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B.N.Khoromskij, G.E.Mazurkevich, E.P.Zhidkov

DOMAIN DECOMPOSITION METHOD FOR SOLVING ELLIPTIC PROBLEMS IN UNBOUNDED DOMAINS

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INTRODUCTION [1]

The domain decomposition method (DD-method) for solving elliptic boundary value problems have been under fixed attention during the last few years, see e.g. [1],[4],[5],[6],[9],[11],[16], [19]-[24],[26]. This is explained by the fact that DD-method has an intrinsic parallelism, a complex problem can be represented as a number of simple problems and in iterations over subdomains only boundary unknowns can be involved. In this way DD-method can be considered as one of the incarnations of the boundary element method (BEM).

This paper concerns the computational aspects of the DD-method for elliptic problems in an unbounded domain. In the framework of the combined method such problems are reduced to elliptic boundary value problem (linear or nonlinear) in an auxiliary bounded domain with an integral (nonlocal) boundary condition,see e.g. [8],[15],[19],[25]. In the process of solving the combined problem there are two main laborious steps: the solution of interior Dirichlet problem in an auxiliary domain and the solution of exterior Neumann problem outside this domain. The solution of the first problem can be effectively obtained by using the DD-method. The second one can be solved in the framework of integral or differential approach. In the first case the exterior problem is reduced to the boundary integral equation of the second kind defined on the surface of the auxiliary domain. This boundary equation can be effectively solved by using either panel clustering techique [12] or techique based on an application of special surfaces [28]. On the other hand in differential approach a boundary value elliptic problem with an artificial boundary conditions approximating the behavior of the unknown function at infinity have to be solved.

Below we consider the application of the DD-method for solving two- and three-dimensional exterior problems in the framework of differential approach.

We suggest a variant of the DD-algorithm for elliptic problems in an unbounded domain in which under the appropriate choice of mesh sizes in dif-

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ferent subregions based on the information available on the asymptotic decay of the solution we can deal with a small number of the subdomains (Lemma 1). Since only unknowns on the boundaries of subdomains take part in iterative processes effective methods of a partial solution of the subproblems can be used [3]. Using effective preconditioners in preconditioned conjugate gradient (PCG) method to solve arising interface problem the solution can be obtained with computational work proportional to the number of unknowns on the boundary of auxiliary domain with only a logarithmic factor (Theorem 2). In this way the suggested DD-algorithm can be considered as an alternative to using boundary integral equations for solving exterior problems.

This paper is organized as follows. In Part 1 we briefly discuss the boundary equations of the DD-method and their fast solvers. We shall not go deep into the construction of preconditioners for the arising boundary operators which are well known, see e.g. [4],[5],[17],[20],[19],[21],[24],[26]. We simply give the estimates of the numerical work to solve the boundary equations of the DD-method by using PCG method with corresponding interface preconditioners. In Part 2 we briefly describe the known variant of the combined method to solve elliptic problems in unbounded domains, earlier suggested in [19],[27], and corresponding computational expenditures. In Part 3 we consider the substructuring scheme for an unbounded domain adapted to the given asymptotic decay of a harmonical function at infinity. We also present the estimates of the computational work to solve exterior elliptic problems using the DD-method with the suggested partitioning of an unbounded domain. In Part 4 we outline an application of the new method to solve a nonlinear problem. We also present numerical experiments demonstrating the effectiveness of suggested algorithm.

1.BOUNDARY INTERFACE PROBLEMS OF DD-METHOD

Let us consider a parallelepiped $\Omega = \{x_k \, : \, 0 \le x_k \le A_k; \, k = 1 \div 3\}$ partitioned by $n_k - 1$, $k = 1 \div 3$, planes parallel to Cartesian system planes into $M = n_1 n_2 n_3$ subdomains $\Omega_i = \{x_k \, : \, a_{i_k - 1} < x_k < a_{i_k}, \, k = 1 \div 3\}$ where $i = (i_1, i_2, i_3)$, $i_k = 1 \div n_k$, $a_0 = 0$, $a_{n_k} = A_k$, $k = 1 \div 3$. We also denote by $\Gamma_i = \partial\Omega_i$ and by $\Gamma = (\cup_{i=1}^M \Gamma_i) \backslash \partial\Omega$ - the union of interior boundaries.

Let the arbitrary positive constants $\mu_i > 0$ be given for any i. We consider

the following elliptic problem:

Problem A. Find the function $w \in H_0^1(\Omega)$ such that

$$\sum_{i=1}^{M} \mu_i \int_{\Omega_i} \nabla w \nabla \eta dx = \int_{\Gamma} \psi \eta ds \tag{1}$$

for all $\eta \in H_0^1(\Omega)$.

