of the dependent parameters can be replaced by the asymptotical relations for some researched solutions, we can speak about the boundary value problem original statement in which the equations depend not only on the solution but on the boundary conditions directly

$$[\mathbf{I}\frac{d^2}{dx^2}\vec{y} + \mathbf{F}(x, \vec{y}, \vec{y}', \vec{y}(0), \vec{y}(\infty), \vec{a})] = 0, \tag{1}$$

$$0 \leq x < \infty, \quad \|\mathbf{A}_L\vec{y}\| < \infty \quad if \quad x \to 0, \quad \|\mathbf{A}_R\vec{y}\| < \infty \quad if \quad x \to \infty,$$

$$\Psi(\mathbf{A}_L\vec{y}, \mathbf{A}_R\vec{y}) = 0\,. \tag{2}$$

Here $\mathbf{I}$ is a unit matrix and $\mathbf{A}_L, \mathbf{A}_R$ are degenerate matrixes, $\mathbf{F}$ is a nonlinear matrix-function, Ψ is a functional and $\vec{a}$ is a parameter vector of a physical model.

The iterative scheme based on the combination of Continuous analogue of the Newton's method and Continuation method was developed in ref.[1]. In this paper we consider the general statement for the boundary value problem with the additional conditions. The suggested method was applied for the numerical investigation of the equations of the solvated electron problem [2], of the some bielectron problem [4] and one QCD problem with an increasing potential [3].

1 The Statement of the Boundary Value Problem with the Additional Conditions

Let us consider the system of the differential equations in the vector form

$$\mathbf{R}(\vec{y}) \equiv \vec{y}'' + \mathbf{F}(\vec{y};\vec{y}';x;\vec{a}) = 0, \quad 0 \le x < \infty, \tag{3}$$

where the prime means the differentiation on x, $\vec{y}$ is a N-component vector-function, $\vec{a}$ is a M-component vector of parameters, $\mathbf{F}$ is a continuous vector-function of $\vec{y}, \vec{y}', \vec{a}$, and as the function on x can have a singularity as $1/x^l, l \le 2$, if $x \to 0$. The components of $\vec{a}$ are the physical parameters of the problem (particle mass, charge, the coupling constant and etc.) or their combinations; $\vec{y}(0)$ and $\vec{y}(\infty)$ - the asymptotical limits of the solutions and some integrals on the functions researched.

Let us research the solution of the system (3) (existence of which is assumed) with the asymptotical conditions

$$\mathbf{R}_L(\vec{y}) \equiv \mathbf{G}_L(\vec{y}(x_L);\vec{y}'(x_L);x_L;\vec{a}) = 0, x_L \to 0,$$
$$\mathbf{R}_R(\vec{y}) \equiv \mathbf{G}_R(\vec{y}(x_R);\vec{y}'(x_R);x_R;\vec{a}) = 0, x_R \to \infty, \tag{4}$$

and the additional functional conditions

$$\mathbf{R}_A(\vec{y}) \equiv \mathbf{S}(\vec{y};\vec{y}';x;\vec{a}) = 0, \tag{5}$$

where $\mathbf{G}_L$ and $\mathbf{G}_R$ are the N-component vector-functions, $\mathbf{S}$ is a p-component vector-function, where $p \le M$. If $p < M$, then there are some free parameters.For the fixed parameter we can find the solutions using the Continuous analogue of the Newton's method and the continuation method for the free parameters. Further we suppose that $p = M$.

In the case $p = M$ we can exclude all the parameters from the equations, solving the eq.(4) with respect to $\vec{a}$ (if it is possible) and substitute the solution in eq.(1). In other case it is necessary to solve the boundary value problem (3)–(4) taking in account the additional condition (5). For the numerical solution of the boundary value problem (3)–(4) on $(0 \le x < \infty)$ it is necessary to determine the boundary conditions on the finite interval $(0 \le x \le x_R)$. Thus we have an additional problem connected with the statement of the boundary conditions for the solutions in the finite points $(x = x_L, \quad x = x_R)$ and with the accuracy estimation for such an approximation. In general, such an estimate can be obtained only numerically. We can use the calculations for

the set of the parameters $x_L, \quad x_R$. Then suppose that the approximation of the boundary conditions(11) for the $x_L = 0$ and $x_R < \infty$ are sufficient accurate for the taking into account of the asymptotic behavior of the solutions. We use the Continuous analogue of the Newton method [5] for the numerical solution of the boundary value problem (3)–(4) on the finite interval $(0 \le x \le x_R)$. Newtonian iterations for this problem consist of the solution of a linear problem at the every k-step's

