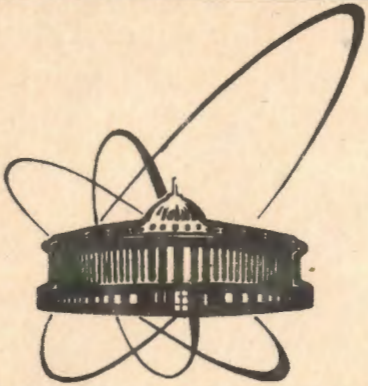

сообщения
объединенного
института
ядерных
исследований
дубна

E11-90-564

V.Yu.Gonchar*, N.A.Chekanov*, B.L.Markovski,
V.A.Rostovtsev, S.I.Vinitsky

THE PROGRAM OF ANALYTICAL CALCULATION OF THE NORMAL BIRKOFF-GUSTAVSON FORM

* Kharkov Institute for Technical Physics, USSR

1990

1. Introduction

Transformation of the initial classical Hamiltonian to a more simple form, called normal form, is one of the universal methods of analyzing and integrating classical equations of motion [1]. In the present paper, we discuss the normal Birkhoff-Gustavson form [2,3] for the Hamiltonians that can be represented as a polynomial near the equilibrium points, i.e.:

$$H(q,p) = H^{(2)}(q,p) + V(q) \tag{1}$$

where

$$H^{(2)}(q,p) = \sum_{\nu} 1/2\ \omega_\nu^2\ (p_\nu^2 + q_\nu^2), \tag{2}$$

$$V(q) = \sum_{|j| \geq 3} V_{j_1 j_2} q_1^{j_1} q_2^{j_2}, \quad q=(q_1,q_2), p=(p_1,p_2). \tag{3}$$

Here we consider the system with two degrees of freedom, however, the method can be applied for Hamiltonian systems with arbitrary number of degrees of freedom.

The procedure of reducing to the normal Birkhoff-Gustavson form and its realization depend on the relation between frequencies ω_ν. Birkhoff has proved [2] that if the frequencies ω_ν of Hamiltonian (1) are incommensurable, there exists a canonical transformation $(q,p) \to (\xi,\eta)$ such that in the new variables the Hamiltonian $\Gamma(\xi,\eta)$ will be a function of only the following variables $I_\nu = 1/2(\xi_\nu^2 + \eta_\nu^2)$, $\nu = 1,2$. In other words, the normal Birkhoff form is an expansion of the initial Hamiltonian in terms of two harmonic oscillators

$$H(q,p) \to \Gamma(\xi,\eta) = \omega_1 I_1 + \omega_2 I_2 + \sum \alpha_{\mu\nu} I_\mu I_\nu + \ldots \tag{4}$$

If the frequencies ω_ν are commensurable, i.e. if there exist resonance relations of the type $n\omega_1+m\omega_2= 0,(n,m \neq 0)$ the Hamiltonian (1) can not be reduced to the normal Birkhoff form due to the appearance of zero denominators $n\omega_1+m\omega_2= 0$. Therefore, one should not cancel some terms in the Hamiltonian (1), and the normal form becomes complicated [3] and will contain apart from I_ν other combinations of variables ξ_ν and η_ν as well. For the two-dimensional system with the resonance condition $n\omega_1+m\omega_2= 0$ these combinations are the following $\xi_1\eta_2 2\ \xi_2\eta_1\ ,\eta_1\eta_2+\ \xi_1\xi_2$. Such an extended normal form is used to be called the Birkhoff- Gustavson form (B-G).

The first rather quick programs in FORTRAN for making cumbersome calculations of the normal form were realized in ref.[3,4]. We shall point out other way of constructing canonical transformations (see formula (5)) suggested by Hori[5] and Deprit [6] whose practical realization is based on the use of Lie transformations [7-9]. The computer program for normalization of Hamiltonian systems in this approach were realized in ref. 10).

In the present paper, we expound two REDUCE programs for analytical construction of the normal B-G form according to the Gustavson algorithm [3] which is briefly represented in sec.2. The first program GITA gives the normal form of a given order s_{max} if one introduces the initial Hamiltonian (1) and resonance relations between frequencies ω_1, ω_2 or frequencies themselves if they are incommensurable. The second program GITN is applicable exceptionally for the resonance case $\omega_1= \omega_2$, i.e. for obtaining a modified normal form whose definition will be given in sec.2,3.