We suppose that for the given function ψ such that $\psi \mid_{\Gamma_i} \in H^{-\frac{1}{2}}(\Gamma_i)$ the equation (1) has the unique solution $w \in H_0^1(\Omega)$.

We denote the space of traces $u_i = \gamma_i w$ on Γ_i of functions $w \in H_0^1(\Omega)$ by $Y_i \subset H^{\frac{1}{2}}(\Gamma_i)$ and the space of traces $u = \gamma w$ on Γ of those functions by Y. The space Y is equipped with the norm

$$\|u\|_Y^2 = \Sigma_i \|\gamma_i w\|^2_{H^{\frac{1}{2}}(\Gamma_i)}.$$

Further we use the Poincaré-Steklov operators

$$S_i^{-1} \, : \, H^{\frac{1}{2}}(\Gamma_i) \to H^{-\frac{1}{2}}(\Gamma_i), \quad S_i^{-1} u = v,$$

where $v = (\partial z / \partial n)$ on Γ_i and z is a harmonic continuation of $u \in Y_i$ into Ω_i such that

$$\int_{\Omega_i} \nabla z \nabla \eta dx = 0, \quad \forall \eta \in H_0^1(\Omega_i), \quad u = \gamma_i z.$$

Note that the operator S_i^{-1} is continuous, symmetric and positive definite on $Y_i / Ker S_i^{-1}$ [1]:

$$(S_i^{-1} u, v)_{L_2(\Gamma_i)} = (u, S_i^{-1} v)_{L_2(\Gamma_i)}, \quad (S_i^{-1} z, z) \ge \alpha \|z\|^2_{H^{\frac{1}{2}}(\Gamma_i)}, \; \alpha > 0$$

for all $u, v \in H^{\frac{1}{2}}(\Gamma_i)$, $z \in H^{\frac{1}{2}}(\Gamma_i)/Ker S_i^{-1}$. Besides

$$Ker \; S_i^{-1} = \{u \in Y_i : \, u = const, \; x \in \Gamma_i\}.$$

According to [17] determine an operator $\mathcal{A} : Y \to Y'$ such that

$$< \mathcal{A} u, v > = \sum_{i=1}^{M} \mu_i (S_i^{-1} u_i, v_i)_{L_2(\Gamma_i)}, \quad \forall u, v \in Y.$$

Then a boundary equation of the DD-method for the decomposition determined above and connected with the Problem A takes the following form [17]:
Problem B. Find the function $u \in Y$ such that

$$< \mathcal{A}u, \eta >= (\psi, \eta), \quad \forall \eta \in Y. \tag{2}$$

The operator $\mathcal{A}$ is symmetric and positive definite on the Hilbert space Y and the equivalent norm in Y can be given as $\|u\|_Y^2 =< \mathcal{A}u, u >$ [17]. Note that for the unique solution of the Problem B we have $u = w \mid_\Gamma$, where w is the solution of the Problem A.

Let $Y_h \subset Y$ be a family of a finite element spaces on $\Gamma = \cup_i \Gamma_i$ determined by a regular triangulation of every facet of Ω_i and corresponding regular families of finite element spaces on Γ_i in sense of Babuska and Aziz [2]. Let H be a subdomain size, h be a mesh size and $N = H/h$ be a maximum number of unknowns in one direction in Ω_i. Let $p = n_k$, $k = 1,2,3$. Then we have $dimY_h = O(p^3 N^2)$, where $p^3 = M$ is the total number of subdomains.

We consider the Galerkin-type boundary element scheme:
Problem C. Find the element $u_h \in Y_h$ such that

$$< \mathcal{A}u_h, \eta >= (\psi, \eta), \quad \forall \eta \in Y_h. \tag{3}$$

Remark 1. Analogous constructions can be obtained if we define the Poincaré-Steklov operators S_i^{-1} by means of "h-harmonic" continuation of a boundary function for some finite difference or finite element schemes in Ω_i. Besides, the boundary reduction (3) can be also constructed for elliptic operators with varying coefficients in subdomains. In any case one can represent (3) in a matrix form as $\mathcal{A}_h U_h = \psi_h$.