$$\vec{v}_k'' + \mathbf{F}'_{\vec{y}'}\vec{v}_k' + \mathbf{F}'_{\vec{y}}\vec{v}_k = -\mathbf{R}(\vec{y}) - \vec{\mu}_k^T\mathbf{F}'_{\vec{a}} \tag{6}$$

with the boundary conditions

$$\mathbf{G}'_{L\vec{y}'}\vec{v}_k'(0) + \mathbf{G}'_{L\vec{y}}\vec{v}_k(0) = -\mathbf{R}_L(\vec{y}) - \vec{\mu}_k^T\mathbf{G}'_{L\vec{a}}$$
$$\mathbf{G}'_{R\vec{y}'}\vec{v}_k'(x_R) + \mathbf{G}'_{R\vec{y}}\vec{v}_k(x_R) = -\mathbf{R}_R(\vec{y}) - \vec{\mu}_k^T\mathbf{G}'_{R\vec{a}} \tag{7}$$

and the additional equation

$$\mathbf{S}'_{\vec{y}'}\vec{v}_k' + \mathbf{S}'_{\vec{y}}\vec{v}_k = -\mathbf{R}_A(\vec{y}) - \mu_k^T\mathbf{S}'_{\vec{a}} \tag{8}$$

with respect to iteration correction $\vec{v}_k = \Delta\vec{y}_k, \vec{\mu}_k = \Delta\vec{a}_k$ to the known approximation $\{\vec{y}(x)_k, \vec{a}_k\}$ of the solution. Let us consider the solution of the system (6) in the form

$$\vec{v}_k = \vec{v}_k^{(1)} + \vec{\mu}_k^T\vec{v}_k^{(2)}. \tag{9}$$

Substitution of eq.(9) into eq.(6) and eq.(7) leads to the problems

$$(\vec{v}_k^{(1)})'' + \mathbf{F}'_{\vec{y}'}(\vec{v}_k^{(1)})' + \mathbf{F}'_{\vec{y}}\vec{v}_k^{(1)} = -\mathbf{R}(\vec{y}) \tag{10}$$

$$\mathbf{G}'_{L\vec{y}'}(\vec{v}_k^{(1)}(0))' + \mathbf{G}'_{L\vec{y}}\vec{v}_k^{(1)}(0) = -\mathbf{R}_L(\vec{y})$$
$$\mathbf{G}'_{R\vec{y}'}(\vec{v}_k^{(1)}(x_R))' + \mathbf{G}'_{R\vec{y}}\vec{v}_k^{(1)}(x_R) = -\mathbf{R}_R(\vec{y}) \tag{11}$$

and

$$(\vec{v}_k^{(2)})'' + \mathbf{F}'_{\vec{y}'}(\vec{v}_k^{(2)})' + \mathbf{F}'_{\vec{y}}\vec{v}_k^{(2)} = -\mathbf{F}'_{\vec{a}} \tag{12}$$

$$\mathbf{G}'_{L\vec{y}'}(\vec{v}_k^{(2)}(0))' + \mathbf{G}'_{L\vec{y}}\vec{v}_k^{(2)}(0) = -\mathbf{G}'_{L\vec{a}}$$
$$\mathbf{G}'_{R\vec{y}'}(\vec{v}_k^{(2)}(x_R))' + \mathbf{G}'_{R\vec{y}}\vec{v}_k^{(2)}(x_R) = -\mathbf{G}'_{R\vec{a}}\,. \tag{13}$$

For the functions $\vec{v}_k^{(1)}$ and $\vec{v}_k^{(2)}$ determinating we have boundary value problems (10)-(11) and (12)-(13). Substituting eq.(9) to eq.(8) we receive the system of algebraic equations for the $\vec{\mu}_k$ determination. Thus it is necessary to solve the boundary value problems (10)-(11), (12)-(13) and the system of the linear algebraic equations (8) for the $\vec{\mu}_k$ determination on every iteration. The next approximations were received by means of the formulas

$$\begin{aligned} \vec{y}_{k+1} &= \vec{y}_k + \tau_k \vec{v}_k \\ \vec{a}_{k+1} &= \vec{a}_k + \tau_k \vec{\mu}_k, \end{aligned} \tag{14}$$

where τ_k is a iteration parameter whose appropriate choice may provide an optimal conditions for the iterations convergence[6,7]. Iterations stop if

$$\delta_k < \epsilon,$$

where δ_k is a discrepancy and can be determined as

$$\delta_k = \max_j \max_i \max_{x \in [x_L, x_R]} | \mathbf{R}_i(y_{ik}(x), c_{jk}) |, \quad i = 1, 2, \cdots, N, \quad j = 1, 2, \cdots, M,$$

where $\epsilon > 0$ is a small constant.