Note that since $I_1(\xi,\eta)$ and $\Gamma_s(\xi,\eta)$ are independent integrals of motion in the S approximation, then expressing ξ and η through the initial coordinates q,p one can obtain an approximate integral of motion $I_1(q,p)$. It can be used in different applications, for instance, in ref.[11] for elucidating the criterion of negative curvature. However, the range of applicability of

the normal B-G form goes beyond the limits of classical description of the system: it has successfully been applied for quantizing the classical Hamiltonian as well [12-16].

2. The method of reducing the Hamiltonian to the normal form

2.1. The normal Birkhoff-Gustavson form

As it has been point out above, the Hamiltonian (1) can always be reduced to the normal form making successive canonical transformations

$$\xi_\nu= q_\nu+ \frac{\partial W^{(j)}(q,\eta)}{\partial\eta_\nu};\ p_\nu= \eta_\nu+ \frac{\partial W^{(j)}(q,\eta)}{\partial q_\nu}\ ,\ \nu= 1,2 \qquad (5)$$

with the help of the generating function, or more exactly, the polynomial $W^{(s)}$ of the power s

$$F_2(q,\eta)= q_\nu\eta_\nu + W^{(s)}(q,\eta). \qquad (6)$$

The integer numbers means the power of that part of the Hamiltonian which is reduced to the normal form by using the polynomial $W^{(s)}$. In this case, the parts of $H^{(i)}$ with $i < s$ do not change under the canonical transformation (5). Then under the canonical transformation the Hamiltonian $H(q,p)$ turns into a new one $\Gamma(\xi,\eta)$

$$H(q,\eta+ \frac{\partial W^{(s)}}{\partial q}\ )= \Gamma(q + \frac{\partial W^{(s)}}{\partial \eta}\ ,\eta). \qquad (7)$$

Expanding this expression in the Taylor series near $q=\xi,\eta=p$ one equates the terms of the same power. For i=2 we get the known results that the Hamiltonian (2) is still in a normal form

$$\Gamma^{(2)}(\xi,\eta)= H^{(2)}(q,p);\ \ q=\xi,p=\eta\ . \qquad (8)$$

For the terms of power i=s=3, by which the procedure is starting the reducion to the normal form, we derive the generic equation in the form

$$D(q,\eta)W^{(s)}(q,\eta)= -H^{(s)}(q,\eta)\ + \Gamma^{(s)}(q,\eta), \qquad (9)$$

where the operator of the normal form

$$D(\xi,\eta)\ := \sum \omega_\nu(\eta_\nu \frac{\partial}{\partial\xi_\nu} - \xi_\nu\frac{\partial}{\partial\eta_\nu}\ ) \qquad (10)$$

is introduced. At each subsequent (S-2) step of reducing the Hamiltonian to the normal form we should solve eq.(9). So we have to find two unknown functions: the generating polynomial $W^{(s)}$ and

Hamiltonian $\Gamma^{(s)}$ in the normal form, satisfying condition $D\Gamma^{(s)}=0$

Before discussing the way of solving eq.(9) we first write down the relations resulting from the higher powers i=s+1, s+2, that will be necessary for the next step of reducing

$$\Gamma^{(j)}(\xi,\eta)= H^{(j)}(\xi\ \eta)+\sum_{k}\frac{1}{K!}\left\{\left(\frac{\partial W^{(S)}}{\partial\ \xi}\right)^{k}\left(\frac{\partial^{k}H^{(1)}}{\partial\ \eta^{k}}\right)-\left(\frac{\partial W^{(S)}}{\partial\ \eta}\right)^{k}\frac{\partial^{k}\Gamma^{(1)}}{\partial\ \xi^{k}}\right\}. \tag{11}$$

Here the two-dimensional integer-valued vector K satisfies the conditions

$$1-|K|+|K|(s-1)=j,\ 1\le|K|\le 1<j,\ 1\ge 2,\ s\ge 3$$

and the notations $K!= k_1!\ k_2!$ and $|K| = k_1+k_2$ are introduced. Having calculated at the (S-2) step, beginning with S=3, and generating polynomial $W^{(s)}$, we find the next higher power contributions to the Hamiltonian (11) to be used in performing the above procedure of reducing to the normal form. The number of iterations determines the order of approximations s_{max}.