Recent developments in preconditioning techniques for the efficient solution of elliptic boundary value problems allow one to costruct a family of symmetric and easily invertible interface preconditioners $\mathcal{B}_0, \mathcal{B}_1$ spectrally close to $\mathcal{A}_h$ such that for piecewise linear finite elements we have the following estimates for the condition number of the preconditioned operator, see e.g. [4],[5],[9],[19],[20],[21],[24]

$$\kappa(\mathcal{B}_1^{-1}\mathcal{A}_h) = O(1 + ln^2(H/h)), \quad \kappa(\mathcal{B}_0^{-1}\mathcal{A}_h) = O(1),$$

where the latter is bounded independently of a substructure size H, a mesh size h and a number M of subdomains.

Remark 2. In order to reduce the residual $\mathcal{A}_h u_h^0 - \psi_h$ in (3) by a factor $\varepsilon = O(N^{-\sigma})$, $\sigma > 0$, using PCG method with the preconditioners $\mathcal{B}_1$ or $\mathcal{B}_0$ one needs

$$Q_C = O(p^3 N^2 ln^4 N)$$

arithmetical operations and

$$Q_M = O(p^3 N^2)$$

space of computer memory. These estimates are based on the use of the algorithm with complexity $O(N^2 ln^2 N)$ proposed in [3] for solving the Dirichlet partial problems in subdomains. For easily invertible preconditioner $\mathcal{B}_1$ we have $O(lnN \; ln\varepsilon^{-1})$ global iterations of PCG method with $O(p^3 N^2 ln^2 N)$ arithmetical operations for evaluation of the residual $\mathcal{A}_h u_h^n - \psi_h$ on any iteration step. For the spectrally equivalent operator of $\mathcal{B}_0$-type we have only $O(ln\varepsilon^{-1})$ global iterations but the inverting of $\mathcal{B}_0$ by means of PCG method has the complexity of the order $O(p^3 N^2 ln^2 N ln\varepsilon^{-1})$ where $O(p^3 N^2 lnN)$ is the computational work for evaluation of $\mathcal{B}_0 \psi_h$.

2.BOUNDARY REDUCTION FOR ELLIPTIC PROBLEMS IN COMBINED FORMULATION

Let $\Omega_0 \in I\!R^3$ be a bounded Lipschitz domain, $\Gamma = \partial\Omega_0$. We consider the following nonlinear problem of the combined method [19]:
Problem D. Find the function $w \in H^1(\Omega_0)$ such that

$$\int_{\Omega_0} \sum_{i=1}^{3} \mu(x, \mid \nabla w \mid) \frac{\partial w}{\partial x_i} \frac{\partial \eta}{\partial x_i} dx + \beta(S_E^{-1} u, \eta)_{L_2(\Gamma_0)} = \int_{\Gamma_1} \psi \eta ds, \; u = w \mid_{\Gamma_0},$$

for all $\eta \in H^1(\Omega_0)$.

Here $\psi \in H^{-\frac{1}{2}}(\Gamma_1)$ on the closed curve $\Gamma_1 \subset \Omega_0$; $\beta > 0$ is a given constant; $S_E^{-1} = L^{-1}(I + K) \; : \; H^{\frac{1}{2}}(\Gamma_0) \to H^{-\frac{1}{2}}(\Gamma_0)$ is the Poincaré-Steklov operator defined by the exterior boundary value problem for the Laplacian in $\Omega_E = I\!R^3 \backslash \Omega_0$ and for functions such that $\mid w(x) \mid = O(\mid x \mid^{-\nu})$, $\mid x \mid \to \infty$, $\nu \geq 2$. Here K and L are the integral operators of the theory of potential (for doubleand simple-layer potentials). Besides $\mu(x,t)$ is a given Lipschitz-continuous and monotone function of $t \in [0,\infty)$ for almost all $x \in \Omega_0$.

Since u is a trace of an unknown function w on the boundary $\Gamma_0 = \partial\Omega_0$ the Problem D is equivalent to the nonlinear boundary operator equation on the surface Γ_0 [19],[27]:

$$Ru \equiv S_I^{-1}u + S_E^{-1}u = 0, \quad x \in \Gamma_0, \tag{4}$$

where S_I is the nonlinear Poincaré-Steklov operator corresponding to a quasi-linear elliptic operator on the domain Ω_0; $u \in \tilde{H}^{\frac{1}{2}}(\Gamma_0) = \{v \in H^{\frac{1}{2}}(\Gamma_0) : (v, g_0)_{L_2(\Gamma_0)} = 0,\ K'g_0 = g_0\}$. Here we have

$$\|u\|^2_{\tilde{H}^{\frac{1}{2}}(\Gamma_0)} = (S_E^{-1}u, u)_{L_2(\Gamma_0)}.$$