Difference approximation by Numerov method for the problems (10)-(11) and (12)-(13) is applied on a uniform grid with step h, which gives an accurate approximation of these systems up to $O(h^4)$ and the same order of difference solution's convergency for our problem.

2 Statements of the Boundary Value Problems for some Physical Models

The method proposed we apply to the solution of the next three nonlinear boundary value problems

- **The solvated electron**
  This problem is formulated as a system of the nonlinear equations [1]

$$\begin{cases} \xi''(x) - \xi(x) + \xi(x)\dfrac{\eta_1(x) - \eta_2(x)}{x} = 0 \\ \eta_1''(x) + \dfrac{1}{x}\xi^2(x) = 0 \\ \eta_2''(x) - \dfrac{\alpha^2}{d} x \sinh\left(\dfrac{d\eta_2(x)}{x}\right) + \dfrac{b}{x}\xi^2(x) = 0 \end{cases} \tag{15}$$

with the boundary conditions

$$\xi(0) = \xi(\infty) = \eta_1(0) = \eta_1'(\infty) = \eta_2(0) = \eta_2(\infty) = 0, \tag{16}$$

$$\eta_1(\infty) = \int_0^\infty \xi^2(x)dx,$$

and the simplest form of the asymptotical behavior of the desired solution researched

$$\xi(x) \sim e^{-x}, \eta_1(x) \sim const, \eta_2(x) \sim e^{-\alpha x}, \quad when \quad x \to \infty \tag{17}$$

where α, b, d are the "physical" parameters.

In our paper [1] we used the additional conditions of the connection between the parameters

$$\begin{aligned} m &= \frac{dT_1^2}{5.2629 \cdot 10^2}, \\ n &= m\frac{d\alpha^2}{b \cdot 6.2125 \cdot 10^2}, \end{aligned} \tag{18}$$

where

$$T_1 = \int_0^\infty \xi^2(x)dx = \eta_1(\infty), \tag{19}$$

m is a electron effective mass, n is a ion concentration, and they can be determined experimentally. In this case the nonlinear boundary value problem with the three parameters can be formulated as a nonlinear boundary value problem with one parameter and the additional conditions (19)

$$\begin{cases} \xi''(x) - \xi(x) + \xi(x)\dfrac{\eta_1(x) - \eta_2(x)}{x} = 0 \\ \eta_1''(x) + \dfrac{1}{x}\xi^2(x) = 0 \\ \eta_2''(x) - \dfrac{A_2}{A_1^2} x \eta_1^4(\infty) \sinh\left(\dfrac{A_1 \eta_2(x)}{x\eta_1^2(\infty)}\right) + \dfrac{b}{x}\xi^2(x) = 0 \end{cases} \tag{20}$$

where

$$\begin{aligned} dT_1^2 &= A_1, \\ d\alpha^2 &= A_2, \end{aligned} \tag{21}$$

$$A_1 = m \cdot 5.2629 \cdot 10^2,$$
$$A_2 = \frac{bn}{m} \cdot 6.2125 \cdot 10^2 \quad (22)$$

are known constants.

- **The bielectron problem**
  This problem is formulated as a system of equations [4]

$$\begin{cases} \xi''(x) - \xi(x) - z_1 \frac{e^{-\alpha x}}{x}\xi(x) + z_2 \frac{\eta_1(x) - \eta_2(x)}{x}\xi(x) = 0 \\ \eta_1''(x) + \frac{1}{x}\xi^2(x) = 0 \\ \eta_2''(x) - \alpha^2\eta_2(x) + \frac{\xi^2(x)}{x} = 0 \end{cases} \quad (23)$$

with the boundary conditions

$$\xi(0) = \xi(\infty) = \eta_1(0) = \eta_1'(\infty) = \eta_2(0) = \eta_2(\infty) = 0, \quad (24)$$

$$\eta_1(\infty) = \int_0^\infty \xi^2(x)dx,$$

and the simplest form of asymptotics of the desired solution

$$\xi(x) \sim e^{-x}, \eta_1(x) \sim const, \eta_2(x) \sim e^{-\alpha x}, \quad when \quad x \to \infty, \quad (25)$$

where α, z_1, z_2 are "physical" parameters.