2.2 The method of solving the basic equation

The linear operator D is determined in the functional space of all homogeneous polynomials $\Phi^{(s)}(\xi,\eta)$ of the power s of four variables (ξ,η) . In this space(dim= m=(s+3)!/3! s!) the set of monomials of the type $\xi^{k}\eta^{l}\equiv\xi_1^{k_1}\xi_2^{k_2}\eta_1^{l_1}\eta_2^{l_2}$, ($|k|+|l|\ \equiv\ k_1+k_2+l_1+l_2$= s) define the basis. In our (two-dimensional) case m=20,35, 56,84, 120 and 165, respectively, for s = 3,4,5,6, 7, and 8.

To solve the equation (9) the canonical transformation to the complex variables x, y is made

$$q_\nu= 1/\sqrt{2}\ (x_\nu+iy_\nu),\ \eta_\nu= i/\sqrt{2}\ (x_\nu- iy_\nu) \tag{12}$$

in which the operator of the normal form will be diagonal

$$\tilde{D}(x,y) = i\sum\omega_\nu\left(x_\nu\frac{\partial}{\partial x_\nu}-y_\nu\frac{\partial}{\partial\ y_\nu}\right), \tag{13}$$

and the basic equation becomes

$$\tilde{D}\tilde{W}^{(s)}(x,y)= -\tilde{H}(x,y)+\tilde{\Gamma}^{(s)}(x,y). \tag{14}$$

The basis of functional space consists of monomials of the form

$$\Phi^{(s)}= x_1^{l_1}x_2^{l_2}\ y_1^{m_1}\ y_2^{m_2} \tag{15}$$

which, as one can easily verify, are eigenfunctions of the operator D ,i.e.

$$\tilde{D}\ \Phi^{(s)}= \left[i\sum\omega_\nu(m_\nu-l_\nu)\right]\Phi^{(s)}. \tag{16}$$

Then ,the solution of the basic equation can be written as

$$\tilde{W}^{(s)}= \tilde{D}^{-1}[\tilde{\Gamma}^{(s)}- \tilde{H}^{(s)}] \tag{17}$$

where the inverse operator D^{-1} is determined by the relation

$$\tilde{D}\ \Phi^{(s)}= \Phi^{(s)}/[i\sum\omega_\nu(m_\nu-l_\nu)]. \tag{18}$$

Since $\Gamma^{(s)}$ is not yet determined, we choose it so that the quantity $\tilde{W}^{(s)}$would have no poles $\sum\omega_\nu(m_\nu-l_\nu)$ .The Hamiltonian $\Gamma^{(s)}$ should be chosen so as to compensate exactly those terms $H^{(s)}$ which lead to the resonance condition $\sum\omega_\nu(m_\nu-l_\nu)= 0$.. But if this condition is fulfilled for some terms of the Hamiltonian $H^{(s)}$ this implies the normal form and consequently $\tilde{D}\tilde{H}^{(s)}=0$. Thus, the quantity $\tilde{H}^{(s)}$ can unambiguously be represented as a sum of polynomials $\tilde{H}^{(s)}=\tilde{N}^{(s)}+\tilde{R}^{(s)}$ with $\tilde{D}\tilde{N}^{(s)}= 0$.Choosing $\tilde{\Gamma}^{(s)}= \tilde{N}^{(s)}$ (the normal form) we see that the terms of the Hamiltonian $\tilde{H}^{(s)}$ leading to the condition $\sum\omega_\nu(m_\nu-l_\nu)= 0$ are canceled by the same terms in the Hamiltonian $\Gamma^{(s)}$,the generating function $W^{(s)}$being finite and determined from the algebraic equation $\tilde{D}W^{(s)}=R^{(s)}$ Moreover, relation $\sum\omega_\nu(m_\nu-l_\nu)= 0$ may holds if $m_\nu=l_\nu$ and ω_ν are incommensurable (the normal Birkhoff form) or if $m_\nu\neq l_\nu$ but there are resonance relations of $\sum n_\nu\omega_\nu= 0$ (n_ν being integers). In the last case, the structure of the normal form Γ becomes more complicated, as ithas been mentioned in the Introduction.

Finally going back from the complex variables x and y to variables ξ,η we find the normal Hamiltonianform $\Gamma^{(s)}(\xi,\eta)$ and the generating function $W^{(s)}$ in the s order of approximation in terms of the initial variables.

