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A PRECONDITIONING TECHNIQUE FOR THE SOLUTION OF 3-D ELLIPTIC PROBLEMS BY SUBSTRUCTURING WITH CROSS-LINES

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Introduction

This paper is a direct continuation of papers /1,2/. Here we consider the problem of cost-effective solution of finite-element system of equations occurring in approximation of elliptic boundary value problem with varying coefficients in the parallelepiped Ω by the Galerkin method. The domain Ω is partitioned into subdomains Ω_i by three groups of planes (with internal cross-lines) parallel to the planes *YOZ*, *XOZ* and *YOX*, correspondingly. In each of the subdomains Ω_i the equation coefficients are equal to some constants $\mu_i>0$. Then the initial problem is transformed to a boundary equation defined at the inner boundaries of subdomains Ω_i. The class of preconditioning operators is constructed which are "spectrally close" to the initial linear boundary operator regardless of the scattering range of the coefficients μ_i. It is shown that the boundaries of preconditioned operator spectrum depend only on the dimension of the basic function space of subdomains Ω_i and don't depend on their number. Preconditioning operators are easily invertible both for parallel and traditional computers since every step of iterations consists of independent solution of the partial boundary value problem for each of the subdomains.

In §1 boundary equations of the domain decomposition method for the "checkerboard" subdivision are formulated which are defined only at the inner boundaries. In §2 the family of preconditioners is constructed coinciding with the block-diagonal part of the initial operator, where blocks correspond to some splitting (at first into direct sum of subspaces $X_1 \oplus X_2 \oplus X_3$ and then splitting of the component X_3) of the basic space of finite elements, corresponding to some triangulation of the inner boundaries of the subdomains Ω_i. The subspace $\mathfrak{X}_1$ defined on the coarse grid with elements Ω_i provides the global data transfer between substructures, other components of the direct sum are defined only for the functions corresponding to separate subdomains. In §3,4 estimates of the spectrum boundaries of the preconditioned operator for the subspaces $\mathfrak{X}_2$, $\mathfrak{X}_3$ of the general form are obtained which are expressed in terms of characteristics of some functionals defined on finite-element subspaces of separate subdomains Ω_i. In §5 the results are applied to some concrete families of the first order finite elements characterized by the maximum number N of unknowns for one variable for all subspaces Ω_i. The method of constructing operators "spectrally equivalent" to the initial boundary operator in two- and three-dimensional cases, based on the idea of deformation of subdomains Ω_i is discussed in §6. Further in §7 it is defined that for uniform partitioning of the domain Ω by the "serendipity type" elements with the total number of unknowns on one direction N_0 and the number of subdomains p^3 (in this case $N_0 = N \cdot p$) for arbitrary $\mu_i > 0$ the asymptotics of the computational work (for the solution of the initial problem by the PCG method with accuracy $\varepsilon = N^{-\nu}$) accounts

$$Q = O\left(N^{1/2}\ln^3 N\left[N_0^3 + N^{1/2}\ln^{1/2}N(p^{7/2}+p^3N^{3/2}\ln N)\right]\right)$$

operations.

Note that the algorithm of the similar type for the problem in the finite-difference formulation have been developed in /3/. The preconditioners constructed here (for the parallelepiped case) provide the effective solution by the PCG method of the equations with varying coefficients in the domain Ω_c with topologically equivalent decomposion into subdomains each being a convex quadrangular prism, as well as of the divergent type quasi-linear and incomplete-nonlinear /1/ elliptic equations in the domain Ω (or Ω_G).

The problems of constructing block preconditioners including those corresponding to the "strip" type of subdivision have been investigated in /4-17/. Preconditioning operators for two-dimensional finite-element elliptic systems in the case of "box" type of decomposition have been sufficiently completely investigated in /18-22,1-3/. Preconditioners for the three-dimensional problems for the case of cuts with cross-lines have been considered in /23,24,1,3/. The multigrid domain decomposition methods have been considered in /25-27/, the problems associated with the local grid refinement have been studied in /28-31/.

§1 Formulation of the equation for the domain decomposition method

Let $\Omega = \{(x,y,z):\ 0\le x\le a\ ,\ 0\le y\le b\ ,\ 0\le z\le c\}$ be a parallelepiped. Consider the decomposition $\Omega = \bigcup \Omega_i$ into subdomains Ω_i formed by three groups of $n_x-1,\ n_y-1,\ n_z-1$ planes , parallel to the planes YOZ, XOZ and YOX, correspondingly. If necessary, denote by $n_x=n_1,\ n_y=n_2,\ n_z=n_3$. Designate by $M=n_1n_2n_3$. Assume $i = (i_1,i_2,i_3)$, $|i| = i_1+i_2+i_3$, $\|i\| = i_1i_2i_3$, where $1\le i_k\le n_k$, $k = 1,2,3$. Define the checkerboard subdivision of the domain Ω as

$$\Omega = \Omega_B \cup \Omega_W\ ,$$
$$\Omega_B = \bigcup_{i\in I} \Omega_i\ ,\qquad \Omega = \bigcup_{i\in I} \Omega_i\ , \tag{1.1}$$

where $I_B = \{\ i:\ |i| = 2l>0,\ l\in\mathbb{N}\ \}$, $I_W = \{\ i:\ |i|=2l+1,\ l>0,\ l\in\mathbb{N}\ \}$. Here $\mathbb{N}$ is a set of positive integers. Denote by $\Gamma = \partial\Omega$, $\Gamma_i = \partial\Omega_i$, and also by $\Gamma_I = \bigcup_{i\in I_B} \Gamma_i\backslash\Gamma \equiv \bigcup_{i\in I_W} \Gamma_i\backslash\Gamma$ - the integration of internal boundaries of the subdomains Ω_i.

For any function $u \in \mathring{\mathbb{H}}^1(\Omega)$ we define traces of this function $u_i=\gamma_i u$ at the internal boundaries $\tilde{\Gamma}_i = \Gamma_I \cap \Gamma_i$ of subdomains Ω_i, $\|i\| \le M$. The space of functions $u_i(\xi)$, $\xi \in \tilde{\Gamma}_i$ (with the $\mathbb{H}^{1/2}(\Gamma_i)$ norm) we denote by $\mathbb{V}_i^{1/2} = \tilde{\mathbb{H}}^{1/2}(\Gamma_i)$. Here and in what follows the index $\sim$ denotes the subspace of functions from $\mathbb{H}^{1/2}(\Gamma_i)$, differing from zero only on $\tilde{\Gamma}_i$ and defined according to the position of boundaries Γ_i and Γ. Similarly the space of traces $v_i = \gamma_i^1(u) \equiv \dfrac{\partial u}{\partial n}\Big|_{\Gamma_i}$ of normal derivatives on $\tilde{\Gamma}_i$ /33/ for $u \in \mathring{\mathbb{H}}^1(\Omega)$ denote by $\mathbb{V}^{-1/2} = \tilde{\mathbb{H}}_i^{-1/2}(\Gamma_i)$, where

$$\tilde{\mathbb{H}}_i^{-1/2}(\Gamma_i) = \left\{ v \in \tilde{\mathbb{H}}^{-1/2}(\Gamma_i) ;\ (v,1)_{\Gamma_i} \stackrel{\bullet}{=} 0 \right\}.$$

Further define the space of the traces of functions from $\mathring{\mathbb{H}}^1(\Omega)$ at the inner boundaries Γ_I as :

$$\mathbb{X}_B = \underset{i \in I_B}{\oplus} \mathbb{V}_i^{1/2} , \qquad \mathbb{X}_W = \underset{i \in I_W}{\oplus} \mathbb{V}_i^{1/2} .$$

According to definition of spaces $\mathbb{X}_B$ and $\mathbb{X}_W$ there exists a permutation operator $\mathbb{T}$, $\mathbb{T}\mathbb{T}^* = \mathbb{E}$ performing one-to-one mapping of $\mathbb{X}_B$ into $\mathbb{X}_W$ which is defined by the following way: let the element $u_B \in \mathbb{X}_B$ be the trace of the function $u \in \mathring{\mathbb{H}}^1(\Omega)$ on Γ_I so that $u_B = \underset{i \in I_B}{\oplus} u_i$, $u_i \in \mathbb{V}_i^{1/2}$ and $u_W \in \mathbb{X}_W$ is a trace of the same function u but having the presentation $u_W = \underset{i \in I_W}{\oplus} z_i$, $z_i \in \mathbb{V}_i^{1/2}$, then $\mathbb{T}u_B = u_W$. The spaces conjugate to $\mathbb{X}_B$ and $\mathbb{X}_W$ are of the form

$$\mathbb{X}_B^* = \underset{i \in I_B}{\oplus} \mathbb{V}_i^{-1/2} , \qquad \mathbb{X}_W^* = \underset{i \in I_W}{\oplus} \mathbb{V}_i^{-1/2} .$$

Let the arbitrary function defined as $\Psi(\xi) \in \mathbb{X}_B^*$, $\xi \in \Gamma_I$, where $\Psi = \underset{i \in I_B}{\oplus} \Psi_i = \mathbb{T}^* \nu$, $\nu \in \mathbb{X}_W^*$ and also arbitrary positive constants $\mu_i > 0$, $\|i\| \le M$ are given. Our aim is to develop cost-effective numerical methods for solving the following problem:

Find the function $u \in \mathring{\mathbb{H}}^1(\Omega)$ such that

$$\sum_{\|i\| \le N} \mu_i \int_{\Omega_i} (\nabla u, \nabla \eta) dx = \int_{\Gamma_I} \Psi(\xi) \gamma_0(\eta) d\xi \qquad (1.2)$$

for all $\eta \in \mathring{\mathbb{H}}^1(\Omega)$.