- **The QCD model with the decreasing potential**
  The problem of the barion states investigations in the framework of $1/N$ QCD-decomposition was reduced to the solving of the Schrödinger equation with the nonlocal potential [3]

$$-\frac{1}{2M}\Delta\Psi(\vec{r}) - \Psi(\vec{r}) \int V(|\vec{r} - \vec{r}'|) \, |\Psi(\vec{r}')|^2 \, d\vec{r}' = E\Psi(\vec{r}) \quad (26)$$

$$\int |\Psi(\vec{r})|^2 \, d\vec{r} = 1 \quad (27)$$

where M is a quark mass, E is a quark energy of the bound state, V is a potential for quarks pair interactions.

In refs.[8,9] spherically-symmetrical solutions of eq.(26) for the potential

$$V = (\alpha_1 r^{-1} - \alpha_2 r) \quad (28),$$

where α_1 and α_2 are constants, was reduced to the initial value problem for the nonlinear differential equation of the 6-th order (if the $\alpha_2 = 0$, then we have the 4-th order).

Here we use the equation (26) and the potential (28) too, but try to apply our method. Let us consider this problem as a system of equations

$$\begin{cases} -\frac{1}{2M}\Delta\Psi(\vec{r}) - [\alpha_1 V_1(\vec{r}) - \alpha_2 V_2(\vec{r})]\Psi(\vec{r}) = E\Psi(\vec{r}) \\ \Delta V_1(\vec{r}) + 4\pi \, |\Psi(\vec{r})|^2 = 0 \\ \Delta V_2(\vec{r}) - 2V_1(\vec{r}) = 0 \, . \end{cases} \quad (29)$$

Thus we use the spherically-symmetrical solution in the form

$$\Psi(\vec{r}) = \frac{\Psi(r)}{r}Y_{00}, \quad V_1(\vec{r}) = \frac{V_1(r)}{r}Y_{00}, \quad V_2(\vec{r}) = \frac{V_2(r)}{r}Y_{00}, \quad (30)$$

where $Y_{00} = 1/\sqrt{4\pi}$ is a spherical function $Y_{lm}(\theta, \psi)$ for $l = 0, m = 0$. Substituting the expressions eq.(30) in eq.(29) and replacing

$$E = -\varepsilon, \quad \varepsilon > 0, \quad \bar{V}_1 = \frac{V_1}{\sqrt{4\pi}}, \quad \bar{V}_2 = \frac{V_2}{\sqrt{4\pi}},$$

we obtain

$$\begin{cases} \frac{1}{2M}\Psi(r)'' - \varepsilon\Psi(r) + \frac{1}{r}[\alpha_1 \bar{V}_1(r) - \alpha_2 \bar{V}_2(r)]\Psi(r) = 0 \\ \bar{V}_1'' + \frac{\Psi(r)^2}{r} = 0 \\ \bar{V}_2'' - 2\bar{V}_1 = 0 \, . \end{cases} \quad (31)$$

The transformation

$$r = \lambda x, \quad \Psi = \gamma\xi, \quad \eta_1 = \bar{V}_1, \quad \eta_2 = \bar{V}_2$$

reduce the system(31) and the normalization (27) to the system

$$\begin{cases} \xi'' - \xi + \frac{1}{x}(\eta_1 - z\eta_2)\xi = 0 \\ \eta_1'' + \frac{\xi^2}{x} = 0 \\ \eta_2'' - 2\eta_1 = 0 \end{cases} \quad (32)$$

where

$$\lambda = \frac{1}{\sqrt{2M\varepsilon}}, \quad \gamma = \sqrt{\frac{\varepsilon}{\alpha_1}}, \quad z = \frac{\alpha_2\lambda}{\alpha_1},$$

$$\varepsilon = \frac{2M\alpha_1^2}{N^2}, \quad N = \int_0^\infty dx\xi^2(x) \tag{33}$$