2.3. The modified normal form of Birkhoff-Gustavson

In this section we present a certain modification [10] of the procedure of reducing the Hamiltonian to the normal B-G.Before reducing the initial Hamiltonian (1) to the normal form according to the Gustavson algorithm [3],(Sec2.1 and 2.2) we first do the canonical change $(q,p)\Rightarrow(\mathcal{P},Q)$ which would transform the operator form Eq.(10) into the diagonal one. For the Hamiltonian (1) with the

resonance relation $\omega_1 = \omega_2$ the above transformation has the form

$$q_1 = i/2(-Q_1+Q_2+\mathcal{P}_1+\mathcal{P}_2),\ p_1=1/2(Q_1-Q_2+\mathcal{P}_1-\mathcal{P}_2),$$
$$q_2 = 1/2(Q_1+Q_2+\mathcal{P}_1+\mathcal{P}_2),\ p_2=i/2(Q_1+Q_2-\mathcal{P}_1-\mathcal{P}_2). \quad (20)$$

The Hamiltonian (1) in the new variables looks like

$$K(Q,\mathcal{P}) = K^{(2)}(Q,\mathcal{P}) + \sum_{j>2} K^{(j)}(Q,\mathcal{P}), \quad (21)$$

were

$$K^{(2)}(Q,\mathcal{P}) = i(Q_1\mathcal{P}_1+Q_2\mathcal{P}_2)$$

and the canonically conjugate coordinates and momenta are $Q=(Q_1,Q_2)$ $,\mathcal{P}=(\mathcal{P}_1\ \mathcal{P}_2)$. Note that transformation (20) are canonical with " valence" equal to imaginary unit [17].

Each term $K^{(j)}$ of Hamiltonian (21), which is a homogeneous polynomial in variables Q and $\mathcal{P}$ of power j ,is reduced to the normal form by making a series of canonical transformations $(Q,\mathcal{P}) \to (\xi,\eta)$ with the generating function (6). Reducing to the normal form is simplified by a solution of the basic equation

$$DW^{(s)} = \Gamma^{(s)} - K^{(s)} \quad (22)$$

owing to the diagonality of the operator D. The introduction of $(Q,\mathcal{P})$ variables simplifies the algorithm providing an useful tool for quantization of (1) for $\omega_1 = \omega_2$.

3. Description of the program in REDUCE

The GITA program consists of 6 basic and 4 auxiliary blocks, some of them being created as procedures.

Block 1.Transformation of the coordinates $p_\nu \to \sqrt{\omega_\nu}p_\nu, q_\nu \to q_\nu/\sqrt{\omega_\nu}$ for reducing $H^{(2)}(p,q)$ to the NF (2). The nonlinear part of the Hamiltonian (3) is simultaneously transformed too.

Block 2.The beginning of the cycle s=3,. ..s_{max} where s_{max} is the maximal order of reducing to the normal form. The procedure SUPQXY performs transformation to the complex variables (12) of the homogeneous part of the s-order $\tilde{H}^{(s)}(x,y) = H^{(s)}(p,q)$.

Block 3. The SEPA procedure makes division of $\tilde{H}^{(s)}(x,y)$ into monomials (15).

Block 4.T The BASIS procedure solves the basic equation (16) and finds the generating function $\tilde{W}^{(s)}(x,y)$ and the sth order NF of the Hamiltonian $\tilde{\Gamma}^{(s)}(x,y)$.

Block 5. The inverse transformation is fulfilled from the complex variable x, y to the initial variables q,p .

Block 6. It implies the transformation of the remaining part of the Hamiltonian containing all higher than s orders which are necessary for the next reduction step. At the end of the cycle one goes back to block 2.

Upon completion of the cycle with respect to s one gets a Hamiltonian in the NF up to order and the "abnormal" part of the order higher than s_{max}.

Additional blocks.

Block 7. One transforms the NF $\Gamma^{(s)}(q,p)$ from the Cartesian coordinates to the action-angle variables I, φ. If the case is nonresonant one the NF $\Gamma^{(s)}(I)$ depending only on the action I (i.e.,it is angle-independent).

Block 8. The basic results are collected into output file for further convenience.

The auxiliary procedures FAC(N) and CNM(N,M) calculate the factorials and binomial coefficients correspondingly. The GITN program calculating the modified B-G form has the same structure as GITA.

