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GRÖBNER BASIS TECHNIQUE, HOMOGENEITY AND SOLVING POLYNOMIAL EQUATIONS

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1. Introduction

Nonlinear algebraic equations arise in many fundamental and applied problems. The analysis and solving such systems is one of central directions of computer algebra development [1]. The most universal and developed algorithmic method for analysis and solution of systems of nonlinear algebraic equations is that based on Gröbner basis construction [2, 3] The computation time for Gröbner basis strongly depends on a term ordering to be chosen. In practice the pure lexicographical ordering and the total degree then inverse lexicographical (degree-reverse-lexicographical) one are used more often. The corresponding algorithms and packages are implemented in all modern computer algebra systems.

Unfortunately, in many practical cases in order to construct a Gröbner basis, extremely tedious algebraic computations should be performed which are caused by exponential complexity of the Buchberger algorithm [2] in a number of variables. In the case of finitely many solutions, i.e. a zero-dimensional ideal, the algorithm complexity is $O(d^{n^2})$ in polynomials degree d and a number of variables n for the lexicographical ordering and d^n for the degree-reverse-lexicographical one, respectively [4]. In this case the optimal strategy is to compute Gröbner basis for the degree-reverse-lexicographical ordering and then to recompute it to the lexicographical one according to algorithm [4], which also has polynomial complexity in d^n. As a result we obtain a "triangular" Gröbner basis with "separated" variables.

For positive dimensional ideals, i.e. for the systems with infinitely many solutions, both theoretical analysis of algorithms and a Gröbner basis construction become much more difficult. In this case the complexity of Buchberger algorithm is estimated to be $O(d^{n*2^n})$. That is why the most of computer algebra packages are designed mainly for zero-dimensional ideals and have not the necessary built-in facilities for the case of positive-dimensional ones. From the other hand, the systems of nonlinear algebraic equations with infinitely many solutions arise in practice quite often, for example, in analyzing integrability of nonlinear evolution equations [5, 6, 7] and isomorphism verification of Lie algebras [8].

In this paper we describe an approach to simplifying and often to solving polynomial systems with infinitely many solutions implemented in the form of REDUCE package ASYS. In contrast to many other packages based on the Gröbner basis technique, it allows to compute the independent sets of variables, which can be treated as free parameters, and due to them to split an initial polynomial into a set of triangular ones over rational function field.

In the case if a polynomial system possesses nontrivial homogeneity properties, it is transformed by another way into smaller subsystems with reduced number of variables and, therefore, necessary computational time is decreased drastically. Homogeneity is typical, in particular, for the problems considered in [5, 6, 8]. Some examples [4, 5, 7, 9], including those with finite many solutions for completeness, and results of comparison with other packages are given.

2. Basic definitions and notations

2.1. Dimension of polynomial ideal and sets of independent variables

The dimension of the algebraic variety defined by a polynomial system can be determined in terms of the Hilbert polynomial, which, in its turn, can be computed via Gröbner basis. This method is used, for example, in the GROEBNER package of the REDUCE computer algebra system. [10, 11].

However, the knowledge of dimension is unsufficient to analyze a structure of a variety for a positive dimensional ideal that is often necessary to find solutions in an algebraic form. One would like to find these solutions in the form of the explicit parametrization of different irreducible subvarieties of the whole variety for a given system.

In paper [12] the algorithm was proposed for finding a complete set of different maximal sets of variables, which are (algebraically) independent modulo given polynomial ideal. This algorithm is based on the knowledge of the Gröbner basis and the following statement.

For minimal or reduced [2] Gröbner basis G a set

$$S = \{x_{i_1}, \ldots, x_{i_r}\} \subseteq X = \{x_1, \ldots, x_n\}$$

is (strongly) *independent* modulo $I = Ideal(G)$, if and only if

$$T(S) \cap LT(G) = \{\emptyset\}, \qquad (1)$$

where $T(S)$ means a set of all possible monomials (terms) depending only on variables $x_i \in S$, and $LT(G)$ - a set of leading, w.r.t. chosen ordering, monomials in a Gröbner basis.