The researched solutions of the system(32) satisfy the asymptotic conditions

$$\xi(0) = 0, \quad \eta_1 = 0, \quad \eta_2 = 0$$

$$\xi(x) \to \xi_A(x), \quad \eta_1(x) \to \eta_{1A}(x), \quad \eta_2(x) \to \eta_{2A}(x) \quad if \quad x \to \infty \tag{34}$$

where

$$\xi_A = \frac{1}{2}\zeta^{-\frac{1}{4}}e^{-\frac{2}{3}\zeta^{3/2}}, \quad \zeta = (zN)^{\frac{1}{3}}(x + \frac{1}{zN})$$

$$\eta_{1A} = N, \quad \eta_{2A} = x^2N + \frac{1}{3}N_2 \tag{35}$$

$$N_2 = \int_0^\infty x^2\xi^2(x)dx \tag{36}$$

3 Numerical investigations and some results

For the numerical investigations the computer program was developed. This program was tested by means of the specially constructed example with the known analytical solution

$$\begin{cases} y_1''(x) - y_1(x) + \dfrac{y_2(x) - y_3(x)}{x}y_1(x) - R_1(x) = 0 \\ y_2''(x) + \dfrac{1}{x}y_1^2(x) - R_2(x) = 0 \\ y_3''(x) - \dfrac{A_2}{A_1^2}xC^4 \sinh\dfrac{A_1y_3(x)}{xC^2} + \dfrac{by_1^2(x)}{x} - R_3(x) = 0 \end{cases} \tag{37}$$

where

$$\begin{cases} R_1(x) = Be^{-x}(-2 + C - Ce^{-x} - Dx^2e^{-2x}) \\ R_2(x) = -Ce^{-x} + B^2xe^{-2x} \\ R_3(x) = 2De^{-2x}(1 - 4x + 2x^2) - \dfrac{A_2}{A_1^2}xC^4 \sinh\dfrac{A_1Dxe^{-2x}}{C^2} + bB^2xe^{-2x} \end{cases} \tag{38}$$

$0 < x < x_R$, and the boundary conditions

$$\begin{cases} y_1(0) = y_2(0) = y_3(0) = 0 \\ y_1'(x_R) + (1 - \dfrac{1}{x_R})y_1(x_R) = 0 \\ y_2'(x_R) + y_2(x_R) = C \\ y_3'(x_R) + (2 - \dfrac{2}{x_R})y_3(x_R) = 0 \end{cases} \tag{39}$$

with the additional condition

$$\int_0^{x_R} y_1^2(x)dx - C = 0. \tag{40}$$

Fig.1 shows the convergency of the iterations to the exact solutions for this example. The discontinuous in asymptotic region function was simulated for the initial approximation. The smoothing of this discontinuite as a result of the iterations are obtained. The initial approximation discrepancy is ~ 0.1, the iterations stop if the discrepancy $\sim 10^{-9}$. Tab.1 shows the solution convergency for the sequence of the twice compressible grids for the test example with $h = 0.132$. The relation

$$\sigma_i = \frac{y_i(h) - y_i(\frac{h}{2})}{y_i(\frac{h}{2}) - y_i(\frac{h}{4})} \cong 16$$

shown in Tab.1, confirms the 4th order of the convergency. For calculations the constants: $B = 1.0$, $C = 0.25$, $D = 1.0$, $n = 1.0$, $b = 0.5$, $m = 1.0$ were used.

Tab. 1

x	y_i(anal.)	y_i(in.apr.)	$y_i(h)$	$y_i(\frac{h}{2})$	$y_i(\frac{h}{4})$	$\frac{y_i(h)-y_i(\frac{h}{2})}{y_i(\frac{h}{2})-y_i(\frac{h}{4})}$
2.11	0.255543	0.281097	0.255550	0.255543	0.255543	15.87
	0.219751	0.263701	0.219765	0.219752	0.219751	15.88
	0.065302	0.071832	0.065269	0.065301	0.065303	15.83
3.04	0.145809	0.160390	0.145813	0.145809	0.145809	15.86
	0.237993	0.285592	0.238008	0.237994	0.237993	15.87
	0.021260	0.023386	0.021232	0.021259	0.021261	15.80
4.09	0.068360	0.000000	0.068362	0.068360	0.068360	15.86
	0.245824	0.294988	0.245838	0.245824	0.245823	15.87
	0.004673	0.000000	0.004650	0.004672	0.004674	15.76
5.02	0.033261	0.000000	0.033262	0.033261	0.033261	15.85
	0.248342	0.298011	0.248357	0.248343	0.248342	15.87
	0.001106	0.000000	0.001087	0.001106	0.001107	15.73
6.07	0.014005	0.000000	0.014006	0.014005	0.014005	15.84
	0.249423	0.299308	0.249439	0.249424	0.249423	15.87
	0.000196	0.000000	0.000180	0.000196	0.000197	15.69

The next table shows the dependence of the solution from the choice of the right boundary x_R if grid step $h = 0.03$($\delta_i = y_i - y_{ianal}$ is a difference of numerical and analytical solutions) . It is obvious that the error of the approximations of asymptotical conditions can be done much smaller than the error of the discrete approximation of the problem.