4. Examples

4.1 The normal B-G form. The resonance case.

The potential function of quadruple vibrations of the surface of a spherical liquid drop of matter can be parameterized as[18-19]

$$V(q_1,q_2) = \frac{1}{2}(q_1^2+q_2^2) + b(q_1^2q_2 - \frac{1}{3}q_2^3) + c(q_1^2+q_2^2)^2. \quad (23)$$

As is known[19-20], the invariant polynomials of the discrete C_{3v} group can be polynomialy expressed in terms of two basis invariants $(q_1^2+q_2^2)$ and $(q_1^2q_2 - 1/3q_2^3)$. Therefore, the potential function (23) is the most general among the 4-order polynomials obeying symmetry C_{3v}[21]. Note that for c=0 the expression (23) is reducing to the well known Henon-Heiles potential[22].

The normal Birkhoff-Gustavson form for the Hamiltonian (1)

with potential (23) , in 4th approximation order equals to

$$\Gamma_4(\xi,\eta)= 1/2(\xi_1^2+ \xi_2^2 + \eta_1^2+ \eta_2^2) + \qquad (24)$$
$$+(- \frac{5b^2}{48} + \frac{3c}{8})(\xi_1^2 + \xi_1^2+ \eta_1^2+ \eta_2^2) +(- \frac{c}{2} + \frac{7b^2}{12})(\xi_1\eta_2- \xi_2\eta_1)^2.$$

In the action-angle variables

$$\xi_\nu= \sqrt{2\ \bar{I}_\nu}\ cos\ \bar{\varphi}_\nu\ ,\eta_\nu= \sqrt{2\bar{I}_\nu}\ sin\ \bar{\varphi}_\nu\ ,\nu=1,2 \qquad (25)$$

one have

$$\Gamma_4= \bar{I}_1 + \bar{I}_2+\left(- \frac{5b^2}{12} + \frac{3c}{2}\right)(\bar{I}_1^2+\bar{I}_2^2)+ (1/3b^2+2c)\bar{I}_1\bar{I}_2+ \qquad (26)$$
$$+(- \frac{7b^2}{6} + c)\bar{I}_1\bar{I}_2 cos\ 2(\bar{\varphi}_2- \bar{\varphi}_1).$$

or in the new canonically conjugate variables

$$2I_1= \bar{I}_1+ \bar{I}_2,\ 2\varphi_1= \bar{\varphi}\ + \bar{\varphi},\ I_2= \bar{I}_1- \bar{I}_2,\ \varphi_2= \bar{\varphi}_1- \bar{\varphi}_2 \qquad (27)$$

the normal form becomes

$$\Gamma_4=2I_1+(- \frac{5b^2}{4}+ \frac{9}{2}c)I_1^2+(- \frac{7b^2}{24} + \frac{c}{4})I_2^2 + \frac{1}{4}(\frac{7b^2}{6} -c)I_2^2\ cos\varphi_2\ . \qquad (28)$$

As the variable φ_1 does not explicitly enter the Hamiltonian (28), the corresponding canonically conjugate momentum I_1 is the integral of motion. Another integral of motion is the Hamiltonian Γ_4 itself.Thus,the Hamiltonian Γ_4 describes the integrable system.

4.2 The modified normal form

For comparison we present the modified NFfor the same Hamiltonian as in Sec. 4.1 in the 4th approximation obtained by GITN

$$\Gamma_4=i(\xi\ \eta\ +\xi\ \eta\ \ +(\frac{b^2}{6} +c)(\xi_1^2\eta_1^2+\xi\ \eta\ )-(2b^2- 2c)\xi_1\xi_2\eta_1\eta_2)\ . \qquad (29)$$

One can easily verify that independent integrals of motion here are $N=\xi_1\eta_1+ \xi_2\eta_2$ and $M=\xi_1\eta_1-\xi_2\eta_2$, and the NF equals to

$$\Gamma_4= i(N+N^2(- \frac{11b^2}{24}+ \frac{5c}{4}) + M^2(\frac{13b^2}{24} - \frac{3}{4}\ ))\ . \qquad (30)$$

As it is seen, the separation of variables takes place in coordinates N, M unlike I_1 and I_2 though in the 6-th approximation there is no separation.