For solving (1.2) we use the equation of the domain decomposition method (for the decomposition (1.1)) defined at the inner boundaries Γ_I with respect to unknown function being a trace of the solution $u(x)$ of the equation (1.2) on Γ_I. We define at the boundary $\tilde{\Gamma}_i$ the linear Poincare-Steklov operator /8,9/ $\mathbb{S}_{\Delta,i}$: $\mathbb{V}_i^{-1/2} \Rightarrow \mathbb{V}_i^{1/2}$, corresponding to the Laplace operator in Ω_i and also the inverse operator $\mathbb{S}_{\Delta,i}^{-1}$, so that $\mathbb{D}(\mathbb{S}_{\Delta,i}) = \mathbb{V}_i^{-1/2}$, $\mathbb{D}(\mathbb{S}_{\Delta,i}^{-1}) = \mathbb{V}_i^{1/2}$, $Ker\ \mathbb{S}_{\Delta,i}^{-1} = \{ u \in \mathbb{V}_i^{1/2} :$ $u = const,\ x \in \Gamma_i \}$. The operator $\mathbb{S}_{\Delta,i}$ maps the function $g \in \mathbb{V}_i^{-1/2}$ into the element $\gamma_i(u) \in \mathbb{V}_i^{1/2}$ which is a trace of the function $u(x)$, $x \in \Omega_i$ on $\tilde{\Gamma}_i$, satisfying the equation

$$\int_{\Omega_i} (\nabla u, \nabla \eta) dx = \int_{\Gamma_i} g(\xi) \gamma_i(\eta) d\xi \qquad (1.3)$$

for all $\eta \in \mathring{\mathbb{H}}^1(\Omega)$. Here the equality holds

$$(\mathbb{S}_{\Delta,i} g, \eta) = (\gamma_i(u), \eta), \qquad \forall \eta \in \mathbb{V}_i^{-1/2}.$$

Further define operators

$$\mathbb{S}_{B,\Delta}^{-1} = \bigoplus_{i \in I_B} \mathbb{S}_{\Delta,i}^{-1} , \qquad \mathbb{S}_{B,\Delta}^{-1} \in \mathcal{L}(\mathbb{X}_B \Rightarrow \mathbb{X}_B^*)$$

$$\mathbb{S}_{W,\Delta}^{-1} = \bigoplus_{i \in I_W} \mathbb{S}_{\Delta,i}^{-1} , \qquad \mathbb{S}_{W,\Delta}^{-1} \in \mathcal{L}(\mathbb{X}_W \Rightarrow \mathbb{X}_W^*) ,$$

and also diagonal operators of the type

$$\mathbb{M}_B = \bigoplus_{i \in I_B} \mu_i \mathbb{E}_i , \qquad \mathbb{M}_W = \bigoplus_{i \in I_W} \mu_i \mathbb{E}_i ,$$

where $\mathbb{E}_i$ are identity operators on $\mathbb{V}_i^{1/2}$.

Let the function $u_I \in \mathbb{X}_B$ be a trace of the solution $u \in \mathring{\mathbb{H}}^1(\Omega)$ of the equation (1.2) on Γ_I. Then the equation of the domain decomposition method for the subdivision (1.1) takes form

$$(\mathbb{A}\, u_I,\ \eta) = (\Psi,\eta), \qquad \forall \eta \in \mathbb{X}_B, \tag{1.4}$$

where $\mathbb{A} = \mathbb{A}_1 + \mathbb{A}_2$ and

$$\mathbb{A}_1 = \mathbb{M}_B \cdot \mathbb{S}^{-1}_{B,\Delta}, \qquad \mathbb{A}_2 = \mathbb{T}^{*} \cdot \mathbb{M}_W\, \mathbb{S}^{-1}_{W,\Delta}\mathbb{T}. \tag{1.5}$$

According to /1/ the operator $\mathbb{A} \in \mathcal{L}(\mathbb{X}_B \Rightarrow \mathbb{X}^*_B)$ is symmetric and positively defined ; the equivalent norms in the spaces $\mathbb{X}_B$ and $\mathbb{X}^*_B$ can be given as

$$\| u \|^2_{X_B} = (\ \mathbb{A}u,u\ ), \qquad \| v \|^2_{X^*_B} = (\ \mathbb{A}^{-1}v,v\ ), \tag{1.6}$$

and for any function $\Psi \in \mathbb{X}^*_B$ there is a unique solution $u_I \in \mathbb{X}_B$ of (1.4) coinciding with the trace on Γ_I of the solution $u(x)$ of the equation (1.2) . Note that the boundary operator $\mathbb{A}$ has properties of both integral and differential operators (pseudo-differential operator) with the domain of definition Γ_I. In the simplest case when Γ_I is a rectangular domain, partitioning the parallelepiped Ω into two subdomains we obtain that $\mathbb{A}$ performs a one-to-one mapping:

$$\mathbb{A}\ :\ \tilde{\mathbb{H}}^{1/2}(\Gamma_I) \Rightarrow \tilde{\mathbb{H}}^{-1/2}(\Gamma_I),$$

where $\tilde{\mathbb{H}}^{1/2}(\Gamma_I)$ is a space of traces on Γ_I of functions $u \in \mathring{\mathbb{H}}^1(\Omega)$ and $\tilde{\mathbb{H}}^{-1/2}(\Gamma_I)$ is a space of traces on Γ_I of their normal derivatives.

Now let us construct the family of effective preconditioners for some class of Galerkin approximations of the operator $\mathbb{A}$.

§2 Construction of preconditioners

Let the subdivision of the domain $\Omega = \bigcup \Omega_i$ considered above form a grid defining the "serendipity type" finite elements family of the first order /34/. We suppose that the subdivision satisfies the condition $d/r_i \le c_0$ with some constant $c_0>0$ which doesn't depend on $d=\max_i(\mathrm{diam}\ \Omega_i)$, where r_i are the radii of spheres inscribed in Ω_i. Dimension of the corresponding space of basic functions $\mathbb{X}_i$ equals to $KI=\prod_{k=1}^{3}(n_k-1)$ of the inner vertexes of subdomains Ω_i. We introduce the notation $\Gamma_i=\bigcup_{k=1}^{6}\Gamma_i^k$, and $\partial\Gamma_i^k=\bigcup_{m=1}^{4}\Gamma_i^{km}$, where Γ_i^k are sides and Γ_i^{km} are edges of the parallelepiped Ω_i. Following /1,2/ introduce next definitions.

Definition 2.1. For every edge Γ_i^{km} we define a finite-dimensional function space $\mathbb{X}_i^{km} \subset \overset{\circ\circ}{\mathbb{H}}{}^{1/2}(\Gamma_i^{km})$. Each function $u\in\mathbb{X}_i^{km}$ (we call it a generatrix function) is associated with four functions $\bar{u}_j$, $j=1,\dots,4$ defined on four sides having common edge Γ_i^{km}. These functions are equal to u at the edge Γ_i^{km} and to zero at the other remaining three edges while at the inner points of the sides they are defined depending on their orientation. Consider, for example, a group of edges Γ_i^{km} which are parallel to the axis OZ. Then we define $\bar{u}_j$ for adjacent to Γ_i^{km} sides parallel to the plane ZOY as a linear continuation on Γ_i^k of the function $u\in\mathbb{X}_i^{km}$ and for the sides parallel to the plane ZOX the function $\bar{u}_j$ is harmonic in the domain Γ_i^k. Two groups of edges parallel to axes OX and OY are treated in a similar way. If we suppose the function $\bar{u}_j$, $j=1,\dots,4$ equal to zero for the rest of the boundary Γ_I, then so obtained functions will form, according to definition, the space $\bar{\mathbb{X}}_i^{km}$. Its elements have nonzero harmonic continuation only for four subdomains Ω_i having a common edge Γ_i^{km}. Further assume

$$\mathbb{X}_2 = \oplus\ \bar{\mathbb{X}}_i^{km} \qquad (2.1)$$

where the sum is extended on all inner edges $\Gamma_i^{km} \in \Gamma_I$.