Tab. 2

x	$\delta_1(x_R=3)$	$\delta_1(x_R=9)$	$\delta_1(x_R=15)$	$\delta_1(x_R=30)$	$\delta_1(x_R=45)$	$\delta_1(x_R=60)$
0.0	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
0.3	.2540E-02	.7764E-07	-.9610E-08	-.9391E-08	-.9315E-08	-.9276E-08
0.6	.4455E-02	.1366E-06	-.1613E-07	-.1574E-07	-.1560E-07	-.1553E-07
0.9	.5435E-02	.1661E-06	-.1914E-07	-.1864E-07	-.1846E-07	-.1838E-07
1.2	.5611E-02	.1703E-06	-.1937E-07	-.1882E-07	-.1862E-07	-.1853E-07
1.5	.5274E-02	.1584E-06	-.1790E-07	-.1734E-07	-.1714E-07	-.1705E-07
1.8	.4682E-02	.1388E-06	-.1563E-07	-.1509E-07	-.1491E-07	-.1481E-07
2.4	.3358E-02	.9637E-07	-.1080E-07	-.1035E-07	-.1020E-07	-.1012E-07
2.7	.2773E-02	.7798E-07	-.8720E-08	-.8323E-08	-.8185E-08	-.8115E-08
3.0	.2274E-02	.6239E-07	-.6957E-08	-.6612E-08	-.6492E-08	-.6431E-08
x	$\delta_2(x_R=3)$	$\delta_2(x_R=9)$	$\delta_2(x_R=15)$	$\delta_2(x_R=30)$	$\delta_2(x_R=45)$	$\delta_2(x_R=60)$
0.0	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
0.3	.3140E-02	.8031E-07	-.8341E-08	-.7656E-08	-.7417E-08	-.7297E-08
0.6	.5963E-02	.1509E-06	-.1542E-07	-.1408E-07	-.1361E-07	-.1338E-07
0.9	.8361E-02	.2085E-06	-.2091E-07	-.1895E-07	-.1827E-07	-.1792E-07
1.2	.1037E-01	.2540E-06	-.2498E-07	-.2244E-07	-.2155E-07	-.2110E-07
1.5	.1207E-01	.2903E-06	-.2798E-07	-.2488E-07	-.2380E-07	-.2326E-07
1.8	.1357E-01	.3203E-06	-.3023E-07	-.2660E-07	-.2534E-07	-.2470E-07
2.4	.1618E-01	.3691E-06	-.3347E-07	-.2882E-07	-.2720E-07	-.2638E-07
2.7	.1739E-01	.3907E-06	-.3474E-07	-.2960E-07	-.2781E-07	-.2690E-07
3.0	.1857E-01	.4112E-06	-.3591E-07	-.3027E-07	-.2831E-07	-.2732E-07
x	$\delta_3(x_R=3)$	$\delta_3(x_R=9)$	$\delta_3(x_R=15)$	$\delta_3(x_R=30)$	$\delta_3(x_R=45)$	$\delta_3(x_R=60)$
0.0	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00	.0000E+00
0.3	-.1021E-01	-.3373E-06	.3245E-07	.3214E-07	.3202E-07	.3197E-07
0.6	-.2225E-01	-.7380E-06	.6886E-07	.6816E-07	.6791E-07	.6779E-07
0.9	-.2756E-01	-.9138E-06	.8484E-07	.8398E-07	.8367E-07	.8352E-07
1.2	-.2695E-01	-.8935E-06	.8284E-07	.8199E-07	.8170E-07	.8154E-07
1.5	-.2316E-01	-.7674E-06	.7110E-07	.7038E-07	.7013E-07	.7000E-07
1.8	-.1832E-01	-.6071E-06	.5624E-07	.5567E-07	.5547E-07	.5537E-07
2.4	-.9828E-02	-.3255E-06	.3016E-07	.2985E-07	.2974E-07	.2969E-07
2.7	-.6831E-02	-.2262E-06	.2096E-07	.2074E-07	.2067E-07	.2063E-07
3.0	-.4630E-02	-.1533E-06	.1423E-07	.1408E-07	.1403E-07	.1401E-07

The same numerical tests were performed for all the real problems.