A maximal, in number of its elements, set corresponds to the highest dimensional variety. Therefore this number gives the dimension of the ideal. All other maximal (in the sense that the addition of any other variable violates condition (1)) independent sets correspond [12] to the isolated prime ideal associated with I and a number of elements in the set is a dimension of this prime ideal.

Therefore, if one considers elements of any maximal independent set as free parameters, then in other variables the original polynomial ideal is a zero-dimensional one and its lexicographical Gröbner basis has a "triangular" form with "separated" variables. The solving of such a triangular system may be done step by step starting with the last univariate polynomial in the lowest variable w.r.t. the chosen ordering. At each step the problem is eventually reduced to a univariate one. To find the roots or to split this polynomial into lower degree ones, one can try to factorize or to decompose it. In

addition to the REDUCE factorizator, a user of ASYS may use a simple decomposition algorithm of paper [13].

2.2. Homogeneity

Let us consider a system of polynomial equations

$$f_m = \sum_{(i)} a_{m,(i)} x^{(i)} = 0, \quad m = 1, 2, \ldots, M, \qquad (2)$$

where $(i) = (i_1, \ldots, i_n)$, $a_{m,(i)} = a^{(m)}_{i_1,\ldots,i_n}$, $x^{(i)} = x_1^{i_1} \cdots x_n^{i_n}$.

Definition. The system of polynomial equations (2) possesses a property of *homogeneity* if under the scale transformations

$$x_i \longrightarrow \alpha_i x_i, \quad x^{(i)} \longrightarrow \alpha^{(i)} x^{(i)}, \quad \alpha_i > 0, \qquad (3)$$

with at least one factor $\alpha_i \neq 1$, each of monomial of any given polynomial $f_m(x_1, \ldots, x_n)$ in (2) takes the same scale factor. In other words, for two arbitrary monomials (i) (j) of polynomial f_m the equality $\alpha^{(i)} = \alpha^{(j)}$ or $\sum_{k=1}^{n} (i_k - j_k)\tilde{\alpha}_k = 0$, where $\tilde{\alpha}_k = \log \alpha_k$, takes place. It should be noted, that in general case scale factors $\alpha^{(i)}$, corresponding to different polynomials f_m might be different.

This definition is closely connected with the concept of Γ - homogeneity [14], which is very useful for computation of homogeneous Gröbner bases, and already used, for example, in [15]. We, however, assume $\tilde{\alpha}_k$ to take any value from the coefficient field of system (2) but not only integer one as in [14], and use homogeneity not for the Gröbner basis construction for system (2) but for splitting the latter into a set of simpler subsystems (see section 3.2).

Equating the scale factors arising from different terms of each polynomial, we obtain a system of linear equations with integer coefficients

$$\sum_{k=1}^{n} z_{ik} \tilde{\alpha}_k = 0, \quad z_{ij} \in \mathbf{Z}. \qquad (4)$$

It is clear, that system (4) always has the trivial solution $\tilde{\alpha}_k = 0$ or $\alpha_k = 1$, $k = 1, 2, \ldots, n$.

From the other hand, if there is a nontrivial solution, then (4) has infinitely many solutions. In this case one can consider a part the variables $(\tilde{\alpha}_i)$ as free parameters. A maximal set of such free variables is a maximal independent one (section 2.1) modulo ideal generated by the left hand sides of system (4).

Definition. Variables x_i corresponding to an arbitrary scale factors $\tilde{\alpha}_i$ are called *homogeneous variables* for system (2), and their number is called its *homogeneity degree.* It is clear, that homogeneity degree is less or equal to dimension of the corresponding polynomial ideal.