The following theorem holds true

THEOREM 1.[19] The stationary Richardson method (with the preconditioner S_E^{-1}) for solving the boundary equation (4) with the transition operator

$$u_{n+1} = ((1-\tau)I - \tau S_E S_I^{-1})u_n, \quad 0 < \tau \leq 1, \quad n = 0, 1, \ldots$$

converges with the rate

$$\|u_n - u\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_0)} \leq \frac{\tau q^n}{1-q} \|Ru_0\|_{H^{-\frac{1}{2}}(\Gamma_0)},$$

where $q = max\,(1 - \tau m_0, 1 - \tau M_0)$ and the constants m_0, M_0 are defined by the following inequalities:

$$(Ru - Rv, u - v) \geq m_0(S_E^{-1}(u - v), u - v),$$

$$\|Ru - Rv\|^2_{H^{-\frac{1}{2}}(\Gamma_0)} \leq M_0^2(S_E^{-1}(u - v), u - v),$$

for all $u, v \in \tilde{H}^{\frac{1}{2}}(\Gamma_0)$.

One of the most laborious stages of this iterative process is the evaluation of the function

$$\phi = S_E\psi, \quad x \in \Gamma_0, \quad \psi \in H^{-\frac{1}{2}}(\Gamma_0) \tag{5}$$

which is equivalent to the solution of the exterior Neumann problem for the Laplace operator with $\psi = (\partial w/\partial n)\,|_{\Gamma_0}$.

Using the explicit representation $S_E = (I + K)^{-1}L$ one can iteratively solve by $O(N^2 ln^6 N)$ operations the equation

$$(E + K)\phi = f, \quad f = L\psi,$$

by the panel clustering technique for multiplication of the stiffness matrix K_h on vector with accuracy $O(N^{-1})$ [12] or by methods oriented on special surfaces [28] which need $O(N^3 ln^2 N)$ arithmetical operations. Besides, there is another approach to the problem (5) which uses the hypersingular integral equations of the first kind [14],[16] with multigrid solvers [13].

Now, we consider the new approach based on the boundary interface equations of DD-method (see Part 1) defined on some artificial surfaces (coarse mesh) which contain as a part the boundary Γ_0 and form the decomposition of a space-extensive domain.

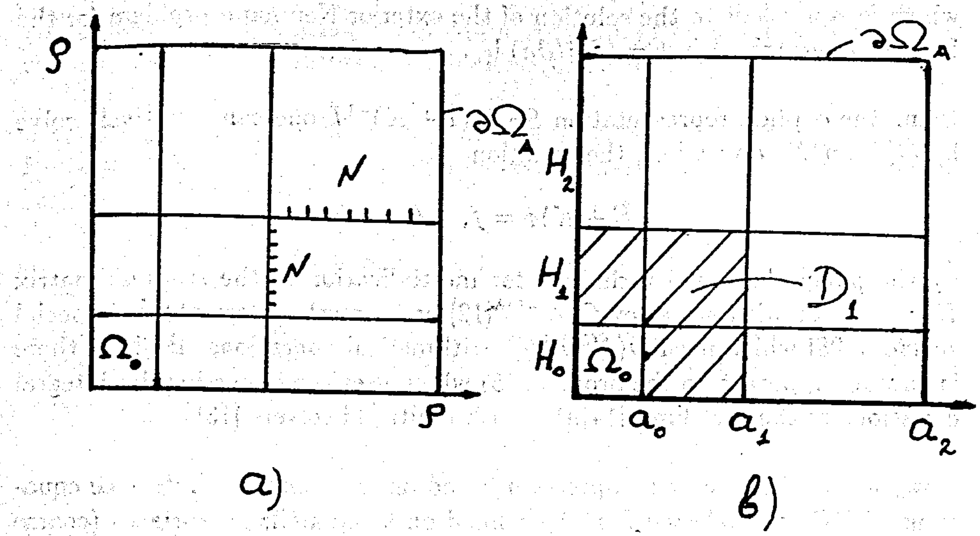
3.MULTI-DOMAIN DECOMPOSITION FOR UNBOUNDED REGION

We introduce an auxiliary domain Ω_A, $\Omega_0 \in \Omega_A$, $\Omega_A = \{x_k : | x_k | \leq \rho,\ k = 1 \div 3\}$, decomposed into $p^3 = (2m+1)^3$ subdomains and put homogeneous Dirichlet or Neumann conditions on the boundary $\partial\Omega_A$. In partitioning of Ω_A we choose the following imbedding domains:

