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APPROXIMATION FORMULAS FOR FUNCTIONAL INTEGRALS IN $P(\phi)_2$ - QUANTUM FIELD THEORY

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1. INTRODUCTION

Functional integration is one of the important means of investigation in many branches of contemporary science [1,2]. Functional integrals in quantum field theory became the convenient tool for solution of the wide spectrum of problems which are inaccessible for numerical study by the other methods [3]. The approach based on the mathematically rigorous definition of measure for functional integrals in quantum physics is intensively developed now [4]. This approach is of high importance both for theory and for numerical computations. Significant progress in development of methods for numerical integration in functional spaces has been achieved last years [5]. In the framework of the mentioned approach we derived for the functional integrals with respect to Gaussian measures in separable Fréchet spaces $\mathfrak{X}$ some new approximation formulas exact on a class of polynomial functionals of a given degree [8,9]. In the case when $\mathfrak{X}$ is a space of continuous functions with conditional Wiener measure, we obtained the family of approximation formulas with a weight [10]. The derived formulas do not need initial space-time discretization like lattice one. The employment of the formulas in the problems of quantum mechanics [9-12] show that these formulas have advantages over the other known methods of computation of functional integrals including the wide-spread lattice Monte Carlo method. In particular, our formulas allow one to use not probabilistic but the more preferable deterministic methods in computations and to obtain the more precise results with essentially smaller computational expences.

In the present work the approximation formulas in functional spaces are applied to the functional integrals of quantum field theory for the first time. We consider the

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two-dimensional theory with polynomial self-interaction of boson fields - the $P(\varphi)_2$ - model [13]. This model enables one to study, in particular, such processes as phase transitions, critical phenomena, interaction of particles, scattering and bound states. $P(\varphi)_2$ - theory is an object of investigations of many authors (see, e.g. [14]). The mathematically rigorous construction of Gaussian measure for the functional integrals in this theory is given in [4]. The important result of [4] is the proof of the existence of measure in infinite volume as a limit of finite volume measures. In this case the space $\mathbb{X}$ is the Schwartz distribution spase $\mathbb{D}'$. In paper [15] we have found some characteristics of $P(\varphi)_2$ - measure which are necessary for numerical computations. In the present paper we derive for the functional integrals in $P(\varphi)_2$ - model some new approximation formulas exact on a class of polynomial functionals. Under determined conditions the convergence of approximations to the exact value of integral is proved, the speed of convergence is estimated. The employment of the formulas is demonstrated in numerical examples.

2. BASIC DEFINITIONS

As shown in [4], the physical quantities in $P(\varphi)_2$-theory which are defined as the averages over the vacuum state of interacting fields φ, can be obtained by evaluation of the functional integral

$$\int_{\mathbb{D}'(\Lambda)} \exp\left\{ -\int_{\Lambda} :P[\varphi(x)]:_{K_\emptyset} d^2x \right\} F[\varphi]\, d\mu(\varphi),$$

where P is a given polynomial which determines the type of interaction; Λ is a bounded domain in $\mathbb{R}^2$ (the following turn to the limit $\Lambda\uparrow\mathbb{R}^2$ is assumed); $F[\varphi]$ is a real functional which corresponds to the given physical quantity, F is defined on the Schwartz distribution space: $\varphi(x)\in\mathbb{D}'(\Lambda)$,

$x=(x_1,x_2)\in\Lambda\subset\mathbb{R}^2$; Gaussian measure $d\mu(\varphi)$ is determined by its correlation functional $K(f,g)$

$$K(f,g) = \int_\Lambda\int_\Lambda K(x,y)\, f(x)\, g(y)\, d^2x\, d^2y$$

and the mean value $\xi(f)$ in a unique way [5]. Here $f,g\in\mathbb{D}(\Lambda)$ - the test function space $\mathbb{C}_0^\infty$. In the sequel we shall assume $\xi(f)=0$ without limitation of generality. The Wick ordering :: is defined with respect to the free covariance $K_\emptyset$ [4]. In paper [15] we obtained the expression of integral kernel $K(x,y)$ for an arbitrary region $\Lambda\subset\mathbb{R}^2$. In the present paper we consider Λ to be a rectangular: $\Lambda=\{[0,2a]\times[0,2a]\}$. In this case $K(x,y)$ is written as follows [15]:

$$K(x,y)= \frac{1}{a^2} \sum_{k_1,k_2=1}^{\infty} \frac{1}{1+\left(\frac{\pi k_1}{2a}\right)^2+\left(\frac{\pi k_2}{2a}\right)^2} \sin\left(\frac{\pi k_1}{2a} x_1\right) \sin\left(\frac{\pi k_2}{2a} x_2\right) \sin\left(\frac{\pi k_1}{2a} y_1\right) \sin\left(\frac{\pi k_2}{2a} y_2\right). \qquad (1)$$