Note that the effect of the Poincare-Steklov operator on the function $\bar{u} \in \bar{\mathbb{X}}_l^{km}$ (at the boundary Γ_l) is easily computed by means of the generatrix function $u \in \mathbb{X}_l^{km}$.

Definition 2.2. Let the partitioning $\Gamma_l = \bigcup_{k=1}^{6} \Gamma_l^k$ is given. For every k_0, $1 \le k_0 \le 6$ we define a finite-dimensional subspace $\mathbb{G}(\Gamma_l^{k_0}) \subset \overset{\circ}{\mathbb{H}}{}^{1/2}(\Gamma_l^{k_0})$ of functions equal to zero on all sides Γ_l^k except $\Gamma_l^{k_0}$ (we denote by $\overset{\circ}{\mathbb{H}}{}^{1/2}(\Gamma_l^{k_0})$ the space of functions $u \in \mathbb{H}^{1/2}(\Gamma_l)$, such that $u(\xi)=0$ for all $\xi \notin \Gamma_l^{k_0}$). Then define

$$\mathbb{X}_3 = \underset{l \in I_B}{\oplus} \mathbb{G}_l \ , \quad \mathbb{G}_l = \underset{\substack{k=1 \\ \Gamma_l^k \cap \ =\varnothing}}{\overset{6}{\oplus}} \mathbb{G}(\Gamma_l^k) \ . \tag{2.2}$$

Construct the finite-dimensional space

$$\mathbb{Y}_0 = \mathbb{X}_1 \oplus \mathbb{X}_2 \oplus \mathbb{X}_3 \subset \mathbb{X}_B \ . \tag{2.3}$$

Consider the following problem: find the function $u_0 \in \mathbb{Y}_0$, satisfying the equation

$$(\mathbb{A}u_0, \eta) = (\psi, \eta) \ , \ \forall \eta \in \mathbb{Y}_0 \ . \tag{2.4}$$

Let us construct preconditioning operators $\mathbb{B}$ for the finite-dimensional operator $\mathbb{A}_0$, defined by *(2.4)*. In the subspace $\mathbb{G}_l \subset \mathbb{X}_3$ we represent the operator $\mathbb{S}_{\Delta,l}^{-1}$ in a block form:

$$\mathbb{S}_{\Delta,l}^{-1} = \{ \mathbb{S}_l^{km} \}, \qquad k,m = 1,\dots,6$$

according to representation of the function $u \in \mathbb{G}_l$ in the form $u = (u_1,\dots,u_6)^T$, $u_k \in \mathbb{G}(\Gamma_l^k)$, $k=1,\dots,6$. In the subspace $\mathbb{X}_3$ define the operator

$$diag\ \mathbb{A}_3 = \mathbb{M}_B \underset{l \in I_B}{\oplus} \Big(\underset{\substack{k=1, \\ \Gamma_l^k \cap \Gamma = \varnothing}}{\overset{6}{\oplus}} \mathbb{S}_l^{kk} \Big) + \mathbb{T} \cdot \mathbb{M}_W \underset{l \in I_W}{\oplus} \Big(\underset{\substack{k=1, \\ \Gamma_l^k \cap \Gamma = \varnothing}}{\overset{6}{\oplus}} \mathbb{S}_l^{kk} \Big) \cdot \mathbb{T} \tag{2.5}$$

We represent an arbitrary function $u \in \mathbb{V}_0$ in a form $u = u_1 + u_2 + u_3$, $u_l \in \mathbb{X}_l$, $l=1,\dots,3$. Consider the family of operators $\mathbb{B}_k u$, $u \in \mathbb{V}_0$, $k=1,\dots,5$, defined by the equalities:

$$(\mathbb{B}_1 u,\ v) = (\mathbb{A}\ u_1, v_1) + (\mathbb{A}\ u_2, v_2) + (diag\ \mathbb{A}_3 u_3, v_3), \tag{2.6}$$

$$(\mathbb{B}_2 u,\ v) = (\mathbb{A}[u_1+u_2],\ v_1+v_2) + (diag\ \mathbb{A}_3 u_3, v_3), \tag{2.7}$$

$$(\mathbb{B}_3 u,\ v) = (\mathbb{A}\ u_1, v_1) + (diag\ \mathbb{A}_3 u_3, v_3), \tag{2.8}$$

for all $v \in \mathbb{V}_0$. Consider also the operator $\mathbb{B}_4 u$, $u \in \mathbb{X}_1 \oplus \mathbb{X}_2$ defined by

$$(\mathbb{B}_4 u,\ v) = (\mathbb{A}\ u_1, v_1) + (\mathbb{A}\ u_2, v_2) \tag{2.9}$$

for all $v \in \mathbb{X}_1 \oplus \mathbb{X}_2$. For solving the auxiliary problems we use the preconditioning operator $\mathbb{B}_5 u$, $u \in \mathbb{X}_2$ defined by the equality

$$(\mathbb{B}_5 u,\ v) = (diag\ \mathbb{A}_2 u, v), \tag{2.10}$$

for all $v \in \mathbb{X}_2$, where $diag\ \mathbb{A}_2$ is the block-diagonal part of the operator $\mathbb{A}_0$ in the subspace $\mathbb{X}_2$, corresponding to the following presentation of the element

$$u = \bigoplus_{u_i \in \bar{\mathbb{X}}_i^{km}} u_i, \qquad u \in \mathbb{X}_2,$$

where the sum is spread for all inner edges $\Gamma_i^{km} \in \Gamma_I$. The block dimension of the operator $\mathbb{A}_0$ in the subspace $\mathbb{X}_2$ equals to 13. Note that the problem of inverting operators $\mathbb{B}_k$, $k=1,\dots,5$ is essentially easier than the one for the operator $\mathbb{A}_0$ (at the corresponding subspace $\mathbb{D}(\mathbb{B}_k)$). Solution of the equation

$$(\mathbb{B}_1 u,\ v) = (f, v), \qquad \forall\ v \in \mathbb{V}_0 \tag{2.11}$$

is reduced to consecutive solution of problems: find functions u_1, u_2, u_3 from the equations

$$(diag\ \mathbb{A}_3 u_3, v) = (f, v), \qquad \forall\ v \in \mathbb{X}_3 \tag{2.12}$$

$$(\mathbb{A} u_2, v) = (f, v), \qquad \forall\ v \in \mathbb{X}_2 \tag{2.13}$$

$$(\mathbb{A} u_1, v) = (f, v), \qquad \forall\ v \in \mathbb{X}_1 . \tag{2.14}$$

From (2.6) it follows that the function $u = u_1 + u_2 + u_3$ is the solution of the equation (2.11). The problem (2.14) is the finite-element system of equations for the "serendipity type" elements of the first order, formed by the subdivision $\Omega = \bigcup \Omega_i$. Dimension of this SLAE equals to KI. The problem (2.12) is equivalent to independent (and partial) solution of mixed problems for the Laplace operator in Ω_i , $\|i\| \le M$ with the Neumann condition on one side and the homogeneous Dirichlet condition for the rest five ones. With appropriate choice of the subspace $\mathbb{X}_3$ partial problems in Ω_i are solved by the FFT method.

For solving the equation (2.13) with unknown functions defined only at the inner edges the PCG method with the preconditioner $\mathbb{B}_5$ can be used. Inverting the operator $\mathbb{B}_5$ from (2.11) is a trivial problem. The estimate of the convergence rate for the PCG method will be given below. Above considerations are similarly applied to the problems of solving equations (2.8), (2.9). The equation resulting from projection of (2.7) to the subspace $\mathbb{X}_1 \oplus \mathbb{X}_2$ is solved by the PCG method using the preconditioner $\mathbb{B}_4$.

Further consider estimates of the condition number of operators $\mathbb{B}_k^{-1}\mathbb{A}_0$, $k=1,\dots,5$ which are reduced to estimates of the constants c_2^k , $c_1^k > 0$ from inequalities

$$c_1^k(\mathbb{A}u,u) \le (\mathbb{B}_k u,u) \le c_2^k(\mathbb{A}u,u), \qquad \forall\ u \in \mathbb{D}(\mathbb{B}_k) \qquad (2.15)$$

§3 General estimates of condition numbers for the operators $\mathbb{B}_k^{-1}\mathbb{A}_0$

In analyzing the spectral closeness of operators $\mathbb{B}_k$, $k=1,\dots,5$ and $\mathbb{A}_0$ we use the following characteristics $\alpha(\mathbb{Z},\mathbb{Z}_i)$ of the direct sum of Hilbert spaces /1/ $\mathbb{Z} = \overset{p}{\underset{k=1}{\oplus}} \mathbb{Z}_k$:

Definition 3.1. For the given spaces $\mathbb{Z}$ and $\mathbb{Z}_k$ we define real numbers $\alpha(\mathbb{Z},\mathbb{Z}_k)$ according to

$$\alpha(\mathbb{Z},\mathbb{Z}_k) = \sup \alpha_k \geq 0 \ ; \quad k = 1,\dots,p \ ,$$

where numbers $\alpha_k \geq 0$ satisfy the conditions

$$\sum_{k=1}^{p} \alpha_k \ \|v_k\|^2 \leq \|v\|^2$$

for all $v_k \in \mathbb{Z}_k$, such that $v = \bigoplus_{k=1}^{p} v_k$.