$$\Delta_{j-1} \subset \Delta_j,\ \Delta_j = \{x_k : | x_k | < a_j,\ k = 1 \div 3\}, \quad j = 1 \div m,$$

$$\Delta_m \equiv \Omega_A, \quad a_{j-1} < a_j, \quad a_m = \rho,$$

and define the layers D_j of the subdomains as $D_j = \Delta_j \backslash \Delta_{j-1}$, $j = 1 \div m$. The numbers a_j describe the coordinates of the boundaries of the subdomains. So, the number of subdomains in each k-th direction, which define the box decomposition of the whole domain $\Omega_A = \cup_{i=1}^{p^3}\Omega_i$ as described in Part 1, is equal to $p = 2m+1$, see Fig.1a. Then in each subdomain we introduce a rectangular uniform mesh with "displacement by h/2" relatively the boundaries of subdomains [19] and assume that the number of mesh nodes in each subdomain in each of the three directions is N.



One quarter of Ω_A, $m = 2$

Fig.1

Because of the behavior of the unknown function w at infinity $\mid w(x) \mid = O(\mid x \mid^{-\nu})$, $\mid x \mid \to \infty$, we choose

$$\rho = CN^q, \quad q = \begin{cases} 2/\nu \ \textit{for Dirichlet conditions on } \partial\Omega_A \\ 2/(\nu+1) \ \textit{for Nuemann conditions on } \partial\Omega_A \end{cases} \tag{6}$$

So, the computational work necessary for the evaluation of (5) based on the DD-method (solving the finite difference analogue of the problem (2) where the Poincaré-Steklov operators are approximated by using the standard seven point stencil on the introduced mesh [19]) is estimated by $O(p^3N^2ln^4N)$ (see Remark 2).

Now we choose the coordinates a_j of the subdomain boundaries so that the number p is minimal. For this purpose we require that average errors of the approximate solutions in each subdomain are almost the same and equal to that in the initial domain Ω_0.

Here we must do necessary explanations. First of all we suppose that for the solution w^0 of the linear (or nonlinear) elliptic problem in Ω_0 we have a finite

element or a finite difference approximate solution w_h^0 such that

$$\|w^0 - w_h^0\|_{C(\Omega_0)} \le C_0 h^2, \quad h = O(1/N) \tag{7}$$

holds. Then, since in the domain $\Omega_A \backslash \Omega_0$ we solve the Laplace equation and know a priori that the solution is smooth then in each subdomain Ω_i the estimate

$$\|w - w_h\|_{C(\Omega_i)} \le \|u - u_h\|_{C(\Gamma_i)}, \quad u = w \mid_{\Gamma_i},$$

holds. Using standard seven point stencil for approximation of the Laplacian on the mesh with "displacement by h/2" we have the estimate [10]

$$\|u - u_h\|_{C(\Gamma_i)} \le C_i h^2 \tag{8}$$

for the trace u_h of the solution of a discrete Neumann problem in Ω_i. The constants C_0 and C_i in (7) and (8) depend linearly on the second partial derivatives of the solution in the corresponding subdomains. Here we suppose that the estimate (8) holds for the solution of the finite difference analogue of the problem (2) with $\mu_i = 1$ on the whole boundary $\Gamma = (\cup_i \Gamma_i) \backslash \partial\Omega_A$ which is confirmed in numerical experiments (see Appendix 1).

Since we require at infinity

$$\mid w(x) \mid = O(\mid x \mid^{-\nu}), \quad \mid \frac{\partial^2 w}{\partial x_k^2} \mid = O(\mid x \mid^{-\nu-2}), \ k = 1 \div 3, \quad \mid x \mid \to \infty \tag{9}$$

we define a mesh size in each of the three directions as a function of $\xi = \mid x \mid$ by

$$\frac{1}{N^2} C_0 = h^2(\xi)[\delta(\xi)]^{-\nu-2}, \quad h(\xi) = C_0[\delta(\xi)]^{1+\frac{\nu}{2}} \frac{1}{N}, \tag{10}$$

where the function $\delta(\xi)$, $\xi \in (0, a_m]$ is defined by

$$\delta(\xi) = a_j, \quad if \ \xi \in (a_{j-1}, a_j], \ j = 1 \div m.$$

Here the constant $C_0 \ge 1$ characterizes the average value of the second partial derivatives of the solution in Ω_0. We define the subdomain size H_j, $j = 0 \div m$, for the layers D_j and the numbers a_j , (see Fig.1b) according to

$$\begin{aligned} &a_0 = 1, \quad H_0 = 1, \quad H_j = Nh(a_{j-1}) = C_0 a_{j-1}^{1+\frac{\nu}{2}} \\ &a_j = a_{j-1} + H_j, \quad j = 1 \div m. \end{aligned} \tag{11}$$

Then the following lemma holds true

LEMMA 1. Let the sequence $\{a_j\}$, $j = 0 \div m$, be defined by (11). If in addition

$$a_m = O(N^q), \quad q > 0,$$

then we have the estimate

$$m < C_1 lnlnN,$$

where C_1 depends only on q, ν and C_0.