The next figures show some results, which we reached by our method. There are plots of the functions calculated for the zero and first mode of solvated electron model (eqs.(15)-(16)) for the set of parameters on Fig.2-3.

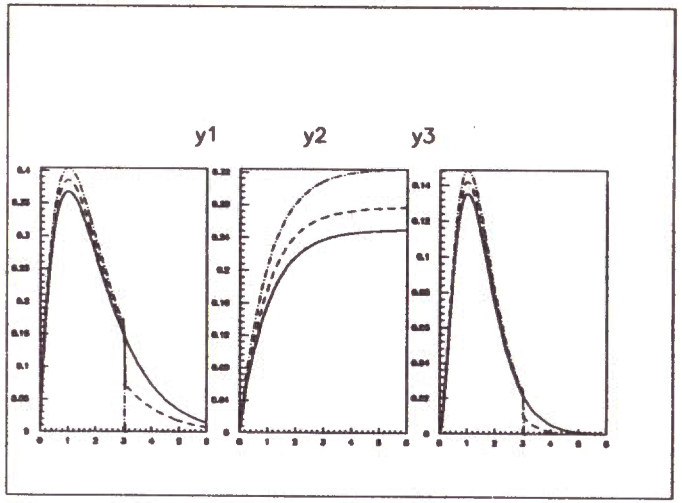

Figure 1: The solution of the example problem. The dot-dashed line is an initial approximation, dashed line is a first iterations and solid line is a solution which coincides with the analytical solution.

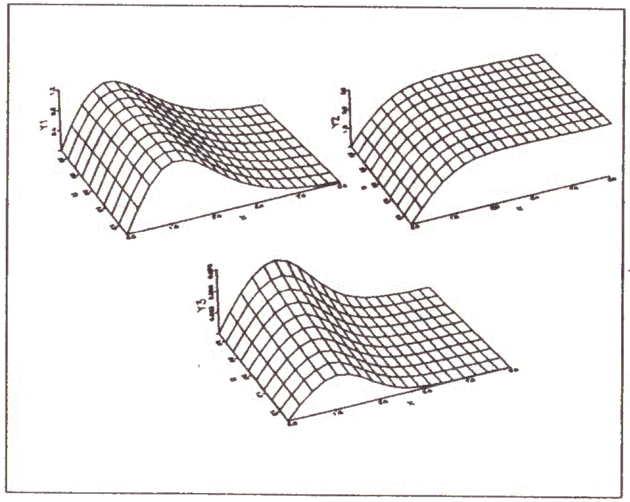

Figure 2: The zero mode solution of the polaron problem. b=1.0, m=2.0, n=0.5 (0.25) 3.0, $Y1 = \xi, Y2 = \eta_1, Y3 = \eta_2$.

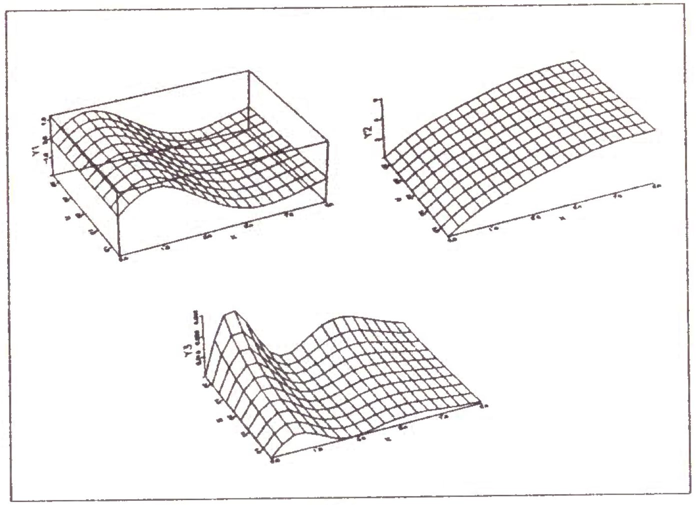

Figure 3: The first mode solution of the polaron problem. b=1.0, m=2.0, n=0.5 (0.25) 3.0, $Y1 = \xi, Y2 = \eta_1, Y3 = \eta_2$.