3. Reduction of polynomial systems with infinitely many solutions

3.1. Reduction by maximal independent sets

Computation of a Gröbner basis allows one to verify compatibility of the initial system of polynomials and to determine whether it has finitely or infinitely many solutions [2]. In the latter case one can apply the method of reduction to zero-dimensional polynomial subsystems over rational function field implemented in ASYS package in the form of the following sequential steps

- Computation of a Gröbner basis for the original system according to Buchberger algorithm [2, 3] for some chosen ordering, typically lexicographical one.
  If $1 \in$ Gröbner basis, this means that the ideal generated by the input polynomials is improper one, or equivalently, that the input polynomials have no common roots [2] and the computation process is terminated.
- Computation of all the maximum independent sets using leading terms of Gröbner basis according to (1) and paper [12].
- Sequential sorting out all maximal independent sets obtained at the previous step and, treating the variables of each set as free parameters, then computing a lexicographical Gröbner basis in remaining variables.

As an output of this algorithm a set of triangular subsystems over rational function field is obtained. Their solutions give the parametrization of the subvarieties corresponding to the isolated prime ideals associated with the initial ideal. To overcome loss of solutions due to denominators of rational functions as coefficients in the final Gröbner bases, all those cases should be carefully analyzed. It can be done properly in the framework of Gröbner basis technique as well.

3.2. Homogeneity Reduction

Let us show, that if the system of polynomial equations under consideration possesses nontrivial homogeneity properties (section 2.2), it can be transformed into an equivalent, in terms of generic zeros, set of subsystems with a reduced number of variables by another way than in previous section.

We describe such a reduction in the form of the following sequential steps, recursively implemented in the ASYS package

1. **Generation and solving system of linear equations (4)**
   For the system to be solved, one can use any suitable method, for example, Gauss elimination. In the ASYS package we use the Gröbner basis technique on this step too. In this linear case both Buchberger algorithm and Gauss elimination method are equivalent [2].
   If this system has only trivial solution $\tilde{\alpha}_i = 0$, i.e. the original system is not

homogeneous one at all, the reduction process terminates, leaving initial polynomial system (2) unchanged.
   Otherwise, at this step the set of $l > 0$ arbitrary values $\tilde{\alpha}_i$ is obtained. Let those arbitrary scale factors and the corresponding homogeneous variables are $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_l$ and $x_1, \ldots, x_l$, respectively. A solution of system (2) is given by

$$\tilde{\alpha}_j = \sum_{k=1}^{l} q_{jk}\tilde{\alpha}_k, \quad q_{ij} \in \mathbf{Q}, \quad j = l+1, \ldots, n. \tag{5}$$

2. **Transformation of variables**
   From equations (5) taking into account that $\alpha_i = e^{\tilde{\alpha}_i}$ one obtains expressions for scale factors α_j, $j > l$ of transformation (3)

$$\alpha_j = \prod_{k=1}^{l} \alpha_k^{q_{ik}}$$

   in terms of arbitrary $\alpha_1, \ldots, \alpha_l$.
   Under transformation of non-homogeneous variables x_j, $j > l$

$$x_j = (\prod_{k=1}^{l} x_k^{q_{jk}})\tilde{x}_j \tag{6}$$

   each monomial in m-th equation of system (2) transforms multiplicatively

$$x^{(i)} = K_m \tilde{x}^{(i)}$$

   with the same factor

$$K_m(x_1, \ldots, x_l) = \prod_{k=1}^{l} x_k^{i_k + \sum_{j=l+1}^{n} q_{jk} i_j},$$

   depending only on the homogeneous variables. Correspondingly, system (2) transforms to the form

$$f_m(x_1, \ldots, x_n) = K_m(x_1, \ldots, x_l)\tilde{f}_m(\tilde{x}_{l+1}, \ldots, \tilde{x}_n). \tag{7}$$

3. **Reductions by non-zero homogeneous variables**
   Let all $x_i \neq 0$, $i = 1, \ldots, l$. Then multiplicative factors K_m in (7) can be omitted and the system of equations (2) in n variables reduces to the subsystem

$$\tilde{f}_m(\tilde{x}_{l+1}, \ldots, \tilde{x}_n) = 0, \quad (m = 1, \ldots, M) \tag{8}$$

   in $n - l$ variables. Compatibility of this subsystem can be verified by construction of its Gröbner basis and, in the case of its compatibility, the further analysis and derivation of the solutions may be done within this technique. Return to "old" variables is performed by inverse transformation to (6). Because homogeneous variables are treated as free parameters, the special analysis of their zero values should be done according to the next step.