4.3. The Birkhoff-Gustavson NF- nonresonance case.

As a potential function we choose the function

$$V(q_1,q_2)=1/2(\omega_1^2q_1^2+\omega_2^2q_2^2) + \sqrt{2}\ q_2^3+ q_1^2q_2 + 1/4q_2^4 \qquad (31)$$
$$\omega_1= \sqrt{2(2+\sqrt{2})} \cong 2.613125,\ \omega_2= 2.$$

The normal form obtained by GITA in the 4th approximation equals

$$\Gamma_4= \omega_1I_1+\omega_2I_2- 0.016735\ I_1^2 -0.375\ I_2^2 -0.219363\ I_1\ I_2\ . \qquad (32)$$

References

1. Arnold V.I.*Mathematical methods of classical mechanics.* M. Atomizdat, 1974,. p. 431 (in Russian)
2. Birkhoff G. *Dynamical systems.* M.OGIZ,1941, p.384 (in Russian)
3. Gustavson F.G., Astron.J. v. 71, p. 670.
4. Giorgilli A.,Comput. Phys.Commun.1979, v. 6, p. 331.
5. Hori G.I., Astron.Soc. Japan. 1966, v.18, p. 287.
6. Deprit A., Cel. Mech. 1969, v.1, p. 12.
7. Campbell J., Jeffris W.H. Celest. Mech. 1970, v. 2, p. 467.
8. Henrad J., Rcels J. Celest. Mech. !974, v.10, 497.
9. Mersman W.A., Celest.Mech. 1970, v. 30, p. 81.
10. Markeev A.P., Sokolsky A.G., *Some calculation algorithms for the normalization Hamiltonian systems.* Preprint IPM AN SSSR,1976, N 31.
11. Bolotin Yu.L. et al Phys.Elem.Part.At.Nucl.,1989, v.20. part 4, p.878.
12. Swimm R.T., Delos J.B.,J.Chem.Phys. 1979,v. 71, p.1706.
13. Jaffe C., Reinhart W.P. J.Chem. Phys. 1982,v.77,p.5191.
14. Robnik M., J.Phys.A: Math.Gen. 1984, v.17, p.109.
15. Uzer T., Marcus R.A., J.Chem.Phys. 1984, v. 81,p. 5013.
16. Chekanov N.A. Jad.Fiz., 1989, v.50. p.344.
17. Gantmaher F.R. *Lectures on analytical mechanics.* M. Nauka,1966 (in Russian)
18. Eisenberg I.,Greiner V. *Model of nuclei. Collective and one particle states.* M. Atomisdat (Russian transl),1975.
19. Bolotin Yu.L.,Vinitsky S.I.,Gonchar V.Yu., Chekanov N.A., Yad. Fiz. 1989, 6(12), p.1563.
20 Bolotin Yu.L et al ,Yad. Phys. 1990,v.51,1818.
21. Sleter Dj. *Electron structure of molecules.* Mir,1965.
22. Henon M., Heiles C. Astron.J., v.69, p. 73.

```
      Appendix
%   ========== The start Conditions==========;
% file Gita;
%-----------Pri ...=1 tables for terms neq 0,  =2 all terms----;
   SMAX:=4; IMAX:=SMAX; PRISEP:=0;PRIBAS:=0;IU:=1;
%BLOKSIZE]*:=80;off nat;
  %off echo;
%-----------------------------------------;
 Procedure ptim(N);
     begin integer N; real TN,DT;
if N=0 then begin TST:=time();NST:=N;write "start-time" end;
tn:=time(); DT:=TST-TN;% if N neq 0 then
write
  "<*<*<*<*<*   range= ",NST,"-",N,"time=",DT,"msec *>*>*>*>*>";
if N neq 0 then write "---------time(",NST,"-",N,")=",DT,"msec--
---------";
       NST:=N; TST:=TN     end;
lisp operator time;  ptim(0);
%*****************************************;
 Procedure SUPQXY(A);
   begin real B;
   let SQ2**2=2;
   B:=sub(Q1=I/SQ2*(y1-i*X1),P1=(Y1+I*X1)/SQ2,
   Q2=I/SQ2*(Y2-I*X2), P2=(Y2+I*X2)/SQ2,A);
clear SQ2 ; return B
       end;
%-----------------------------------------;
     Procedure SUXYPQ(A);
    begin real B;
   let SQ2**2=2;
   B:=sub(Y1=1/SQ2*(P1-I*Q1), X1=1/SQ2*(-I*P1+Q1),
   Y2=1/SQ2*(P2-I*Q2), X2=1/SQ2*(-I*P2+Q2),A);
clear SQ2 ; return B      end;
-----------------------------------------;