The eigefunctions of the kernel $K(x,y)$ are

$$\beta_{k_1,k_2}(x)= \frac{1}{a} \sin\left(\frac{\pi k_1}{2a} x_1\right) \sin\left(\frac{\pi k_2}{2a} x_2\right),$$

the corresponding eigenvalues

$$\lambda_{k_1,k_2}= \frac{1}{1+\left(\frac{\pi k_1}{2a}\right)^2+\left(\frac{\pi k_2}{2a}\right)^2} .$$

Gaussian measure μ on a separable Fréchet space $\mathbb{X}$ generates the separable Hilbert subspace $\mathbb{H}$. We shall denote the conjugated to $\mathbb{H}$ space by H. The closure of H is the support of the measure μ and it is dence almost everywhere in

$\mathbb{X}$ [5]. The spaces $\mathbb{H}$ and H are important for construction of approximation formulas for the functional integrals with Gaussian measures.

The various expressions of functional integrals as limits of n-dimensional integrals when n tends to the infinity are known from the literature [5,6]. Particularly, the correspondong theorem for the integrals with Gaussian measure in separable Fréchet space $\mathbb{X}$ is proved in [5]. Although the space $\mathbb{D}'(\Lambda)$ is not a separable Frechet space, for any $\varphi\in\mathbb{D}'(\Lambda)$ the following expansion holds

$$\varphi = \sum_{k=1}^{\infty} (e_k,\varphi)_H e_k, \tag{2}$$

where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis in H [5]. This expansion converges in topology of the space $\mathbb{D}'(\Lambda)$ for almost all $\varphi\in\mathbb{D}'(\Lambda)$. In this case the proof of the mentioned theorem remains valid. Due to this fact we can formulate the following theorem:

<u>Theorem</u> 1.

Let $F[\varphi]$ be a continuous almost everywhere on $\mathbb{D}'(\Lambda)$ functional satisfying

$$|F[\varphi]| < \Phi[\varphi],$$

where $\Phi(\varphi)$ is a non-negative integrable functional and

$\Phi[\sum_{k=1}^{\infty}(e_k,\varphi)_H e_k]$ does not decrease as n tends to the infinity.

Then

$$\int_{\mathbb{D}'(\Lambda)} F[\varphi]d\mu(\varphi) = \lim_{n\to\infty} (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left\{-\sum_{k=1}^{\infty}\frac{u_k^2}{2}\right\} F[\sum_{k=1}^{\infty} u_k e_k]du . \tag{3}$$

This theorem enables one to evaluate the functional integrals from the wide class of functionals. The approximations converge to the exact value when the multiplicity of integrals in the right-hand side of eq. (3) approaches the infinity. However, this way of numerical evaluation of functional integrals does not guarantee fast convergence to the exact result (see, e.g. [16]).

There exist the approximation formulas which are exact on some class of functionals. In particular, there are formulas [5,6] exact for polynomial functionals of degree 2m+1, where m is a given positive number. Remember that polynomial functional of degree m is a functional

$$P_m[x] = \sum_{k=1}^{m} p_k[x],$$

where $p_k[x]$ is a continuous on $\mathbb{X}$ homogeneous form of order k. These formulas guarantee the good approximation when the functional F is close to the polynomial functional of degree ≤ 2m+1 [5]. Moreover, using these formulas one can construct new approximation formulas applicable to the larger class of functionals. We derive these formulas in section 3 of the present paper.

<u>Theorem</u> 2 [5].

Let ν be a symmetric probabilistic measure in $\mathbb{R}$, let K be an arbitrary given correlation functional of the measure $d\mu(x)$ on $\mathbb{X}$ and let the function $\rho(v)\colon \mathbb{R}\mapsto\mathbb{X}$ satisfy

$$\rho(v) = -\rho(-v) \tag{4}$$

$$\int_{\mathbb{R}} \langle\xi,\rho(v)\rangle\langle\eta,\rho(v)\rangle d\nu(v) = K(\xi,\eta), \tag{5}$$

$$\prod_{i=1}^{l}\langle\xi_i,\rho(v)\rangle \in \mathbb{L}(\mathbb{R},\nu) \quad \text{for } 1\le l\le 2m+1 \quad \text{and for any } \xi,\eta,\xi_i\in\mathbb{X}'. \tag{6}$$

Then the approximation formula

$$\int_{\mathbb{X}} F[x]\,d\mu(x) \approx \int_{\mathbb{R}} F[\theta_m(v)]\,d\nu^m(v) \qquad (7)$$

is exact if $F[x]$ is a polynomial functional of degree $\le 2m+1$. Here

$$\theta_m(v) = \sum_{k=1}^{m} c_k^{(m)}\,\rho(v_k),$$

$[c_k^{(m)}]^2$ are the roots of $Q_m(t)=\sum_{k=0}^{m}(-1)^k t^{m-k}/k!$ and the measure ν^m in $\mathbb{R}^m$ is a cartesian product of the measures ν.