4. **Reduction by vanishing homogeneous variables**
In their turn all different sets of homogeneous variables when at least one variable vanishes, are considered. All zero variables of each such a set are substituted in initial system (2) and the process of homogeneity reduction is performed again as long as new systems in reduced number of variables are generated.
All inconsistent sets with zero homogeneity variables are rejected at step 3. Those of sets for which all the polynomials of (2) vanish, are immediately added to the solution list and subsequently the next set with zero homogeneous variables is considered.

In general case, vanishing some homogeneous variables may produce a new additional homogeneous variable that, in turn, provides additional homogeneity reduction.
It is worth noting that the reduction by maximal independent sets may be performed after the homogeneity reduction. It often gives the most appropriate final set of subsystems to find solutions (see ref.[8] for example).

4. Description of the ASYS package

4.1. General structure

The ASYS package is written in the symbolic mode language Rlisp of the computer algebra system REDUCE 3.4 [11] and consists of a number of modules providing a user with the following facilities in accordance with the methods described above.

- Gröbner basis constructing by Buchberger algorithm [2, 3];
- determination of the dimension of a variety for a given polynomial system, computation of all sets of independent variables and reduction by these sets;
- verification of homogeneity properties and carrying out homogeneity reduction;
- polynomial decomposition.

Because the basic recursive polynomial representation used in REDUCE does not provide reasonable efficiency of a Gröbner basis construction, the ASYS package much like the REDUCE standard package GROEBNER [10, 11] uses the distributive representation.
Let a polynomial be given in the form $f = \sum_{i=1}^{m} c_i u_i = \sum T_i$, where u_i– power products $x_1^{i_1}, \ldots, x_n^{i_n}$, and c_i are their coefficients. Then, in the distributive representation this polynomial will have the form $((T_1)(T_2)\ldots(T_m))$, where $T_i = (D_i.c_i)$– a dot pair, $D_i = (i_1 i_2 \ldots i_n)$– a list of exponents of power product u_i, $c_i =< s.q. >$ – the coefficient at power product u_i in the form of standard quotient.

4.2. Special switches

The package contains various switches **lexord, setord, setdim, setgb, scaletest, scale** for control over the reduction process and a Gröbner basis construction, where

lexord selects a term ordering (pure lexicographical one if the switch is on and degree-reverse-lexicographical ordering otherwise);

setord generates an heuristically optimal ordering [16];

setdim computes a dimension of a polynomial ideal and maximal independent sets of variables;

setgb performs reduction by maximal independent sets with a Gröbner basis construction for each subsystem;

scaletest verifies of homogeneity properties without doing the homogeneity reduction;

scale performs homogeneity reduction.

By default only **lexord** switch is on and the others are off. In this case the call to the main procedure of ASYS, which has the same syntax as standard REDUCE package GROEBNER, provides nothing more than computation of a lexicographical Gröbner basis for the system under consideration.

5. Examples and comparison with other packages

In this section a number of polynomial systems are considered and comparison of ASYS with high efficient standard REDUCE 3.4 package GROEBNER and the special-purpose systems for algebraists ALPI and FELIX [15] is given.
Examples I–III were taken from [5, 7]. These systems arise in integrability analysis of nonlinear evolution equations which play an important role in modern mathematical physics and applied mathematics. Well-known examples of zero-dimensional ideals IV–V [4, 9] have now become classical benchmark for Gröbner basis computation, are added for completeness. Note that two last examples distinguish from each other in only one term and this leads to drastic distinction in computing time.