      Procedure FAK(N);
FAK:= if N<0 then 0 else if N=0 then 1 else for J:=1:N product J;
%-----------------------------------------;
      Procedure CNM(N,M);
CNM:= if  (N<0 or N<M ) then 0 else FAK(N)/(FAK(N-M)*FAK(M));
%=========================================;
      Procedure SEPA(HT,S);
       begin integer K;
                  real c1,c2,c3,c4;
   for L11:=0:S do begin
   for L22:=0:S do begin
   for M11:=0:s do begin
   M22:=s-(M11+L11+L22);
 if M22 geq 0 then HK(L11,L22,M11,M22):=0;
  end end end;
```



```
  for L1:=0:s do
    begin
          C1:=coeffn(ht,Y1,L1);
          for L2:=0:s do
    begin
          C2:=coeffn(C1,y2,L2);
                for M1:=0:s do
    begin
             C3:=coeffn(C2,X1,m1);
                   M2:= S-(L1+L2+M1);
             if M2 geq 0 then
    begin
   K:=K+1;
                        c4:=coeffn(c3,x2,m2);
%---------------Print table H-components-----------;
 if K=1 and PRISEP>0 then write " K  M1 M2 L1 L2    H ";
 if PRISEP > 0 then
write K, "    ",M1,"    ",M2,"    ",L1,"    ",L2,"    ",C4;
        if C4 neq 0 and M2 geq 0 then
        HK(L1,L2,M1,M2):=c4;
                          end end end end;
 return end;
%-------------------------------------------------;
      procedure basis(S); begin real FF,TF; integer K;K:=0;
      for L1:=0:S do
    begin
      for L2:=0:S do
    begin
      for M1:=0:S do
    begin
   M2:=S-(M1+L1+L2);
 if M2 geq 0 then
    begin K:=K+1;
 if K=1 then write "      Resonances at S=",S;
   FT:=Y1**L1*Y2**L2*X1**M1*X2**M2;
   EID:=I*(W1*(M1-L1)+W2*(M2-L2));TGT:=0;
 if EID =0 then
    begin TGT:=HK(L1,L2,M1,M2); GT:=GT+TGT*FT end else
    begin  TWT:= HK(L1,L2,M1,M2)/EID;
  WT:=WT- TWT*FT    end;
%---------------Print table-------------------------;
 if EID=0 and M1 neq L1 and HK(L1,L2,M1,M2) neq 0 then
write K,"    M1=",M1,"    L1=",L1,"    M2=",M2,"    L2=",L2,
      " M1-L1 =",M1-L1,"M2-L2=",M2-L2,
        "    HK=",HK(L1,L2,M1,M2);
 if PRIBAS >0 then begin
 if K=1 then write " K    M1  L1  M2  L2    *W1    *W2    HK";
 if TGT =0 and TWT =0 and PRIBAS<2 then goto MFB;
write K,"  ",M1," ",M2," ",L1," ",L2," ",FT," ",EID," " G=",TGT,
      "   W=", TWT; MFB:  end  end end end end;
```

```
            return end;
%------------The procedures end---------------------;
    ptim(1);
%=================================================;
   factor EPS;
  on div;
   operator HK,GG,H,W;
       factor W1,W2; on rat;
       for k:=2:8 do H(K):=0;
write  if IU =1 then begin write "Potential Henon-Heiles  ";
write  W1:=1;
write  W2:=1;
write  SQW1:=1;
write  SQW2:=1;
write  H2:=1/2*(P1**2+P2**2)+1/2*((W1*Q1)**2+(W2*Q2)**2);
write  H3:=    (Q1**2*Q2-1/3*Q2**3);
write  H4:=0;       end;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% ;
       if IU=2 then begin write "   Potential anti-Henon-Heiles";
       W1:=1;W2:=1;SQW1:=1;SQW2:=1;
       H2:=1/2*(P1**2+P2**2)+1/2*((W1*Q1)**2+(W2*Q2)**2);
       H3:= (Q1**2*Q2 +1/3*Q2**3);
       H4:=0;
       end;
%************************************************;
      if IU =7 then begin write "Potential quadr. vibrations";