In the present paper for the space $\mathbb{D}'(\Lambda)$ and the correlation operator K with the kernel (1) we construct the function $\rho(v)$: $\mathbb{R}^2 \mapsto \mathbb{D}'(\Lambda)$ and the symmetric probabilistic measure ν in $\mathbb{R}^2$ which satisfy the conditions analogous to (4)-(6). Consequently, we derive the approximation formula similar to (7) for the numerical evaluations of functional integrals

$$\int_{\mathbb{D}'(\Lambda)} F[\varphi]\,d\mu(\varphi).$$

3. CONSTRUCTION OF APPROXIMATION FORMULAS

Consider the integer-valued mesh (n_1,n_2) on $\mathbb{R}^2$. We define the function ν on the manifolds $Q\subset\mathbb{R}^2$ as follows

$$\nu(Q) = \sum_{(n_1,n_2)\in Q} \nu(n_1,n_2), \qquad Q\subset R \qquad (8)$$

where

$$\nu(n_1,n_2) = \frac{1}{\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]^2 4Z},$$

$$Z = \sum_{(n_1,n_2)\in Q} \frac{1}{\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]^2}.$$

Since $\nu(\mathbb{R}^2)=1$, the measure $\nu(x_1,x_2)$ is a symmetric on both arguments probabilistic measure in $\mathbb{R}^2$. The integral with respect to this measure is written as follows:

$$\int_{\mathbb{R}^2} f(x_1,x_2)\,d\nu(x_1,x_2) = \frac{1}{Z}\sum_{n_1,n_2=-\infty}^{\infty} \frac{1}{\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]^2} f(n_1,n_2).$$

We shall take the function $\rho(u)$: $\mathbb{R}^2 \mapsto \mathbb{D}'(\Lambda)$ in the following form:

$$\rho_u(x) = \frac{1}{2a}\sqrt{Z\left[1+\left(\frac{\pi v_1}{2a}\right)^2+\left(\frac{\pi v_2}{2a}\right)^2\right]^2}\ \sin\left(v_1\frac{\pi x_1}{2a}\right)\sin\left(v_2\frac{\pi x_2}{2a}\right), \qquad (9)$$

where $v=(v_1,v_2)\in\mathbb{R}^2$, $x=(x_1,x_2)\in\Lambda$.

$\rho_v(x)$ is an odd function of arguments v_1 and v_2. Let us test the condition (5). For any $\xi,\eta\in\mathbb{D}'(\Lambda)$ we have

$$\int_{\mathbb{R}^2} \langle\xi,\rho_v\rangle\langle\eta,\rho_v\rangle\,d\nu = \iint_{\Lambda\Lambda} \xi(x)\eta(y)\,\frac{1}{Z}\sum_{n_1,n_2=-\infty}^{\infty}\frac{1}{\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]^2}$$

$$\rho_{n_1,n_2}(x)\,\rho_{n_1,n_2}(y)\,dx\,dy =$$

$$= \iint_{\Lambda\Lambda} K(x,y)\xi(x)\eta(y)\,dx\,dy = K(\xi,\eta).$$

Let us check the condition analogous to (6):

$$\prod_{i=1}^{m}\langle\xi_i,\rho(v)\rangle \in \mathbb{L}(\mathbb{R}^2,\nu) \quad \text{for } 1\le l\le 2m+1 \text{ and every } \xi_i\in\mathbb{D}(\Lambda). \qquad (6')$$

In order to do it we shall prove the following lemma.

Lemma.

Let

$$S_{n,m} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\ \sin(nx)\ \sin(my)\ dx\ dy,$$

where $f(x,y)\in D(\Lambda)$.
Then

$$|S| \le \frac{C}{mn},$$

where C does not depend on n and m.

Proof.
Since for the function f(x) with bounded variance the following relation holds [17]

$$\left|\int_{-\pi}^{\pi} f(x)\ \sin(nx)\ dx\right| \le \frac{Var(f)}{2n},$$

for $S_{n,m}=\int_{-\pi}^{\pi} F(y)\ \sin(my)\ dy$, where $F(y)=\int_{-\pi}^{\pi} f(x,y)\ \sin(nx)\ dx$

we have

$$|S_{n,m}| \le \frac{1}{2m}\ Var\big(F(y)\big),$$

$$|F(y)| \le \frac{1}{2n}\ Var_x f(x,y) = \frac{1}{2n}\int_{-\pi}^{\pi}|f'(x,y)|dx.$$

Therefore

$$|S_{n,m}| \le \frac{1}{2m}\int_{-\pi}^{\pi}|F(y)|dy \le \frac{1}{4mn}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f'(x,y)|dxdy.$$

The proof of the Lemma is complete.