Remark 3.1. Let for $p=2$ the estimate $\alpha(\mathbb{Z},\mathbb{Z}_1) \geq a > 0$ is true, then the inequality holds

$$\alpha(\mathbb{Z},\mathbb{Z}_2) \geq (1+a^{-1})^{-1}.$$

In fact, if $\|x_1\|^2 \leq a^{-1}\|x_1+x_2\|^2$, $x_1 \in \mathbb{Z}_1$, $x_1+x_2 \in \mathbb{Z}$, then $\|x_2\|^2 \leq (1+a^{-1})\|x_1+x_2\|^2$.

Remind that according to (1.6) the norm in $\mathbb{V}_0$ is given by

$$\|u\|^2_{V_0} = \sum_{\|i\| \leq N} \mu_i \mathbb{A}_i(u_i), \qquad u_i \in \mathbb{V}_i^{1/2}, \qquad (3.1)$$

where $\mathbb{A}_i(v) = \int_{\Omega_i} |\nabla\tilde{v}|^2 dx$ is the Dirichlet integral from the function $\tilde{v}(x)$ defined on Ω_i such that $\tilde{v}(x)_{\Gamma_i} = v(x)$. Later we shall use the norm of the type (3.1).

Remark 3.2. We consider below only such subspaces $\mathbb{X}_2$, $\mathbb{X}_3$ for which the norms in $\mathbb{H}^{1/2}(\Gamma_i)$ (Γ_i is a surface of a parallelepiped) constructed according to definition from /32/(where we choose a combination of various pairs of adjacent sides as a finite subcovering of the boundary Γ_i) are equivalent to standard norms using subcovering including vertexes of the domain Ω_i. Formulated requirement is a simple consequence of the hypotheses 1-4 (concerning spaces $\mathbb{X}_2$, $\mathbb{X}_3$) used below (see §5).

For the function $v \in \mathbb{G}_i \subset \mathbb{X}_3$ given on sides $\Gamma_i^k = \{x_1, x_2$: $0 \leq x_1 \leq a_i^k,\ 0 \leq x_2 \leq b_i^k\}$, $k=1,\dots,6$ define functionals

$$\bar{f}_{i,k}(v) = \int_0^{a_i^k} \frac{\|v(x_1,x_2)\|^2_{(2)}}{x_1|a_i^k - x_1|}dx_1 + \int_0^{b_i^k} \frac{\|v(x_1,x_2)\|^2_{(1)}}{x_2|b_i^k - x_2|}dx_2 ,$$

where $\|\cdot\|_{(j)}$ is the $\mathbb{L}_2$ norm over the variable x_j, $j=1,2$. For the function $u = \oplus_{k=1}^{6} u^k \in \mathbb{G}_i$ define the functional .

$$f_i(u) = \sum_{k=1}^{6} \bar{f}_{i,k}(u^k). \tag{3.2}$$

The functional f_i, $u \in \mathbb{G}_i$ majorizes the functional used in /32/ in formulation of conditions of pasting functions from spaces $\mathbb{H}^{1/2}(\Gamma_i^k)$, $k=1,\dots,6$ for every pair of adjacent sides.

Denote by $u_{m,i}$ the trace of the function $u_m \in \mathbb{X}_m$, $m=1,\dots,3$ at the boundary $\tilde{\Gamma}_i$. The following theorem is true.

Theorem 1. Let the numbers $\alpha(\mathbb{X}_1,\mathbb{Y}_0)$, $\alpha(\mathbb{X}_2,\mathbb{X}_2 \oplus \mathbb{X}_3)$ are positive, then the estimate holds

$$(\mathbb{B}_1 u,u) \le c_2^1\left[\left(1+\alpha(\mathbb{X}_1,\mathbb{Y}_0)^{-1}\right)\cdot\left(1+\alpha(\mathbb{X}_2,\mathbb{X}_2\oplus\mathbb{X}_3)^{-1}\right)\cdot (\mathbb{A}u,u) + \right.$$
$$\left. + \sum_{\|i\|\le M} \mu_i f_i(u_{3,i})\right] \tag{3.3}$$

for all $u \in \mathbb{Y}_0$ and also the estimate

$$(\mathbb{B}_3 u,u) \le c_2^3\left[\left(1+\alpha(\mathbb{X}_1,\mathbb{Y}_0)^{-1}\right)\cdot (\mathbb{A}u,u) + \sum_{\|i\|\le M} \mu_i f_i(u_{3,i})\right] \tag{3.4}$$

for all $u \in \mathbb{X}_1 \oplus \mathbb{X}_3$.

Let $\alpha(\mathbb{X}_1\oplus\mathbb{X}_2,\mathbb{Y}_0)>0$, then the estimate

$$(\mathbb{B}_2 u,u) \le c_2^2\left[\left(1+\alpha(\mathbb{X}_1\oplus\mathbb{X}_2,\mathbb{Y}_0)^{-1}\right)\cdot(\mathbb{A}u,u) + \sum_{\|i\|\le M} \mu_i f_i(u_{3,i})\right] \tag{3.5}$$

is true for all $u \in \mathbb{Y}_0$ and for $\alpha(\mathbb{X}_1,\mathbb{X}_1\oplus\mathbb{X}_2)>0$ the following expression holds

$$(\mathbb{B}_4 u,u) \le c_2^4 \left(1+\alpha(\mathbb{X}_1,\mathbb{X}_1\oplus\mathbb{X}_2)^{-1}\right)\cdot(\mathbb{A}u,u) \tag{3.6}$$

for all $u \in \mathbb{X}_1\oplus\mathbb{X}_2$. The lower estimate from (2.15) is true, where constants $c_1^n>0$ $n=1,\dots,5$ as well as the constants c_2^k , $k=1,\dots,4$ depend only on the shape of subdomains Ω_i and don't depend on dimension of the space $\mathbb{Y}_0$.

Proof. Assume $u = u_1+ u_2+ u_3$, $u_k\in \mathbb{X}_k$ and use the presentation

$$(\mathbb{A}u,u) = \sum_{\|l\|\le N} \mu_l \mathbb{A}_l(u_1+ u_2+ u_3). \qquad (3.7)$$

According to triangle inequality we obtain

$$(\mathbb{A}u,u) \le c \sum_l \mu_l \left(\sum_{q=1}^{3} \mathbb{A}_l(u_{q,l}) \right)$$

whence again using triangle inequality we find

$$(\mathbb{B}_1 u,u) = \sum_l \mu_l \Big[\mathbb{A}_l(u_{1,l}) + \mathbb{A}_l(u_{2,l}) \Big] + \sum_{l\in I_B} \mu_l \sum_{k=1}^{6} (\mathbf{S}_l^{kk} u_{3,l}^k, u_{3,l}^k) +$$

$$+ \sum_{l\in I_W} \mu_l \sum_{k=1}^{6} (\mathbf{S}_l^{kk}(\mathbb{T} u_{3,l})^k, (\mathbb{T} u_{3,l})^k) \Bigg) \ge c_1^1(\mathbb{A}u,u).$$

The lower estimate for the form $(\mathbb{B}_k u,u)$ for $k=2,\dots,5$ can be performed in a similar way. Consider the upper estimate . According to *Remark 3.1* we have

$$\alpha (\mathbb{X}_2\oplus\mathbb{X}_3, \mathbb{Y}_0) \ge \left(1 + \alpha(\mathbb{X}_1,\mathbb{Y}_0)^{-1}\right)^{-1}$$
$$\alpha (\mathbb{X}_3,\mathbb{X}_2\oplus \mathbb{X}_3) \ge \left(1 + \alpha(\mathbb{X}_2, \mathbb{X}_2\oplus \mathbb{X}_3)^{-1}\right)^{-1} . \qquad (3.8)$$

Therefore, taking into account (3.7) and using inequalities (3.8) we obtain

$$\left(1 + \alpha(\mathbb{X}_1,\mathbb{Y}_0)^{-1}\right)\left(1 + \alpha(\mathbb{X}_2, \mathbb{X}_2\oplus \mathbb{X}_3)^{-1}\right) (\mathbb{A}u,u) \ge$$
$$\ge c \sum_l \mu_l \left(\sum_{q=1}^{3} \mathbb{A}_l(u_{q,l}) \right) . \qquad (3.9)$$

From the other side according to the traces theorem for functions from $\mathbb{H}^1(\Omega_i)$ on the Lipschitz surfaces /32/ we obtain

$$(\mathbb{B}_1 u,u) \le c \sum_l \mu_l \left(\sum_{q=1}^{2} \mathbb{A}_l(u_{q,l})+ \sum_{k=1}^{6} \| u_{3,l}^k \|^2_{\mathring{\mathbb{H}}^{1/2}(\Gamma_l^k)} \right) . \qquad (3.10)$$