Proof: According to (11) we have the following recursion relation for the numbers a_j:

$$a_j = a_{j-1} + C_0 a_{j-1}^{\theta}, \quad \theta = 1 + \frac{\nu}{2}, \quad a_0 = 1,$$

from which it follows that $a_j \geq C_0 a_{j-1}^{\theta}$, $j = 1 \div m$.
Because $C_0 \geq 1$ then the following sequence of inequalities holds true

$$CN^q = a_m \geq C_0 a_{m-1}^{\theta} \geq C_0^{1+\theta} a_{m-2}^{\theta^2} \geq \ldots \geq C_0^{p(m)} a_1^{\theta^{m-1}} \geq (1 + C_0)^{\theta^{m-1}}$$

where $p(m) = \frac{2}{\nu}(\theta^{m-1} - 1)$, and so

$$m < C_1 lnlnN \quad \blacksquare$$

In the above lemma the parameter $\nu \geq 2$ characterizes the behavior of the unknown function at infinity, C_0 - the average value of the second partial derivatives of solution in Ω_0 and q - the type of boundary conditions on the surface $\partial\Omega_A$ of the auxiliary domain Ω_A.

From this lemma follows

THEOREM 2: Let the parameters q, C_0, ν defined in (6),(7),(9) be given. Then the solution of the discrete analogue of the problem (5) by the DD-method with decomposition of Ω_A described in (10),(11) can be obtained using

$$O(N^2 ln^4 N (lnlnN)^3)$$

arithmetical operations and

$$O(N^2 (lnlnN)^3)$$

space of computer memory.

Of course all these estimates are asymptotic ones and the real cost-effectiveness for not too large N depends on the concrete problem. Some numerical experiments for the exterior Dirichlet problem which confirm the estimates of Theorem 2 are given in Appendix 2.

4.SOLUTION OF NONLINEAR PROBLEM

Let us consider the incomplete-nonlinear (IN) formulation [18] for the approximation of nonlinearity which is well suited for efficient evaluation of the vector $\psi = S_I^{-1} u_h$ in the iterative processes for solving (4). This problem is equivalent to the solution of the interior nonlinear Dirichlet problem in $\Omega_0 = \Omega_\mu \cup \Omega_1$ for the same box-type decomposition $\Omega_0 = \cup_{k=1}^{M_0} \Omega_k$ of the domain Ω_0, see Fig.2.

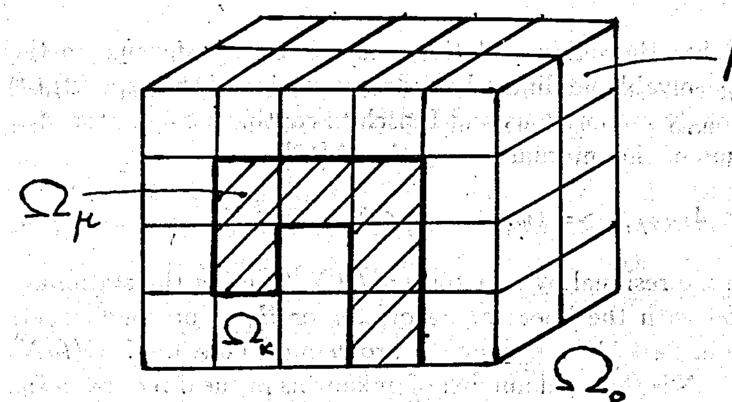


Fig.2

Let us suppose

$$\mu(x, |\nabla w|) = \begin{cases} \mu(|\nabla w|), & x \in \Omega_\mu \\ 1 & x \in \Omega_1 \end{cases}$$

where $\mu(t) > 0,\ t \in [0,\infty)$ is a given function with the properties

$$\mu(t)t - \mu(r)r \geq m(t-r), \quad t \geq r, \quad m > 0$$
$$| \mu(t)t - \mu(r)r | \leq M \mid t - r \mid, \quad M < \infty$$