According to the Lemma,

$$|\langle\xi,\rho_{n_1,n_2}\rangle| = \frac{1}{a}\left|\int_{\Lambda}\xi(x_1,x_2)\sqrt{Z\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]}\ \sin\left(\frac{\pi n_1}{2a}\ x_1\right)\right.$$

$$\left.\sin\left(\frac{\pi n_1}{2a}\ x_2\right)\ dx_1\ dx_2\right| \le \frac{1}{a}\sqrt{Z\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]}\ \frac{C}{n_1 n_2}.$$

Taking into account that

$$\frac{1}{n_1 n_2}\sqrt{1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2} \le G,$$

where G does not depend on n_1 and n_2, we obtain the estimate

$$|\langle\xi_i,\rho_{n_1,n_2}\rangle| \le A_i$$

where the constants A_i, $i=1,\ldots,l$ do not depend on n_1 and n_2.
Therefore

$$\int_{\mathbb{R}^2}\prod_{i=1}^{m}\langle\xi_i,\rho_{n_1,n_2}\rangle d\nu = \frac{1}{Z}\sum_{n_1,n_2=-\infty}^{\infty}\frac{1}{\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]^2}$$

$$\times\prod_{i=1}^{m}\langle\xi_i,\rho_{n_1 n_2}\rangle \le \frac{1}{Z}\sum_{n_1,n_2=-\infty}^{\infty}\frac{1}{\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]^2}\times\prod_{i=1}^{l}A_i.$$

Thus the conditions analogous to (4)-(6) are justified in the case of the space $\mathbb{R}^2$ instead of $\mathbb{R}$, for the measure ν (8) in $\mathbb{R}^2$ and for the function $\rho(v)=\rho_v(x)$: $\mathbb{R}^2\mapsto\mathbb{D}'(\Lambda)$ defined by (9). Since the proof of Theorem 2 remains valid in the case of $\mathbb{R}^2$, the following theorem appears to be proved:

Theorem 3.

Let $\theta_m(\nu) = \sum_{k=1}^{m} c_k^{(m)} \rho(v_k)$, $\nu=(v_1,\dots,v_m)$, $v_k \in \mathbb{R}^2$,

let $[c_k^{(m)}]^2$ be the roots of $Q_m(t)=\sum_{k=0}^{m} (-1)^k t^{m-k}/k!$,

$\nu_m(\nu)$ be the measure in $\mathbb{R}^{2m}$ (cartesian product of measures $\nu(v)$ given by (8)) and the function $\rho(v)$ be determined by (9) Then the approximation formula

$$\int_{\mathbb{D}'(\Lambda)} F[\varphi]d\mu(\varphi) \approx \int_{\mathbb{R}^2} F[\theta_m(\nu)]\, d\nu_m(\nu) \qquad (10)$$

is exact for any polynomial functional of degree $\le 2m+1$.

Function ρ and measure ν satisfying (4)-(6) are often used for construction of approximation formulas for the functional integrals from the functionals which are not close to the polynomial functionals [5,6,8-10]. One of the variants of these formulas is the so-called composite approximation formula. The construction of it is based on the use of the relation which is called "mixed integration formula" and is given by the following theorem:

Theorem 4 [5].
For any functional F[x] which is integrable with respect to measure μ in the separable Fréchet space $\mathbb{X}$ the following equality holds:

$$\int_{\mathbb{X}} F[x]d\mu(x)=(2\pi)^{-n/2}\int_{\mathbb{R}^n} \exp\{-\tfrac{1}{2}(u,u)\}\int_X F[x-S_n(x)+\Psi_n(u)]d\mu du \qquad (11)$$

where $S_n(x) = \sum_{k=1}^{n} (e_k x)_H e_k$, $\Psi_n(u) = \sum_{k=1}^{n} u_k e_k$,

e_k is an orthonormal basis in H; $u \in \mathbb{R}^n$, $(u,u) = \sum_{k=1}^{n} u_k^2$.

In paper [16] using this theorem we derived the composite approximation formula exact for polynomial functionals of degree $\le 2m+1$ in separable Fréchet spaces. Although the space $\mathbb{D}'(\Lambda)$ is not a separable Fréchet space, for any $\varphi \in \mathbb{D}'(\Lambda)$ the expansion (2) holds and the relation (11) remains valid. Consequently, we obtain the following composite approximation formula of 2m+1 degree of accuracy for the integrals in $\mathbb{D}'(\Lambda)$ as a corollary of theorem proved in [16]:

$$\int_{\mathbb{D}'(\Lambda)} F[\varphi]d\mu(\varphi)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \exp\{-\tfrac{1}{2}(u,u)\}\int_{\mathbb{R}^{2m}} F[\rho_m(\nu)-\rho_m^n(\nu)+ \qquad (12)$$

$$+\Psi_n(u)]d\nu(\nu)du + R_m^n(F).$$

Here

$$\rho_m(\nu) = \sum_{k=1}^{m} c_k^{(m)}\rho(v_k), \quad \rho_m^n(\nu)=S_n(\rho_m(\nu)), \quad \nu\in\mathbb{R}^{2m},\ u\in\mathbb{R}^n.$$