All computations using ASYS and GROEBNER as well as FELIX have been performed on an 25 MHz MS-DOS based AT/386 computer with 8 Mb RAM. The results of comparison are given in Tables 1 and 2. Data for AlPI were taken from paper [9] and reduced in factor 16/25 taking into account a difference in computer performance. The comparison was done using two different orderings and the results are collected in Tables 1 and 2 with the following notations

Lex means lexicographical Gröbner basis computation for initial polynomial system,

Lex+Scale means homogeneity reduction and then lexicographical Gröbner basis computation for all the subsystems.
For system FELIX (Table 1) it means taking advantage of homogeneity to optimize the procedure of Gröbner basis construction [14, 15].

DegRevLex means Gröbner basis computation in the degree-reverse-lexicographical ordering.

Each example is supplied with the chosen ordering and dimension of the polynomial ideal. In addition, in examples I–III their homogeneity degree and in examples IV–V a number of their solutions are indicated.

Example I [5]

Ordering - $\lambda_7 > \lambda_6 > \lambda_5 > \lambda_4 > \lambda_3 > \lambda_2 > \lambda_1$.
Dimension of polynomial ideal - 3.
Homogeneity degree - 1.

$$\begin{aligned}
&\lambda_1(\lambda_4 - \lambda_5/2 + \lambda_6) = (2/7\lambda_1^2 - \lambda_4)(-10\lambda_1 + 5\lambda_2 - \lambda_3) = 0,\\
&(2/7\lambda_1^2 - \lambda_4)(3\lambda_4 - \lambda_5 + \lambda_6) = 0,\\
&a_1(-3\lambda_1 + 2\lambda_2) + 21a_2 = a_1(2\lambda_4 - 2\lambda_5) + a_2(-45\lambda_1 + 15\lambda_2 - 3\lambda_3) = 0,\\
&2a_1\lambda_7 + a_2(12\lambda_4 - 3\lambda_5 + 2\lambda_6) = b_1(2\lambda_2 - \lambda_1) + 7b_2 = b_1\lambda_3 + 7b_2 = 0,\\
&b_1(-2\lambda_4 - 2\lambda_5) + b_2(2\lambda_2 - 8\lambda_1) + 84b_3 = 0,\\
&b_1(8/3\lambda_5 + 6\lambda_6) + b_2(11\lambda_1 - 17/3\lambda_2 + 5/3\lambda_3) - 168b_3 = 0,\\
&15b_1\lambda_7 + b_2(5\lambda_4 - 2\lambda_5) + b_3(-120\lambda_1 + 30\lambda_2 - 6\lambda_3) = 0,\\
&-3b_1\lambda_7 + b_2(-\lambda_4/2 + \lambda_5/4 - \lambda_6/2) + b_3(24\lambda_1 - 6\lambda_2) = 0,\\
&3b_2\lambda_7 + b_3(40\lambda_4 - 8\lambda_5 + 4\lambda_6) = 0,
\end{aligned}$$

where

$$a_1 = -2\lambda_1^2 + \lambda_1\lambda_2 + 2\lambda_1\lambda_3 - \lambda_2^2 - 7\lambda_5 + 21\lambda_6, \quad a_2 = 7\lambda_7 - 2\lambda_1\lambda_4 + 3/7\lambda_1^3,$$
$$b_1 = \lambda_1(5\lambda_1 - 3\lambda_2 + \lambda_3), \quad b_2 = \lambda_1(2\lambda_6 - 4\lambda_4), \quad b_3 = \lambda_1\lambda_7/2,$$

Example II [5]

Ordering - $t > x > y > z$.
Dimension of polynomial ideal - 2.
Homogeneity degree - 1.

$$\begin{aligned}
&-2z^3t + (3z^2t - 2z^2 - 6zyt + 6zy + 6y^2t - 6y^2)x - ztx^2 = 0,\\
&18z^3t^2 - 9z^3t - 18z^2yt^2 + 18z^2yt + 18zy^2t^2 - 18zy^2t +\\
&(-27z^2t^2 + 24z^2t - 5z^2 + 63zyt^2 - 78zyt + 15zy - 63y^2t^2 +\\
&78y^2t - 15y^2)x + 9zt^2x^2 = 0,\\
&-8z^4t + (6z^3t - 6z^3 - 12z^2yt + 12z^2y + 12zy^2t - 12zy^2)x +\\
&(5z^2t - 4z^2 - 18zyt + 18zy + 18y^2t - 18y^2)x^2 - 3ztx^3 = 0,\\
&(3t - 5)z^2y - 15(t - 1)zy^2 + 10(t - 1)y^3 +\\
&(zy + 3y^2t - 3y^2)x - ytx^2 = 0.
\end{aligned}$$