Using the results of pasting functions from $\mathbb{H}^r$, $0<r<1$ for two adjacent sides of parallelepiped /32/ we obtain the inequality for $u_{3,i} \in \mathbb{X}_3$

$$\mathbb{A}_i(u_{3,i}) \geq c\| \bar{u}_{3,i} \|^2_{\mathbb{H}^1(\Omega_i)} \geq c\| u_{3,i} \|^2_{\mathbb{H}^{1/2}(\Gamma_i)} \geq$$

$$\geq c \sum_{k=1}^{6} \| u^k_{3,i} \|^2_{\mathbb{H}^{1/2}(\Gamma^k_i)},$$

$$\| u^k_{3,i} \|^2_{\mathring{\mathbb{H}}^{1/2}(\Gamma^k_i)} \leq c \left(\| u^k_{3,i} \|^2_{\mathbb{H}^{1/2}(\Gamma_i)} + f_{i,k}(u^k_{3,i}) \right)$$

for $k=1,\dots,6$. Summing the latter inequality over k we obtain

$$\mu_i \sum_{k=1}^{6} \|u^k_{3,i}\|_{\mathring{\mathbb{H}}^{1/2}(\Gamma^k_i)} \leq c\ \mu_i \left(\|u_{3,i}\|^2_{\mathbb{H}^{1/2}(\Gamma_i)} + \vec{f}_i(u_{3,i}) \right). \qquad (3.11)$$

Summing over i inequalities (3.11) and comparing the result with estimates (3.9), (3.10) we come to inequality (3.3). Inequalities (3.4)-(3.6) can be obtained similarly. The theorem is proved.

Let us formulate as a hypothesis the estimates characterizing some general properties of subspaces $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_3$ which allow one to present results of the Theorem 1 in constructive form:

G1. There is a constant $g_3(\mathbb{V}_0)>0$ such, that for any function $u \in \mathbb{V}_0$ satisfying the condition $u(\xi_0)=0$, for some $\xi_0 \in \Gamma_i$ the inequality holds

$$\| u \|^2_{L^\infty(\Gamma_i)} \leq g_3(\mathbb{V}_0) \int_{\Omega_i} | \nabla \bar{u} |^2 dx$$

G2. For any function $u_i \in \mathbb{X}_3$ the estimate is true

$$\bar{f}_{i,k}(u^k_i) \leq \varepsilon_3(\mathbb{X}_3) \max_{x \in \Gamma^i_k} | u^k_i |^2$$

with a unique constant $\varepsilon_3(\mathbb{X}_3)>0$ for all inner edges Γ^k_i.

G3. There is a constant $\nu(\mathbb{X}_2,\mathbb{X}_3)>0$ such that for any function $u \in \mathbb{X}_2 \oplus \mathbb{X}_3$ the inequality holds

$$\| u_i \|^2_{\mathbb{H}^1(\Gamma_i)} \leq \nu(\mathbb{X}_2,\mathbb{X}_3)\ \| u_i \|^2_{\mathbb{H}^{1/2}(\Gamma_i)}.$$

G4. There is a constant $\eta(\mathbb{X}_2)>0$ such that for any function $u \in \mathbb{X}_1 \oplus \mathbb{X}_2$ having a trace u_i on Γ_i the estimate is true

$$\| u_i \|^2_{\mathbb{H}^1(\Gamma_i)} \le \eta\ (\mathbb{X}_2)\ \| u_i \|^2_{\mathbb{H}^{1/2}(\Gamma_i)}$$

for all subdomains $\Omega_i \subset \Omega$.

Now remind the formulation of hypotheses H1, H2 /1,2/ analogous to G1 and G2 but relating to two-dimensional problems. Denote by u_i^k the trace of arbitrary function $u \in \mathbb{Y}_0$ on the side Γ_i^k and by u_i^{km} the trace of the function $u \in \mathbb{X}_2$ at the edge $\Gamma_i^{km} = \{\ \xi:\ 0 \le \xi \le a\ \}$.

H1. There is a constant $g(\mathbb{Y}_0)>0$, such that for any function u_i^k satisfying the condition $u_i^k(\xi_0) = 0,\ \xi_0 \in \Gamma_i^k$ the inequality holds

$$\| u_i^k \|^2_{L^\infty(\Gamma_i^k)} \le g(\mathbb{Y}_0) \int_{\Gamma_i^k} |\nabla u_i^k|^2 dx\ .$$

H2. There exists a constant $\varepsilon(\mathbb{X}_2)>0$, such that for any function $v(\xi) \equiv u_i^{km}(\xi),\ \xi \in \Gamma_i^{km}$ the inequality holds

$$\int_0^a \frac{v^2(\xi)}{\xi|\xi-a|}\, d\xi \le \varepsilon(\mathbb{X}_2)\ \|v\|^2_{L^\infty(\Gamma_i^{km})}\ .$$

Denote by $d = \max_i\ (diam\ \Omega_i)$. The following Lemmas are true.

Lemma 1. Let the hypothesis G1 be fulfilled, then the estimate is true

$$\alpha(\mathbb{X}_1, \mathbb{Y}_0) \ge c\ \left(1 + d\cdot g_3(\mathbb{Y}_0)\right)^{-1}\ .$$

Lemma 2. Let the hypothesis G3 be fulfilled, then the inequality holds

$$\alpha(\ \mathbb{X}_1 \oplus \mathbb{X}_2, \mathbb{Y}_0) \ge c\ \left(1 + d\cdot \nu(\mathbb{X}_2, \mathbb{X}_3)\right)^{-1}\ .$$

Lemma 3. Let the hypothesis G3 be fulfilled, then the inequality is true

$$\alpha(\ \mathbb{X}_2, \mathbb{X}_2 \oplus \mathbb{X}_3) \ge c\ \left(1 + d\cdot \nu(\mathbb{X}_2, \mathbb{X}_3)\right)^{-1}\ .$$

Lemma 4. Let the hypotheses H1 with the constant $g(\mathbb{X}_2)$ and H2 be fulfilled, then the estimate is true

$$\alpha(\ \mathbb{X}_1, \mathbb{X}_1 \oplus \mathbb{X}_2) \ge c\ \left(1 + g(\mathbb{X}_2)\ \eta(\mathbb{X}_2)\cdot d\ \right)^{-1}.$$

Proofs for Lemmas 1-4 are given in §4. In all Lemmas the constant c depends only on the shape of subdomains Ω_i. From the Theorem 1 and Lemmas 1-4 immediately follows the

Theorem 2. Under hypotheses G1-G4 as well as their two-dimensional analogues H1 and H2 the estimates are true

$$(\mathbb{B}_1 u, u) \le c_2^1 \left[1 + d\cdot g_3(\mathbb{Y}_0)\right]\left(1 + \varepsilon_3(\mathbb{X}_3) + \left(1+\varepsilon(\mathbb{X}_2) g(\mathbb{X}_2\oplus\mathbb{X}_3)\right) *\right.$$

$$\left. \nu(\mathbb{X}_2, \mathbb{X}_3) d \right) (\mathbb{A}u, u),$$

$$(\mathbb{B}_2 u, u) \le c_2^2 \left[1 + d\cdot g_3(\mathbb{Y}_0)\right]\left(1 + \varepsilon_3(\mathbb{X}_3)\right)(\mathbb{A}u, u)$$

for all $u \in \mathbb{Y}_0$ and also the estimate

$$(\mathbb{B}_3 u, u) \le c_2^3 \left[1 + d\cdot g_3(\mathbb{Y}_0)\right]\left(1 + \varepsilon_3(\mathbb{X}_3)\right)(\mathbb{A}u, u)$$

holds for all $u \in \mathbb{X}_1 \oplus \mathbb{X}_3$ and also the estimate

$$(\mathbb{B}_4 u, u) \le c_2^4 \left[1 + d\cdot g_3(\mathbb{Y}_0)\right](\mathbb{A}u, u)$$

is true for all $u \in \mathbb{X}_1 \oplus \mathbb{X}_2$.

The following theorem is also true (see the proof in §4).

Theorem 3. Let the hypotheses H1, H2 and G4 are fulfilled for the space $\mathbb{X}_2$. Then the estimate holds

$$(\mathbb{B}_5 u, u) \le c_2^5 \cdot d\left(1 + g(\mathbb{X}_2)\cdot\varepsilon(\mathbb{X}_2)\right)\cdot\eta(\mathbb{X}_2)(\mathbb{A}u, u)$$

for all $u \in \mathbb{X}_2$.

In §5 we shall obtain using Theorems 2,3 estimates of the condition numbers of operators $\mathbb{B}_k^{-1}\mathbb{A}_0$, $k=1,\dots,5$ via the quantity d/h (here h is a parameter of triangulation of some concrete finite-element subspaces $\mathbb{X}_2$ and $\mathbb{X}_3$) as a consequence of the corresponding estimate of constants g_3, ε_3, ν, η from the hypotheses G1-G4.