4. CONVERGENCE OF APPROXIMATIONS.

In paper [16] we have proved the convergence of approximations obtained using the composite approximation formula to the exact value as $n\to\infty$ in the case if $\mathbb{X}$ is a separable Fréchet space. This proof is valid for the case $\mathbb{D}'(\Lambda)$ as well. Therefore, the following theorem holds:

Theorem 5.
Let $F[\varphi]$ be a continuous on $\mathbb{D}'(\Lambda)$ functional satisfying

$$|F[\varphi]| \le g(A(\varphi,\varphi)),$$

where $A(\varphi,\varphi)$ is a non-negative quadratic functional of the form

$$A(\varphi,\varphi)=\sum_{k=1}^{\infty} \gamma_k(\varphi,e_k)_H^2, \qquad (\sum_{k=1}^{\infty}\gamma_k<\infty,\quad \gamma_k\ge 0), \qquad (13)$$

$g(r)$ is a nondecreasing positive function and let

$$\int_{\mathbb{R}^{2m}} \int_{\mathbb{D}'(\Lambda)} g[A(\rho_m(\nu),\rho_m(\nu))+A(\varphi,\varphi)]d\mu(x)d\nu(\nu) < \infty .$$

Then the remainder of the formula (12) $R_m^n(F)\to 0$ as $n\to\infty$.

In paper [16] for the case of separable Fréchet spaces we obtained the estimate of the remainder $R_m^n(F)$ and determined the order of its convergence to zero. Particularly, in the case of integration with respect to the Wiener measure the order of convergence $R_m^n(F)\to 0$ is $O(n^{-(m+1)})$. Let us estimate the speed of convergence of the remainder of the composite approximation formula in the case $\mathbb{X}=\mathbb{D}'(\Lambda)$. Consider the following linear transformation $Q_{n_1}(\varphi)$: $\mathbb{D}'(\Lambda)\to\mathbb{R}$:

$$\varphi \to \sum_{n_2=1}^{\infty} (\varphi,e_{n_1,n_2})_H e_{n_1,n_2} \qquad (14)$$

and the quadratic form

$$A_1(\varphi,\varphi)=\int_{\Lambda} Q_{n_1}^2(\varphi)\,dx \qquad (15)$$

which can be represented in form (13). Indeed, substituting the expression (14) into (15) and taking into account that

$$\int_{\Lambda} e_{n_1,n_2}^2(x)\,dx = \lambda_{n_1,n_2}$$

we obtain

$$A_1(\varphi,\varphi) = \sum_{n_2=1}^{\infty} (\varphi,e_{n_1,n_2})_H^2\, \lambda_{n_1,n_2};$$

$$\sum_{n_2=1}^{\infty} \lambda_{n_1,n_2} < \infty; \qquad \lambda_{n_1,n_2} > 0.$$

<u>Theorem</u> 6.

Let the integrable with respect to the measure $d\mu$ functional $F[\varphi]$ can be represented in the form

$$F[\varphi+\varphi_0] = P_{2m+1}[\varphi]+r_{2m+1}[\varphi,\varphi_0] , \qquad (16)$$

where P_{2m+1} is a polynomial functional of degree $\le 2m+1$ and the remainder r_{2m+1} is estimated by the expression

$$|r_{2m+1}[\varphi,\varphi_0]|\le[A_1(\varphi,\varphi)]^{m+1}\{L_1\exp[L_2\, A_1(\varphi+\varphi_0,\varphi+\varphi_0)]+ L_3\exp[L_2A_1(\varphi_0,\varphi_0)]\}. \qquad (17)$$

Here φ_0 is a fixed element of $\mathbb{D}'(\Lambda)$, $L_1,L_2,L_3>0$;

$$1-2L_2\lambda_{n_1,n_2} \ge \alpha > 0, \quad n_2=1,2... \qquad (18)$$

Then the order of convergence of approximations obtained by the composite formula (12) to the exact value of integral is

$$|R_m^n(F)| \sim O(n^{-(m+1)}).$$

Proof.