Example III [7]

Ordering - $a_2 > b_2 > a_4 > b_4 > a_1 > b_1 > a_3 > b_3 > a_0 > b_0$.
Dimension of polynomial ideal - 6.
Homogeneity degree - 3.

$$e_k = \hat{e}_k = 0, \quad (k = 1 \div 6),$$

where $\hat{e}_k = e_k\,|_{a_i \Leftrightarrow b_i}$ and

$$\begin{aligned}
&e_1 = a_1\ (a_3 - a_4) - a_4\ (b_3 - b_4)\ ,\\
&e_2 = (2a_3 - a_4)\ y_1 - b_2y_2\ , \qquad y_1 = 6a_0a_3b_2 + (a_0 - b_0)\ (a_1^2 + a_4b_2)\ ,\\
&e_3 = a_2y_1 - (2b_3 - b_4)\ y_2\ , \qquad y_2 = 6a_0a_2b_3 + (a_0 - b_0)\ (a_1a_2 + a_4b_1)\ ,\\
&e_4 = 3a_0\ (a_2b_2 + a_3b_3) + (a_0 - b_0)\ (a_1 + b_3)\ a_4\ ,\\
&e_5 = 2\ (2a_0^2 + 8a_0b_0 - b_0^2)\ a_3b_3 + 2\ (a_0 - b_0)\ (4a_0 - b_0)\ a_3b_4 -\\
&\quad 6a_0\ (a_0 + 2b_0)\ a_2b_2 + (a_0 - b_0)^2\ (5a_1a_3 - 5a_1a_4 + a_4b_4) -\\
&\quad (a_0 - b_0)\ (7a_0 - b_0)\ a_4b_3\ ,\\
&e_6 = 3a_0\ [\ (a_0 - b_0)^3 - 3a_0\ (a_0 + 2b_0)^2\ ]\ (a_2b_2 + a_3b_3) +\\
&\quad (a_0 - b_0)^3\ [\ 3a_0a_1a_3 - 2\ (2a_0 + b_0)a_1a_4\ ] + 9a_0^2\ (a_0 - b_0)\\
&\quad [\ (a_0 - b_0)\ a_4 - (a_0 + 2b_0)\ a_3\ ]\ b_4 - (a_0 - b_0)\ (2a_0^3 - 30a_0^2b_0 + b_0^3)\ a_4b_3.
\end{aligned}$$

As a result of the computations with the ASYS package in **Lex+Scale** mode with **setdim** switch on we obtain 76 subsystems. For instance, one of the subsystems is given by the following output

---

Variables = (A2 B2 A4 B4 A1 B1 A0)
Parameters = (A3 B3 B0) % non-zero homogeneous variables
Zeros = NIL

GROEBNER BASIS

$$G(1) = A2 - \frac{1}{3} * \frac{A0 * A3^2}{B0 * B3} - \frac{5}{3} * \frac{A3^2}{B3}$$

$$G(2) = B2 + \frac{1}{3} * \frac{A0 * B3^2}{A3 * B0} + \frac{2}{3} * \frac{B3^2}{A3}$$

$$G(3) = A4 + \frac{1}{3} * \frac{A0 * A3}{B0} - \frac{1}{3} * A3$$

$$G(4) = B4 - \frac{1}{3} * \frac{A0 * B3}{B0} - \frac{8}{3} * B3$$

$$G(5) = A1 + B3$$

$$G(6) = B1 + A3$$

$$G(7) = A0^2 + 7 * A0 * B0 + B0^2$$

DIMENSION

M = NIL % set of maximal independent sets
D = 0 % dimension of ideal generated by the subsystem

Example IV [4, 9]

Ordering - $x_1 > x_2 > x_3 > x_4 > x_5$.