§4. Proofs for Lemmas 1-4 and the Theorem 3.

Let us prove Lemma 1. Take any one of subdomains Ω_i. Present the function $u_L \in \mathbb{X}_1$ defined in Ω_i in a form $u_L = u_{L,0} + const$, where $u_{L,0}(x_k) = 0$ and x_k is one of the vertices of Ω_i. Since

$\mathbb{A}_l(u \;\; +u_L) \; = \; \mathbb{A}_l(u_{L,0}+ \;\; u)$, we can assume $u_L \equiv \; u_{L,0}$. Simple calculations bring to inequalities /1,2,18/

$$\mathbb{A}_l(u_L) \le c\cdot d \; \| u_L\|^2_{L^\infty(\Gamma_l)} \le c\cdot d \; \| u_L+u_2+u_3\|^2_{L^\infty(\Gamma_l)} \; ,$$

where $u_2 \in \mathbb{X}_2$ and $u_3 \in \mathbb{X}_3$ are some arbitrary elements. Now the necessary statement follows from the hypothesis G1.

Similarly for proving Lemma 4 choose such $u_L \in \mathbb{X}_l$, for which $u_L(\xi_0)=0$ with ξ_0 being the vertex of Ω_l. Again we have (for arbitrary $u_2 \in \mathbb{X}_2$) the inequality

$$\mathbb{A}_l(u_L) \le c\cdot d \; \| u_L+ \, u_2\|^2_{L^\infty(\Gamma_l)} \le c\cdot d \; g(\mathbb{X}_2) \; \|\nabla(u_L+ \, u_2)\|^2_{L_2(\Gamma_l)} \le$$

$$\le c\cdot d \; g(\mathbb{X}_2)\eta(\mathbb{X}_2)\|u_L+ \, u_2\|^2_{\mathbb{H}^{1/2}(\Gamma_l)} \le c\cdot d \; g(\mathbb{X}_2)\eta(\mathbb{X}_2)\cdot\mathbb{A}_l(u_L+ \, u_2)$$

where the latter inequality is a consequence from the Poincare inequality and the traces inequality for functions from $\mathbb{H}^1(\Omega_l)$ on the Lipschitz surfaces /32/ for the case of harmonic continuation inside Ω_l. Lemma 4 is proved.

To prove Lemma 2 use at first the traces inequality for functions from $\mathbb{H}^1(\Omega_l)$ /32/

$$\|u_L+ \, u_2\|^2_{\mathbb{H}^1(\Omega_l)} \le c \;\; \|u_L+ \, u_2\|^2_{\mathbb{H}^{1/2}(\Gamma_l)}$$

then it's easy to obtain the following estimate for every pair of adjacent sides Γ_l^k and Γ_l^m (denote by $G = \Gamma_l^k \cup \Gamma_l^m$)

$$\|u_L+ \, u_2\|^2_{\mathbb{H}^{1/2}(G)} \le c\cdot d \;\; \|u_L+ \, u_2\|^2_{\mathbb{H}^{1/2}(\partial G)} \le$$

$$\le c\cdot d \;\; \|u_L+ \, u_2+ \, u_3\|_{\mathbb{H}^{1/2}(\partial G)} \le c\cdot d \;\; \|u_L+ \, u_2+ \, u_3\|^2_{\mathbb{H}^1(G)}$$

for all $u_3 \in \mathbb{X}_3$. Summing the latter inequality over all adjacent pairs $G \subset \Gamma_l$ we obtain

$$\|u_L+ \, u_2\|^2_{\mathbb{H}^1(\Omega_l)} \le c\cdot d \;\; \|u_L+ \, u_2+ \, u_3\|^2_{\mathbb{H}^1(\Gamma_l)} \le$$

$$\le c\cdot d\ \nu(\mathbb{X}_2,\mathbb{X}_3)\|u_L+u_2+u_3\|^2_{\mathbb{H}^{1/2}(\Gamma_i)} \le c\cdot d\ \nu(\mathbb{X}_2,\mathbb{X}_3)\mathbb{A}_i(u_L+u_2+u_3)$$

Q.E.D.

The proof of Lemma 3 is exactly analogous to that for Lemma 2.

Now let us prove the Theorem 3. Let $u\in\mathbb{X}_2$. Designate the trace of u on Γ_i by u_i. Then the inequality holds

$$(\mathbb{A}u,u) = \sum_i \mu_i\mathbb{A}_i(u_i) \ge \sum_i \mu_i\|u_i\|^2_{\mathbb{H}^{1/2}(\Gamma_i)} . \qquad (4.1)$$

Let $l_i^q = \{\ \xi:\ 0\le\xi\le a_i^q\ \}$, $q=1,\dots,12$ are the edges of the boundary Γ_i and functions $v_i^q(\xi)$ are generatrix functions at these edges corresponding to the given function u_i. Denote by $u_i^q(x)$, $x\in\Gamma_i$ the continuation of the function v_i^q on Γ_i (according to Definition 1 this continuation differs from zero only on two adjacent sides having common edge l_i^q). It is easy to obtain the inequality

$$(\mathbb{B}_5u,u) \le \sum_i \mu_i\left(\sum_{q=1}^{12}\|u_i^q\|^2_{\mathbb{H}^{1/2}(\Gamma_i)}\right) . \qquad (4.2)$$

From (4.1) and (4.2) it follows that it is sufficient to determine the estimate of the Theorem 5 only for one subdomain Ω_i and for an arbitrary function u_i being a trace of some $u\in\mathbb{X}_2$ on Γ_i. Let u_i^k be a trace of the function u_i on the side Γ_i^k, $k=1,\dots,6$. Continue (4.1) for one subdomain Ω_i using the hypothesis G4:

$$\|u_i\|^2_{\mathbb{H}^{1/2}(\Gamma_i)} \ge \frac{c}{\eta(\mathbb{X}_2)}\|u_i\|^2_{\mathbb{H}^{1}(\Gamma_i)} \ge \qquad (4.3)$$

$$\ge \frac{c}{\eta(\mathbb{X}_2)}\sum_{k=1}^{6}\|u_i^k|_{\partial\Gamma_i^k}\|^2_{\mathbb{H}^{1/2}(\partial\Gamma_i^k)} \ge \frac{c}{\eta(\mathbb{X}_2)}\sum_{q=1}^{12}\|v_i^q\|^2_{\mathbb{H}^{1/2}(l_i^q)} .$$

Continue the estimate of the summand from (4.2) with the index i at fixed q. Let Γ_i^k and Γ_i^m are sides having a common edge l_i^q. Direct computations using Definition 2.1 lead to the estimate

$$\| u_i^q \|^2_{\mathbb{H}^{1/2}(\Gamma_i)} = \| u_i^q \|^2_{\mathring{\mathbb{H}}^{1/2}(\Gamma_i^k \cup \Gamma_i^m)} \leq \Sigma_1 + \Sigma_2 ,$$

where

$$\Sigma_1 = (1/d) \int_{\Gamma_i^k} (u_i^k)^2 dx, \qquad \Sigma_2 = d \int_{l_i^q} v_i^q(\xi) \left. \frac{\partial u^k}{\partial n} \right|_{l_i^q} d\xi ,$$

whereas at the side Γ_i^k the function u_i^k is harmonic, while on the side Γ_i^m the function u_i^m is a linear continuation of $v_i^q(\xi)$. Here $\left. \frac{\partial u^k}{\partial n} \right|_{l_i^q}$ is a normal derivative of harmonic function $u_i^k(x)$, $x \in \Gamma_i^k$ on the part of the boundary $l_i^q \in \partial\Gamma_i^k$. Using the Poincare inequality we obtain the estimate

$$\Sigma_1 \leq c \cdot d \, \| \nabla u_i^k \|^2_{L_2(\Gamma_i^k)} \leq c \cdot d \, \| v_i^q \|^2_{\mathring{\mathbb{H}}^{\circ 1/2}(l_i^q)} .$$

According to the Green formula we obtain for the quantity Σ_2 analogous estimate

$$\Sigma_2 \leq c \cdot d \, \| v_i^q \|^2_{\mathring{\mathbb{H}}^{\circ 1/2}(l_i^q)} .$$