According to [16], under the conditions (16)-(18) the remainder $R_m^n(F)$ is estimated as follows

$$|R_m^n(F)| \le G_m\Bigl(\sum_{n_1,n_2=n+1}^{\infty} \lambda_{n_1,n_2}\Bigr)^{m+1}+H_m\Bigl(\sum_{n_1,n_2=n+1}^{\infty} \lambda_{n_1,n_2}a_{n_1,n_2}\Bigr)^{m+1}$$

where G_m and H_m are positive quantities which do not depend on n, and a_{n_1,n_2} are determined by

$$(e_{n_1,n_2},\rho_m(\nu))_H^2 \le a_{n_1,n_2} \quad \text{for all } \nu\in\mathbb{R}^{2m}.$$

Here $\rho_m(\nu)$ is determined by (12).

Since

$$(e_{n_1,n_2},\rho(v))_H = \int_\Lambda \frac{1}{\sqrt{\lambda_{n_1,n_2}}} \beta_{n_1,n_2}(x) \frac{1}{2a} \sqrt{Z\left(1+(\frac{\pi v}{2a}1)^2+(\frac{\pi v}{2a}2)^2\right)}$$

$$\sin\left(\frac{\pi v}{2a}1\ x_1\right) \sin\left(\frac{\pi v}{2a}2\ x_2\right)dx_1dx_2 = \frac{\sqrt{Z}}{2\lambda_{n_1,n_2}} \int_\Lambda \beta^2_{n_1,n_2}(x)dx \le 1$$

and $\sum_{k=1}^{m}\left[c_k^{(m)}\right]^2 = 1$, we have $a_{n_1,n_2}=1$ and

$$\sum_{k=n+1}^{\infty} \lambda_k = \sum_{n_1,n_2=n+1}^{\infty} \frac{1}{1+(\frac{\pi n}{2a}1)^2+(\frac{\pi n}{2a}2)^2} \sim \frac{1}{n}.$$

Then

$$\left[\sum_{n_1,n_2=n+1}^{\infty} a_{n_1,n_2}\ \lambda_{n_1,n_2}\right]^{m+1} \sim O\left(n^{-(m+1)}\right);$$

$$\left[\sum_{n_1,n_2=n+1}^{\infty} \lambda_{n_1,n_2}\right]^{m+1} \sim O\left(n^{-(m+1)}\right).$$

The assertion of the theorem follows from the last relations.

5. NUMERICAL EXAMPLES

1. We shall consider first the integral with respect to measure μ in the spase $\mathbb{D}'(\Lambda)$ from the functional

$$F_1[\varphi] = \langle 1,\varphi\rangle^6, \qquad \text{where} \quad 1(x,y) = \begin{cases} 1, & 0\le x,y\le 1 \\ 0, & \text{otherwise.} \end{cases}$$

This integral is a central moment of order 6 of the Gaussian

measure $d\mu_K(\varphi)$. It's exact value is known [6]:

$$I_1 = \int_{D'(\Lambda)} \langle 1,\varphi\rangle\ d\varphi_K = 15\ K^3(1,1).$$

Since $F_1[\varphi]$ is a polynomial functional of degree 6, the approximation formula (10) with m=3 gives the exact value of integral in the form of expression which coincides with the one presented above. We computed this integral by the simple formulas of the 3-d and 5-th degree of accuracy (formula (10) with m=1 and m=2 correspondingly) and by the composite formula of the 3-d degree of accuracy (formula (12) with n=1, m=1). These formulas in the mentioned cases give the following expressions for approximate evaluation of the integral I_1:

$$I_1^{simple(1)} = \frac{4}{Z} \sum_{n_1,n_2=1}^{\infty} \lambda^2_{n_1,n_2} \langle 1,\rho_{n_1,n_2}\rangle^6;$$

$$I_1^{simple(2)} = 2(c_1+c_2)\ R_6 + 2(15\ c_1-c_2)\ R_4\ R_2$$

$$I_1^{composite(1,1)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\{-u^2/2\}(w_0+w_2u^2+w_4u^4+w_6u^6)du.$$

Here

$$c_1=k_1^6-k_2^6;$$

$$c_2=15(k_1^2k_2^4-k_1^4k_2^2);$$

$$k_1=2^{-1/4}\cos\frac{\pi}{8}; \qquad k_2=2^{-1/4}\sin\frac{\pi}{8};$$

$$R_2=\int_{\mathbb{R}^2} \langle 1,\rho(u)\rangle^2 d\nu(u);$$

$$R_4=\int_{\mathbb{R}^2} \langle 1,\rho(u)\rangle^4 d\nu(u);$$

$$R_6=\int_{\mathbb{R}^2} \langle 1,\rho(u)\rangle^6 d\nu(u);$$

$$<1,\rho(u)> = \frac{1}{2a}\sqrt{\frac{Z}{\lambda_{u_1,u_2}}}\int_0^1 \sin\frac{u_1\pi x_1}{2a}\,dx_1\int_0^1 \sin\frac{u_2\pi x_2}{2a}\,dx_2 \quad ;$$

$$w_0 = \int_{\mathbb{R}^2} r^6(u)\,d\nu(u);$$

$$w_2 = 15\,p^2\int_{\mathbb{R}^2} r^4(u)\,d\nu(u);$$

$$w_4 = 15\,p^4\int_{\mathbb{R}^2} r^2(u)\,d\nu(u);$$

$$w_6 = p^6;$$

$$p = \frac{1}{a}\sqrt{\lambda_{1,1}}\left[\frac{2a}{\pi}\left(1-\cos\frac{\pi}{2a}\right)\right]^2;$$

$$r(u) = <1,\rho(u)>-p<\phi_{1,1},\rho(u)>.$$

The dependence of results on the size a of the region Λ is shown in Fig.1. For every fixed size of the region the simple