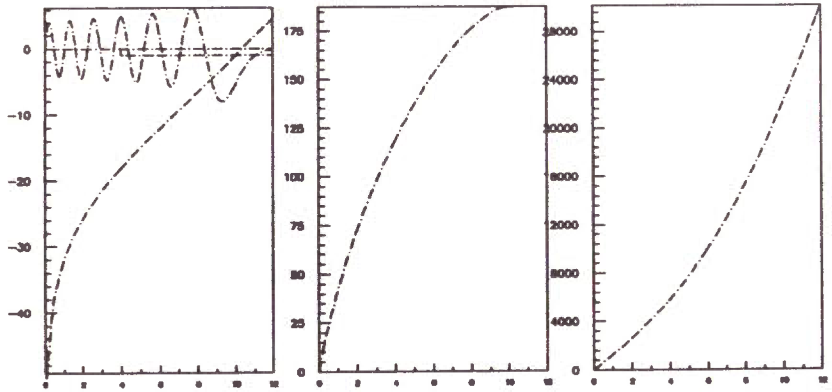

Figure 4: The solution of QCD model for eleventh mode.

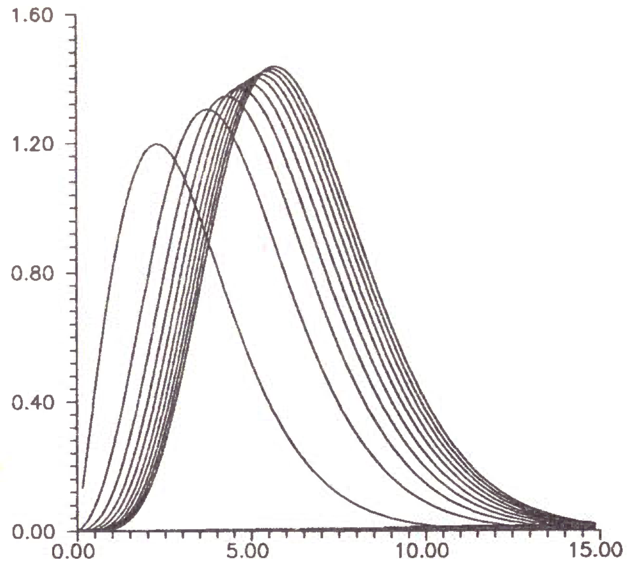

Figure 5: The plot of functions $\xi(x)$ for the $\alpha = 1., z_2 = 1, z_1 = 0.(2.)20$. The case of $z_1 = 0.$ is a polaron solution (the first one on the plot).

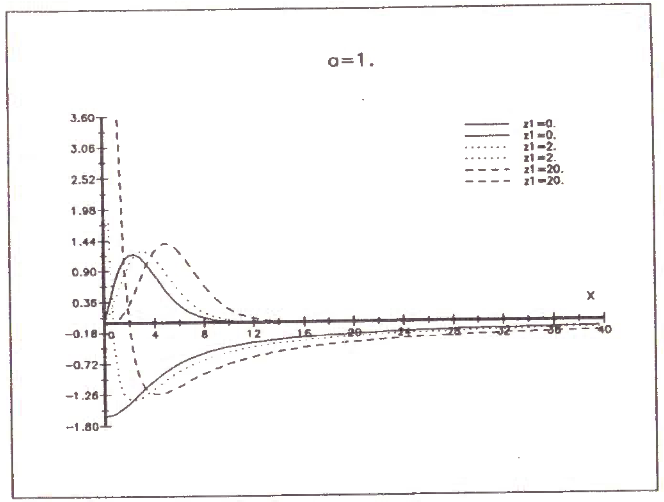

Figure 6: The solution of bielectron problem $\xi(x)$ and the corresponding potential V. The case of $z_1 = 0.$ is a polaron solution (the solid line on the plot).

The QCD-model calculations (eqs.(32)-(36)) were performed for different parameters z_1, z_2. We find the solutions for modes from zero up to eleven. This result is better than those in refs.[8-9]. The plot of the solutions with the 11-zero are drawn together with the potential for the QCD model in Fig.4.

We have preliminary results for the bielectron model too. The zero mode solutions (function $\xi(x)$) for different value of z_1 and $z_2 = 1$ are shown in Fig.5. In Fig.6 you can see these solutions for different z_1 and $z_2 = 1$ together with the function $V = -\left[-z_1\frac{e^{-\alpha x}}{x} + z_2\frac{\eta_1(x) - \eta_2(x)}{x}\right]$.

The physical analysis of these results can be done in the future. The main purpose of this work is the demonstration of possibilities of the suggested method.

All the calculations were performed on PC/AT-386 by using NDP-Fortran codes under MS/DOS. For the graphical illustrations and the graphical interface the HIGZ-package [10] was used.

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