As a result the inequality holds

$$\sum_{q=1}^{12} \| u_i^q \|^2_{\mathbb{H}^{1/2}(\Gamma_i)} \leq c \cdot d \sum_{q=1}^{12} \| v_i^q \|^2_{\mathring{\mathbb{H}}^{\circ 1/2}(l_i^q)} , \tag{4.4}$$

which must be compared with (4.3). For this purpose use hypotheses H1, H2 and results of pasting functions from $H^r(\gamma)$ (where γ is a combination of two segments /32/); as a result we obtain estimates

$$\sum_{q=1}^{12} \| v_i^q \|^2_{\mathring{\mathbb{H}}^{\circ 1/2}(l_i^q)} \leq c \sum_{q=1}^{12} \Big(\| v_i^q \|^2_{\mathbb{H}^{1/2}(l_i^q)} + \tag{4.5}$$

$$+ \int_0^{a_i^q} \frac{[v_i^q(\xi)]^2}{\xi|\xi-a_i^q|}\,d\xi \Bigg) ,$$

$$\sum_{q=1}^{12} \int_0^{a_i^q} \frac{[v_i^q(\xi)]^2}{\xi|\xi-a_i^q|}\,d\xi \le c\cdot\varepsilon(\mathbb{X}_2)\cdot g(\mathbb{X}_2) \sum_{k=1}^{6} \int_{\Gamma_i^k} |\nabla u_i^k|^2 dx \le$$

$$\le c\cdot\varepsilon(\mathbb{X}_2)\cdot g(\mathbb{X}_2)\cdot\eta(\mathbb{X}_2)\| u_i\|^2_{\mathbb{H}^{1/2}(\Gamma_i)} . \qquad (4.6)$$

Comparing estimates (4.5), (4.6) with (4.3) we come to inequality for one subdomain Ω_i :

$$(\mathbb{B}_5 u,u) \le c\cdot d\cdot\Bigg(\sum_{q=1}^{12} \| v_i^q \|^2_{\mathbb{H}^{1/2}(l_i^q)} + \varepsilon(\mathbb{X}_2)\cdot g(\mathbb{X}_2)\cdot\eta(\mathbb{X}_2)\cdot$$

$$\cdot\| u_i\|^2_{\mathbb{H}^{1/2}(\Gamma_i)}\Bigg) \le c\cdot d \Big(1 + \varepsilon(\mathbb{X}_2)\cdot g(\mathbb{X}_2)\Big)\cdot\eta(\mathbb{X}_2)\| u_i\|^2_{\mathbb{H}^{1/2}(\Gamma_i)}$$

Q.E.D.

§5 Estimates of condition numbers for operators $\mathbb{B}_k^{-1}\mathbb{A}_0$ for some classes of finite element subspaces

The choice of the space $\mathbb{X}_2$ is defined by the components $\mathbb{X}_i^{km}$. Consider the subspace of piecewise linear functions on Γ_i^{km} with zeroes at the vertices of the side Γ_i^k. We partition the edge Γ_i^{km} into $n_{i,k}+1$ segments Δ_j, $j=0,\dots,n_{i,k}$ so that for all i,k there exist constants c_o, c_1 which don't depend on $h_2>0$ and $c_0 h_2 \le |\Delta_j| \le c_1 h_2$ for all j and for all edges. Then define

$$\mathbb{X}_{i,h}^{km} = \{ u \in \mathbb{C}(\bar{\Gamma}_i^{km}) : u|_{\Delta_j} \in \mathbb{P}_1(x) ; u(x) = 0, x\in\partial(\Gamma_i^{km})\}, \qquad (5.1)$$

where $\mathbb{P}_1(x)$ is a set of linear polynomials. Spaces $\bar{\mathbb{X}}_{i,h}^{km}$ are constructed according to Definition 2.1

Now consider the space $\mathbb{X}_3$. We construct a regular triangulation /34/ $\Gamma_{i,h}^k(\mathbb{X}_3)$ with the step $h_3=h_3(i,k)$ and elements e_j at every inner side Γ_i^k. We take as $\mathbb{G}(\Gamma_i^k)$

the finite element space

$$\mathbb{V}^k_{i,h}(\Gamma^k_i) = \{ v \in \mathbb{C}(\bar{\Gamma}^k_i):\ v|_{e_j} \in \mathbb{P}_1(x)\ ;\ v(x) = 0,\ x\in\partial(\Gamma^k_i)\}, \quad (5.2)$$

where $\mathbb{P}_1(x)$ is a set of linear polynomials. Along with $\mathbb{V}^k_{i,h}$ one can use the space $\mathbb{U}^k_{i,h}$ constructed on the basis of "serendipity" elements f_j, so that

$$\mathbb{U}^k_{i,h}(\Gamma^k_i) = \{ v \in \mathbb{C}(\bar{\Gamma}^k_i):\ v|_{\partial f_j} \in \mathbb{P}_1(x)\ ;\ \Delta v = 0,\ x \in f_j; \quad (5.3)$$
$$v(x) = 0,\ x\in\partial(\Gamma^k_i)\} \subset \mathring{H}^{1/2}(\Gamma^k_i).$$

Suppose that partitioning of edges Γ^{km}_i (for the space $\mathbb{X}_2$) and triangulation of sides (for the space $\mathbb{X}_3$) are in agreement.

According to Definitions 2.1 and 2.2 functions from spaces $\mathbb{X}_2$ and $\mathbb{X}_3$ are harmonic inside subdomains Ω_i. Besides, traces of functions from $\bar{\mathbb{X}}^{km}_i$ on Γ_i corresponding to generatrix function belonging to $\mathbb{X}^{km}_i$ are also harmonic functions on one of two sides Γ^k_i belonging to domain Ω_i and having a common edge Γ^{km}_i.

Then consider spaces $\mathbb{X}_{2,h}$ and $\mathbb{X}_{3,h}$ being "h-harmonic" analogues of spaces $\mathbb{X}_2$ and $\mathbb{X}_3$, correspondingly, for which the harmonic components in subdomains are substituted by "h-harmonic" ones. For this purpose construct regular triangulation $\Gamma^k_{i,h}(\mathbb{X}_2)$ (with the step h_2) of the side adjacent to the edge Γ^{km}_i (coordinated with decomposition of the edge Γ^{km}_i) and let's consider regular triangulations $\Omega_{i,h}(\mathbb{X}_2)$ (with the step h_2) and $\Omega_{i,h}(\mathbb{X}_3)$ (with the step h_3) in the domain Ω_i coordinated with decompositions $\Gamma^k_{i,h}(\mathbb{X}_2)$ and $\Gamma^k_{i,h}(\mathbb{X}_3)$ for each of spaces $\mathbb{X}_2$ and $\mathbb{X}_3$. In the case of the space $\mathbb{X}_3$ (similarly for $\mathbb{X}_2$) consider the set of the first order finite elements $\mathbb{V}_h(\Omega_{i,h}(\mathbb{X}_3))$ defined on $\Omega_{i,h}(\mathbb{X}_3)$ in the same way as (5.2) or (5.3). We suppose that the function $u_h\in \mathbb{X}_{3,h}$ if the following inequality holds in each of subdomains Ω_i:

$$\int_{\Omega_i} (\nabla u_h, \nabla z)dx = 0 \ , \ u_h(x) = u(x), \ x \in \Gamma_i \qquad (5.4)$$

for all $z \in \mathbb{V}_h(\Omega_{i,h}(\mathbb{X}_3))$, where $u(x) \in \mathbb{V}^k_{i,h}(\Gamma^k_i)$, $k=1,\ldots,6$ (or $u \in \mathbb{U}^k_{i,h}(\Gamma^k_i)$).

According to /34/ (see also /1/) the estimate is true

$$\alpha_1 \cdot \| \bar{u} \|_{\mathbb{H}^1(\Omega_i)} \le \| u_h \|_{\mathbb{H}^1(\Omega_i)} \le \alpha_2 \cdot \| \bar{u} \|_{\mathbb{H}^1(\Omega_i)} \qquad (5.5)$$

where α_1, α_2 don't depend on h_3 and $\bar{u}$ is a harmonic function in Ω_i, such that $\bar{u}(x)=u(x)$, $x \in \Gamma_i$. Analogous statement is true for elements from $\mathbb{X}_{2,h}$. Because of (5.5) we perform all further estimates for elements from $\mathbb{X}_{2,h}$ and $\mathbb{X}_{3,h}$.

Designate the "h-harmonic" continuation of the function $v \in \mathbb{X}^{km}_{i,h}$ on the adjacent edge Γ^k_i by $\bar{v}_h$ ($\bar{v}_h$ is defined from the equation of the type (5.4) on the edge Γ^k_i).

It is sufficient to obtain estimates of the constants g_3, ε_3, ν, η from the hypothesises G1-G4 for spaces $\mathbb{X}_{2,h}$, $\mathbb{X}_{3,h}$ only for some one-subdomain $\Omega = \Omega_i$. Use notation $N_2=[d/h_2]$, $N_3=[d/h_3]$, $h_0= \min(h_2,h_3)$, $N = \max(N_2,N_3)$.