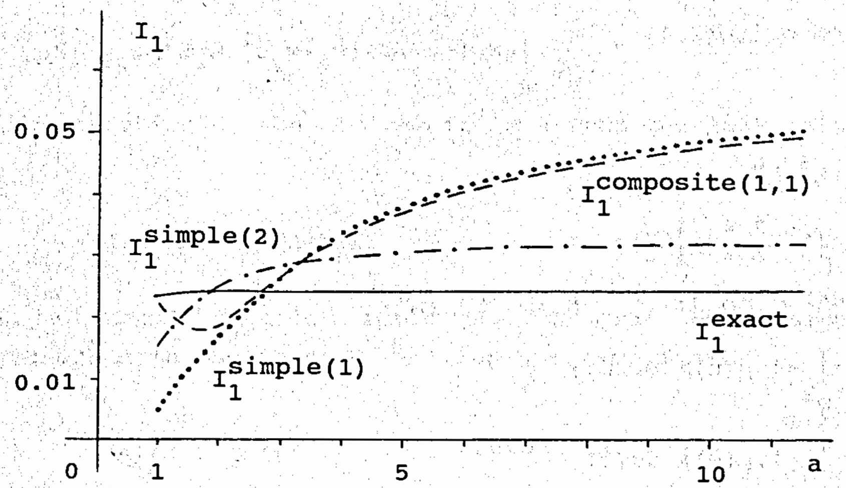

Fig.1

formula of the 5-th degree of accuracy gives the better

approximation than the simple formula of the 3-d degree of accuracy (formulas of the 7-th and higher degree of accuracy give the exact result). The composite formula of the 3-d degree of accuracy for small sizes of the region gives the more precise approximations than the simple formula of the same degree of accuracy.

2. Consider the integral with measure $d\mu$ in the space $\mathbb{D}'(\Lambda)$ from the functional

$$F_2[\varphi]=<1,\varphi>^4.$$

This integral is a central moment of order 4 of the Gaussian measure $d\mu(\varphi)$. It's exact value is[6]:

$$I_2 = \int_{\mathbb{D}'(\Lambda)} F_2[\varphi]\,d\mu(\varphi) = 3K^2(1,1).$$

We computed this integral by the simple formula (10) of the third degree of accuracy

$$I_2^{simple(1)} = \frac{4}{Z}\sum_{n_1,n_2=1}^{\infty}\frac{1}{\left[1+\left(\frac{\pi n_1}{2a}\right)^2+\left(\frac{\pi n_2}{2a}\right)^2\right]^2}<1,\rho_{n_1,n_2}(x)>^6,$$

by the simple formula (3)

$$I_2^{basis(n)} = \prod_{k=1}^{n}\int_{\mathbb{R}} e^{-\frac{u^2}{2}}(u^4g_k^4+6u^2g_k^2)\,du$$

where

$$g_k = \frac{1}{a}\,\frac{1}{\sqrt{1+(\frac{\pi k_1}{2a})^2+(\frac{\pi k_2}{2a})^2}}\left(\frac{2a}{\pi k_1}\right)\left(\frac{2a}{\pi k_2}\right)\left(1-\cos\frac{\pi k_1}{2a}\right)\left(1-\cos\frac{\pi k_2}{2a}\right)$$

and by the composite formula (12) of the 3-d degree of accuracy

$$I_2^{composite(1,1)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{u^2}{2}}(z_4u^4+z_2u^2+z_0)\,du,$$

$$z_0 = \frac{4}{Z} \sum_{n_1,n_2=1}^{\infty} \frac{1}{(1+(\frac{\pi n_1}{2a})^2+(\frac{\pi n_2}{2a})^2)^2} <1,\rho_{n_1,n_2}>^4$$

$$+\frac{4}{Z} \sum_{n_1,n_2=1}^{\infty} \frac{1}{(1+(\frac{\pi n_1}{2a})^2+(\frac{\pi n_2}{2a})^2)^2} \Big[-2<1,\rho_{n_1,n_2}>^3 \frac{\sqrt{Z}}{\lambda_{n_1,n_2}} p_{n_1,n_2}$$