The IN formulation for the Problem D (for $\beta = 0$) reads as follows
Problem IN. Find the function $w \in H_0^1(\Omega_0)$ such that

$$\sum_k \mu_k \int_{\Omega_k} \nabla w \nabla \eta dx - \int_{\Gamma_1} \psi(s)\eta(s)ds = 0, \quad \forall \eta \in H_0^1(\Omega_0). \qquad (12)$$

Here the constants μ_k are defined in Ω_k according to the formulas:

$$\mu_k = \mu(\tau_k(\nabla w)), \quad \tau_k(\nabla w) = ((mes\Omega_k)^{-1} \int_{\Omega_k} | \nabla w |^2 dx)^{\frac{1}{2}},$$

where τ_k is the average value of the module of the gradient $| \nabla w |$ in Ω_k, $k = 1 \div M_0$. If $\Omega_k \subset \Omega_1$ then $\mu_k = 1$.

The equation (12) has the unique solution and can be transformed to the equivalent uniquely solvable nonlinear boundary equation of the type (2),(3) with nonlinear strongly monotonous and Lipschitz continuous operator $\mathcal{A}_{IN}$ defined on the union of the internal boundaries Γ [18]:

$$< \mathcal{A}_{IN} u_h, \eta > = (\psi, \eta), \quad u_h \in Y_h \quad \forall \eta \in Y_h. \qquad (13)$$

In order to reduce the residual by a factor $\varepsilon = O(N^{-\sigma})$ using the stationary Richardson method with the linear operators $\mathcal{A}_h$ or $\mathcal{B}_0$ as preconditioners (which are defined in Part 1) for solving the problem (13) one needs $O(lnN)$ iterations [18]. Here N is the total number of unknowns in one direction in Ω_0. Consequently the above approach needs $O(p_0 N^2 ln^4(N/p_0) lnN)$ arithmetical operations, where $p_0^3 = M_0$ is the total number of subdomains, provided we use the implicit finite difference representation for the Poincaré-Steklov operator in parallelepiped-type subdomains Ω_k [19].

Remark 3. Here we have considered the two-domain reduction (4) for the combined formulation (Problem D) where (for the decomposition $I\!R^3 = \Omega_0 \cup \Omega_E$) evaluation of the element $S_E\psi,\ x \in \Gamma_0$ is performed using the

approach described in Part 3. Besides one can solve the Problem D in framework of box domain decomposition method for the auxiliary space-extensive domain $\Omega_A \supset \Omega_0$, i.e. the interior (in Ω_0) and exterior (in $\Omega_A \backslash \Omega_0$) problems can be solved "simultaneously" by using the global iterative process for solving the boundary equation analogous to (13) and defined on all interior boundaries of substructures in Ω_A.

To conclude this paper we note that the described above approach for solving exterior elliptic problems with a given behavior of the unknown function at infinity can be treated as some procedure inverse to the grid refinement on the coarse mesh level. So for the appropriate choice of the "grid enlargement" for an adequate approximation of the solution at infinity we can deal with the uniformly bounded (with respect to the fixed parameters q, C_0 and ν) number of subdomains, independent of mesh size h.

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APPENDIX 1.

Here we present the numerical experiments demonstrating the estimate (8) for the solution of the finite difference analogue of the problem (2) on the whole interior boundary $\Gamma = (\cup_i \Gamma_i) \backslash \partial\Omega$. For the simplicity we consider the Dirichlet problem for the Laplace equation in the unit square $\Omega \subset I\!R^2$, decomposed into 9 substructures $\Omega = \cup_{i,j} \Omega_{ij}$ with the subdomain size $H = 1/3$. In each subdomain we introduce a uniform mesh with the mesh size $h = H/N$ and "displacement by h/2", i.e. the edges of each subdomain are located in the middle of two layers of the corresponding boundary nodes. Using the standard five-point stencil for the approximation of the Laplacian in Ω_{ij} and approximating boundary values and boundary conormal derivatives on $\partial\Omega_{ij}$ by the midpoint sums and differences, respectively, we have the estimate (8) for sufficiently smooth functions on Ω_{ij}. Table 1 presents the results of numerical experiments for the test function

$$w(x,y) = e^x sin(y), \quad x, y \in \Omega,$$

where $u = w(x,y) |_\Gamma$ is the trace of the test function on Γ, u_h is the vector approximating u on the introduced mesh.

N	4	8	16	32
$\|u - u_h\|_{C(\Gamma)}$	0.2	0.048	0.013	0.0034

Table 1

APPENDIX 2. In this appendix we demonstrate the technique for solving the exterior problem (5) described in Part 3. Since the expressions (10),(11) have quasi-one-dimensional form, i.e. the asymptotic behavior of the parameter m under the increase of the number of grid unknowns N is independent of the dimension of the problem, we consider the two-dimensional case in order to use as much sequences of grids as possible to observe the behavior of the parameter m.