      W1:=1;W2:=1;SQW1:=1;SQW2:=1;
      write H2:=1/2*(P1**2+P2**2)+1/2((W1*Q1)**2+(W2*Q2)**2);
      write H3:=B*(Q1**2*Q2-1/3*Q2**3);
      write H4:=C*(Q1**2+Q2**2)**2;
       end;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%;
write "   Transformation of H(2) to normal form";
     let SQ**2=2;let SQW1**2=W1;
write  H(2):=sub(P1=SQW1*P1,P2=SQW2*P2,Q1=Q1/SQW1,Q2=Q2/SQW2,H2);
write  H(3):=sub(P1=SQW1*P1,P2=SQW2*P2,Q1=Q1/SQW1,Q2=Q2/SQW2,H3);
write  H(4):=sub(P1=SQW1*P1,P2=SQW2*P2,Q1=Q1/SQW1,Q2=Q2/SQW2,H4);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%5;
write  GG(2):=H(2);
     for S:=3:SMAX do
     begin
write "   ------------------------------------------------------";
       H3X:=SUPQXY(H(S));
write  "H(P,Q) to H(X,Y) ==================",ptim(2);
       H3N:=NUM(H3X); H3D:=DEN(H3X);SEPA(H3N,S);
write "Separate H(X,Y) to components=======",ptim(3);
       GT:=0;WT:=0;
       basis(S);
write "Evaluation W(X,Y) and G(X,Y)==========",ptim(4);
       GT:=GT/H3D;WT:=WT/H3D;
```



```
write  GG(S):= SUXYPQ(GT);
write  W(S):= SUXYPQ(WT);
write  "G(X,Y),W(X,Y)  to G(P,Q), W(P,Q)==============";
       ptim(7);
       for II:=S+1:IMAX do
      begin
      SS:=0;
       for J:=1:IMAX do
      begin
 L:=II-(S-2)*J;write " S=",S,"   I=",II,"   J=",J,"    L=",L;
    if L<J then goto FINJ;
       for J1:=0:J do
      begin
 J2:=J-J1; DHL:=DF(H(L),P1,J1,P2,J2); DGL:=DF(GG(L),Q1,J1,Q2,J2);
   if J1>0 then    WQ1:=DF(W(S),Q1)**J1;
   if J1>0 then    WP1:=DF(W(S),P1)**J1;
   if J2>0 then    WP2:=DF(W(S),P2)**J2;
   if J2>0 then    WQ2:=DF(W(S),Q2)**J2;
   if J1=0 then    WQ1:=1;
   if J1=0 then    WP1:=1;
   if J2=0 then    WQ2:=1;
   if J2=0 then    WP2:=1;
      TT:=1/(FAK(J1)*FAK(J2))*(DHL*WQ1*WQ2-DGL*WP1*WP2);
      SS:=SS+TT  end;
FINJ:       end;
   write GG(II):= H(II)+SS; ptim(6)   end;
     for K:=S:IMAX do H(K):=GG(K);  end;ptim (10);
write "============Normal end=========================";
%on gc;
%%%%%%%%%%%% Action-angle variables F1-A1,F2-A2 %%%%%%%%%%;
   HTAF:=0;
      for JS:=2:IMAX do
   begin
   HHH:=GG(JS);
   if (JS>SMAX) then HHH:=H(JS);
   HAF:=sub(Q1=SQ2*SQA1*cos(F1),P1=SQ2*SQA1*sin(F1),
        Q2=SQ2*SQA2*cos(F2),P2=SQ2*SQA2*sin(F2),HHH);
  for all X let cos(X)**2=(1+cos(2*X))/2 ,sin(X)**2=(1-cos(2*X))/2;
  factor sin, cos;
  let SQA1**2=A1,SQA2**2=A2,SQ2**2=2;
  HAF:=HAF;
  for all X,Y  let cos(X)*cos(Y)=(cos(X+Y)+cos(X-Y))/2,
     cos(X)*sin(Y)=(sin(X+Y)-sin(X-Y))/2,
     sin(X)*sin(Y)=(cos(X-Y)-cos(X+Y))/2;
  HAF:=HAF;
  for all X,Y clear cos(X)*cos(Y), cos(X)*sin(Y), sin(X)*sin(Y);
  for all Z  clear cos (Z)**2, sin(Z)**2;
  HTAF:=HTAF + HAF
    end;
on float;
```

```
write HTAF:=HTAF;
%-----------------------------------------------------------;
off nat; off echo;
out gitares;
IMAX:=IMAX;  IU:=IU; SMAX:=SMAX;H2:=H2;H3:=H3;H4:=H4;
W1:=W1;W2:=W2;NU:=NU ;MU:=MU;
for S:=3:IMAX do write "W(",S,"):=",W(S);
for K:=2:IMAX do write "H(",K,"):=",H(K);
on list;
HTAF:=HTAF;
shut gitares;
ptim(20);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%;
end;
```

Received by Publishing Department
on December 18, 1990.