$$+\frac{3}{2}<1,\rho_{n_1,n_2}>^2 \frac{Z}{\lambda_{n_1,n_2}} p^2_{n_1,n_2} - <1,\rho_{n_1,n_2}> \frac{Z\sqrt{Z}}{2\lambda_{n_1,n_2}} p^3_{n_1,n_2}$$

$$+ \frac{Z^2}{16\lambda_{n_1,n_2}} p^4_{n_1,n_2}\Big];$$

$$z_2 = \frac{1}{Z} \sum_{n_1,n_2=1}^{\infty} \lambda^2_{n_1,n_2} <1,\rho_{n_1,n_2}>^2$$

$$+ \sum_{n_1,n_2=1}^{\infty} \lambda^2_{n_1,n_2} \Big(-2<1,\rho_{n_1,n_2}> \frac{\sqrt{Z}}{2\lambda_{n_1,n_2}} p_{n_1,n_2} + \frac{Z}{4\lambda^2_{n_1,n_2}} p^2_{n_1,n_2}\Big);$$

$$z_4 = p^4.$$

with different values of parameter n. The dependence of results on the size a of the region Λ for fixed n is shown in Table 1. It is seen that the approximations obtained using each of these formulas become less accurate as the size of the region increases. However, for every fixed size of Λ the results obtained by the composite formula and by the formula (3) converge to the exact value of the integral as n tends to the infinity. The composite formula provides the higher speed of convergence, which is illustrated by Tables 2 and 3. The simple approximation formula (10) with m=1 does

Table 1

a	I_2^{exact} $\times 10^2$	$I_2^{basis(5)}$ $\times 10^2$	$I_2^{simple(1)}$ $\times 10^2$	$I_2^{composite(1,5)}$ $\times 10^2$
1	4.07	.67	1.98	4.13
2	4.13	3.73	3.83	4.21
3	4.13	1.58	4.76	5.66
4	4.13	0.37	5.28	6.61
5	4.13	0.08	5.61	6.50
10	4.13	0.00	6.30	6.35

Table 2 (a=1)

n	$I_2^{basis(n)}$	$I_2^{composite(1,n)}$
1	6.732E-3	4.133E-2
2	3.520E-2	4.105E-2
3	3.849E-2	4.098E-2
4	3.849E-2	4.098E-2
5	3.903E-2	4.096E-2
10	4.053E-2	4.070E-2
15	4.063E-2	4.069E-2
20	4.065E-2	4.068E-2

$$I_2^{exact} = 4.068E-2$$

$$I_2^{simple(1)} = 1.98E-2$$

Table 3 (a=10)

n	$I_2^{basis(n)}$	$I_2^{composite(1,n)}$
1	4.07E-13	6.29E-2
2	2.19E-10	6.29E-2
10	5.56E-4	7.04E-2
20	2.27E-2	5.52E-2
30	3.86E-2	4.22E-2
40	3.91E-2	4.22E-2
50	3.93E-2	4.22E-2
60	4.03E-2	4.16E-2
70	4.09E-2	4.13E-2
80	4.09E-2	4.13E-2
90	4.09E-2	4.13E-2

$$I_2^{exact} = 4.129E-2$$

$$I_2^{simple(1)} = 6.30E-2$$

not give good approximations in this case because the functional $F_2[\varphi]$ is not close to the polynomial functional of third degree in the sence of [6]. Note that this formula with m=2 gives the exact value of the considered integral.

3. Now we shall compute the integral

$$I_3 = \int_{\mathbb{D}'(\Lambda)} F_3[\varphi]d\mu(\varphi) \qquad \text{where} \qquad F_3(\varphi)=\exp(-\langle 1,\varphi\rangle).$$

We used the simple basis expansion formula (3) with various n, the the simple formula (10) of the 3-d degree of accuracy, the simple formula (10) of the 5-th degree of accuracy and the composite formula (12) of 3-d and 5-th degrees of accuracy with various n. The results are presented in Table 4. It is seen that the values $I_3^{basis(n)}$, $I_3^{composite(1,n)}$ and $I_3^{composite(2,n)}$ converge to the same quantity I=1.060 as n increases. Accordind to Theorem 1 and to theorem on

Table 4

n	$I_3^{basis(n)}$	$I_3^{composite(1,n)}$	$I_3^{composite(2,n)}$
1	1.023	.888	1.35
2	1.055	1.02	1.12
3	1.058	1.046	1.08
4	1.058	1.046	1.08
5	1.059	1.059	1.07
6	1.060	1.060	1.066
15	1.060	1.060	1.061
50	1.060	1.060	1.060

$$I_3^{simple(1)} = 0.74$$

$$I_3^{simple(2)} = 0.622$$

convergence of the composite approximation formulas [16] this quantity is the exact value of integral I_3.

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