Dimension of polynomial ideal - 0 (number of solutions - 70).

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 + x_5 = 0,\\
&x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 0,\\
&x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 = 0,\\
&x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 = 0,\\
&x_1x_2x_3x_4x_5 - 1 = 0.
\end{aligned}$$

Example V [4]

Ordering - $x_4 > x_1 > x_2 > x_5 > x_3$.

Dimension of polynomial ideal - 0 (number of solutions - 64).

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 + x_5 = 0,\\
&x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 0,\\
&x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 = 0,\\
&x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 = 0,\\
&x_1x_2x_3x_4x_5 - 1 = 0.
\end{aligned}$$

Table 1

Computing time for examples I-III of positive dimensional polynomial ideals

Package	Mode	I	II	III
ASYS	Lex	2' 35"	35"	unsufficient memory
ASYS	Lex+Scale	25"	2.5"	2' 45"
GROEBNER	Lex	11"	7"	unsufficient memory
FELIX	Lex	22"	18"	unsufficient memory
FELIX	Lex+Scale	20"	7"	unsufficient memory
AlP*I*	Lex	72"	22"	–

Table 2

Computing time for examples IV-V of zero-dimensional polynomial ideals

Package	Mode	IV	V
ASYS	DegRevLex	42"	14'
GROEBNER	DegRevLex	10"	19' 40"
FELIX	DegRevLex	21"	6' 48"
AlP*I*	DegRevLex	1' 36 "	–

6. Conclusion

Different reduction methods and taking into account the special properties of the polynomial system in the framework of the Gröbner basis technique appears to have considerable promise to increasing its practical importance. Besides the reductions described in this paper, factorization of intermediate multivariate polynomials built-in the GROEBNER package of REDUCE [10] and discrete symmetry analysis of polynomial systems [17] are very fruitful.

In addition to drastic decrease in computing time, such reductions often lead to much more readable output. For example, the computation of the complete Gröbner basis for the polynomial system of example III (section 5) with the fast and effective FELIX system took more than 70 hours on a 33 Mhz 80486 DOS computer with 64 Mb RAM and produced 3 Mb output. As a result of homogeneity reduction ASYS took less than 3 minutes on a smaller machine (Table 1) with 22 Kb output involving 76 subsystems, one of them is shown above.

It should be also noted, that among output subsystems might be identical ones or those which describe eventually the same subvariety. The selection of a minimal set of these subsystems giving the same generic zeros as original system is important practical problem, which is not solved yet.

After reduction of the original system to a set of triangular subsystems the problem of finding common zeros of the original system reduces to subsequent finding zeros of univariate polynomials. In examples I–III the final subsystems have a very simple structure, so one can find an explicit form of solution in the analytical form. This remarkable property of these systems is probably a consequence of integrability of underlying nonlinear evolution equations [5, 6].

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Техника базисов Гребнера, однородность
и решение полиномиальных уравнений

Описан подход к анализу и решению систем нелинейных алгебраических уравнений, имеющих бесконечное число общих корней, и его реализация в виде пакета программ ASYS, написанных на языке аналитических вычислений REDUCE. Показано, как такие системы могут быть, в рамках техники базисов Гребнера, автоматически редуцированы эквивалентному набору подсистем с меньшим числом переменных. Этот метод является особенно эффективным для систем, обладающих нетривиальными свойствами однородности. Рассмотрены некоторые примеры и сравнение ASYS с другими пакетами.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Gröbner Basis Technique, Homogeneity
and Solving Polynomial Equations

An apporach to investigating and solving systems of nonlinear algebraic equations with infinitely many solutions and based on it the REDUCE package ASYS are described. It is shown that in the framework of Gröbner basis technique such systems can be transformed into an equivalent set of subsystems with a reduced number of variables in a completely automatic way. This method appears to be particularly effective for the systems possessing nontrivial properties of homogeneity. Some examples and results of comparison between ASYS and other packages are given.

The investigaiton has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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