As a test function we take

$$w(x,y) = \frac{2xy}{(x^2+y^2)^2}, \tag{14}$$

which has the asymptotics

$$w(x,y) = O(|\, r\,|^{-2}), \quad |\, r\,| = \sqrt{x^2+y^2} \to \infty.$$

at infinity, and consider the evaluation of the trace $\phi = w(x,y)$, $x, y \in \Gamma_0$, by the given conormal derivatives $\psi = \frac{\partial w}{\partial n}(x,y)$, $x, y \in \Gamma_0$.

Using formulas (11) we calculate the locations of the boundaries of the subdomains:

$$a_0 = 1,\ a_1 = 1.7,\ a_2 = 3.7,\ a_3 = 13.3,\ a_4 = 137.1, \ldots$$

(the constant C_0 for the test function can be easily calculated) and choose the number m according to the criterion

$$a_{m-1} < \rho \le a_m,$$

where ρ describes the location of the exterior boundary $\partial\Omega_A$. It is calculated according to (6) by

$$\rho = \frac{1}{C_0} N$$

for the homogeneous Dirichlet boundary conditions on $\partial\Omega_A$. Note that actually for the location of the boundary $\partial\Omega_A$ holds

$$\rho = \left(\frac{1}{C_0} N\right)^{\frac{2}{3}} \tag{15}$$

since the component of the error of the solution on Γ_0 which depends on the approximate boundary condition on $\partial\Omega_A$ decreases as

$$O\left(\frac{1}{r(\partial\Omega_A, \Gamma_0)}\right), \quad r(\partial\Omega_A, \Gamma_0) = inf\ |\, x - y\,|, \quad x \in \partial\Omega_A,\ y \in \Gamma_0$$

Table 2 presents the results of numerical experiments for the test function (14). Here N is the number of mesh nodes in one direction in each subdomain, ρ is the location of the exterior boundary $\partial\Omega_A$ evaluated by (15), IT is the number of iterations of the PCG method with the preconditioner from [19] to decrease the initial residual 10^{-5} times.

m	ρ	N	$\|\phi - \phi_h\|_{L_2(\Gamma_0)}$	IT	AR
2	3.1	4	0.18×10^{-1}	4	1.4
3	5.0	8	0.56×10^{-2}	5	2
3	8.0	16	0.16×10^{-2}	5	4.3
3	12.7	32	0.39×10^{-3}	6	9
4	20.1	64	0.95×10^{-4}	7	10
4	32.0	128	0.24×10^{-4}	8	19

Table 2

Note that a possible disadvantage of the method described in Part 3 is the existence of the subdomains with a large aspect ratio which, in general, can decelerate the convergence of the DD-algorithm. Here we do not analyse the effect of this geometrical factor on convergence properties of iterative processes. Some discussions of that problem can be found in [7], [22]. The last column (AR) in Table 2 show that we have no crucial deterioration of convergence properies of the DD-algorithm from [19] with the growth of the maximum aspect ratio (AR) of the subdomains.

APPENDIX 3. We calculated the magnetic field distribution of the dipole spectrometric magnet [29] using the incomplete-nonlinear formulation [18]. The domain of nonlinearity has a step-type form. The basic decomposition of the domain of nonlinearity Ω_μ into 12 elements is presented in Fig.2.

We present the computing times t_k, $k = 0, 1, 2, 3$, on the computer with 5 mln.op/s. for the solution of the nonlinear problem on a sequence of four grids with a difference of two consecutive approximate solutions $\varepsilon = 10^{-4}$. For grids (N_{xk}, N_{yk}, N_{zk}) with $N_{xk} = 12 \times 2^k$, $N_{yk} = N_{zk} = 14 \times 2^k$, $k = 0, 1, 2, 3$, the finest grid has the dimension (96,112,112) and the corresponding computing times turned out to be $t_0 = 1min$, $t_1 = 2min$, $t_2 = 8min$, $t_3 = 34min$. Note that with each step the computing time raises only by the factor 4 (and not 8) because we solve the boundary equation defined on the interior boundaries of subdomains. To achieve the accuracy of the solution in aperture of the magnet to be 10^{-4} we have partitioned the domain Ω_A into 150 subdomains Ω_i such that only 48 of them contained the domain of nonlinearity. Thus we have only 48 "nonlinear variables" τ_i.

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