Lemma 4. The following estimates are true

$$\nu(\mathbb{X}_{2,h}, \mathbb{X}_{3,h}) \le c \cdot h_0^{-1} \qquad (5.6)$$

$$\eta(\mathbb{X}_{2,h}) \le c \cdot h_2^{-1} \qquad (5.7)$$

Estimates (5.6) and (5.7) are a particualar case of a more general property of inverse assumption /39/ for the spaces of conform finite elements $\mathbb{S}^{t,r}_h(\mathbb{G}) \subset \mathbb{H}^r(\mathbb{G})$, $t > r \ge 0$, which include also our constructed spaces $\mathbb{V}^k_{i,h}$, $\mathbb{U}^k_{i,h}$, ($\bar{v}_h$, $v \in \mathbb{X}^{km}_{i,h}$) (defined on sides) and spaces $\mathbb{X}^{km}_{i,h}$ (defined on edges). Let's recall the definition. The space $\mathbb{S}^{t,r}_h(\mathbb{G})$ satisfies the inverse assumption if there exists $\varepsilon>0$, $r \ge \varepsilon > 0$ such that for any s, $r-\varepsilon \le s \le r$ and for any $\varphi \in \mathbb{S}^{t,r}_h(\mathbb{G})$ the estimate holds

$$\| \varphi \|_{\mathbb{H}^r(\mathbb{G})} \le c \cdot h^{s-r} \| \varphi \|_{\mathbb{H}^s(\mathbb{G})} \tag{5.8}$$

Remark 5.1. Lemma 4 is a consequence of the inverse assumption in a form (5.8) at $r=1$, $s=1/2$, $\varepsilon \ge 1/2$ with one of the sides Γ_i^k chosen as $\mathbb{G}$ and with the estimate /32/

$$\sum_{k=1}^{6} \| u_i^k \|^2_{\mathbb{H}^{1/2}(\Gamma_i^k)} \le c \cdot \| u_i \|^2_{\mathbb{H}^{1/2}(\Gamma_i)} .$$

To prove the inverse assumption we use the following auxiliary statement.

Lemma 5. Let there exist $\varepsilon > 0$, such that for any $\varphi \in \mathbb{S}_h^{t,k}(\mathbb{G})$ it is true:

$$\| \varphi \|_{\mathbb{H}^k(\mathbb{G})} \le c \cdot h^{-\varepsilon} \| \varphi \|_{\mathbb{H}^{k-\varepsilon}(\mathbb{G})} .$$

Then for any $0 \le z \le \varepsilon$ we have

$$\| \varphi \|_{\mathbb{H}^k(\mathbb{G})} \le c \cdot h^{z-\varepsilon} \| \varphi \|_{\mathbb{H}^{k-\varepsilon+z}(\mathbb{G})} \tag{5.9}$$

Proof. We use the scale of Hilbert spaces which connects spaces $\mathbb{H}^k$ and $\mathbb{H}^{k-\varepsilon}$ and consider the representation

$$\varphi = \sum_{k=1}^{\infty} b_k \cdot g_k$$

in eigenbasis of positively defined self-adjoint operator $\mathbb{V}$: $\mathbb{H}^k \Rightarrow \mathbb{H}^{k-\varepsilon}$, which is besides compact in $\mathbb{H}^{k-\varepsilon}$ so that $\mathbb{V} \cdot g_k = \lambda_k g_k$, $\lambda_{k+1} \le \lambda_k$, $0 < \lambda_k \Rightarrow 0$ at $k \Rightarrow \infty$. Under condition of Lemma 5 for any $\varphi \in \mathbb{S}_h^{t,k}(\mathbb{G})$ we have

$$\sum_{k=1}^{\infty} |b_k|^2 \cdot \lambda_k^{-2} \le c \cdot h^{-2\varepsilon} \sum_{k=1}^{\infty} |b_k|^2 . \tag{5.10}$$

Our aim is to obtain the estimate (at $0 \le \vartheta \le 1$)

$$\sum_{k=1}^{\infty} |b_k|^2 \cdot \lambda_k^{-2} \le c \cdot h^{-2\varepsilon\vartheta} \sum_{k=1}^{\infty} |b_k|^2 \lambda_k^{-2\vartheta}, \tag{5.11}$$

which is equivalent to (5.9). Let in $\mathbb{S}_h^{t,k}$ some basis $\{\varphi_m\}$, $m = 1, \ldots, L$ is chosen. Then any function φ can be presented as

$$\varphi = \sum_{k=1}^{L} b'_k \cdot g'_k \ ,$$

where elements g'_k satisfy the equality $(\mathbb{V}g'_k,\ \varphi_m) = \bar{\lambda}_k(g'_k,\ \varphi_m)$, at $m = 1,\dots,\ L$, $\bar{\lambda}_{k+1} \le \bar{\lambda}_k$, $\bar{\lambda}_k \ge \bar{\lambda}_L > 0$, $k=1,\dots,L$. Therefore inequalities (5.10), (5.11) take form

$$\sum_{k=1}^{L} |b'_k|^2 \cdot \bar{\lambda}_k^{\ -2} \le c \cdot h^{-2\varepsilon} \sum_{k=1}^{L} |b'_k|^2, \tag{5.12}$$

$$\sum_{k=1}^{L} |b'_k|^2 \cdot \bar{\lambda}_k^{\ -2} \le c \cdot h^{-2\varepsilon\vartheta} \sum_{k=1}^{L} |b'_k|^2\ \bar{\lambda}_k^{\ -2\vartheta}. \tag{5.13}$$

From (5.12) immediately follows the estimate

$$\bar{\lambda}_L^{\ -1} \le c \cdot h^{-\varepsilon} \tag{5.14}$$

which gives rise to inequality

$$\sum_{k=1}^{L} |b'_k|^2 \cdot \bar{\lambda}_k^{\ -2} \le \bar{\lambda}_L^{\ 2(\vartheta-1)} \cdot \sum_{k=1}^{L} |b'_k|^2\ \bar{\lambda}_k^{\ -2\vartheta} \le$$

$$\le \ c \cdot h^{-2\varepsilon(1-\vartheta)} \cdot \left(\sum_{k=1}^{L} |b'_k|^2\ \bar{\lambda}_k^{\ -2\vartheta} \right). \tag{5.15}$$

Then assume $\vartheta = (\gamma - l_1)/(l_2 - l_1) \equiv (\gamma - k + \varepsilon)/\varepsilon$, where $l_2 = k$, $l_1 = k - \varepsilon$, $\gamma = \vartheta\varepsilon + k - \varepsilon$. In this case we obtain $1 - \vartheta = \varepsilon^{-1}(k - \gamma)$, or $\varepsilon(1-\vartheta) = k - \gamma$. Now (5.15) takes form

$$\| \varphi \|^2_{\mathbb{H}^k} \le c \cdot h^{2(z-\varepsilon)} \| \varphi \|^2_{\mathbb{H}^\gamma}$$

where it is supposed $\gamma = k - \varepsilon + z$. Lemma 5 is proved.

Now it is easy to verify the fulfillment of Lemma 5 condition for each of the spaces $\mathbb{V}^1_{1,h}$, $\mathbb{U}^1_{1,h}$, $\{\bar{v}_h,\ v \in \mathbb{X}^{1m}_{1,h}\}$ and $\mathbb{X}^{1m}_{1,h}$ at $k=1$, $\varepsilon=1$, for $h=h_2$ or $h=h_3$, where under $\mathbb{H}^0(\mathbb{G})$ the space $\mathbb{L}_2(\mathbb{G})$ is supposed. Using Lemma 5 at $k=1$, $\varepsilon=1$, $z=1/2$ and taking into account Remark 5.1 we obtain estimates of Lemma 4.

Remark 5.2. According to /1,2/ for the constants $g(\mathbb{Y}_0)$ and $\varepsilon(\mathbb{X}_2)$ from the hypotheses H1 and H2 the estimates are true

$$g(\mathbb{Y}_0) \le c \cdot \left(1 + ln(d/h_0) \right) \tag{5.16}$$

$$\varepsilon(\mathbb{X}_2) \le c\cdot\left(1 + \ln(d/h_2)\right)\cdot \qquad (5.17)$$

Lemma 6. The following estimate is true

$$\varepsilon_3(\mathbb{X}_3) \le c\cdot\left(1 + \ln(d/h_3)\right)\cdot \qquad (5.18)$$

The estimate (5.18) is obtained exactly in a similar way as (5.17) (see /1/).

Lemma 7. The estimate is true

$$g_3(\mathbb{Y}_0) \le c\cdot\left(1 + \ln(d/h_0)\right)\cdot h_0^{-1}. \qquad (5.19)$$

Proof. Use the hypothesis H1 with estimate (5.16) and also the hypothesis G3 with the estimate (5.6). We obtain inequalities

$$\| u_l \|^2_{L^\infty(\Gamma_l)} \le c\cdot\left(1 + \ln(d/h_0)\right) \sum_{k=1}^{6} \int_{\Gamma_l^k} |\nabla u_l^k|^2 dx \le$$

$$\le c\cdot\left(1 + \ln(d/h_0)\right)\cdot h_0^{-1} \| u_l \|^2_{\mathbb{H}^{1/2}(\Gamma_l)} \le$$

$$\le c\cdot\left(1 + \ln(d/h_0)\right)\cdot h_0^{-1}\cdot \int_{\Omega_l} |\nabla \bar{u}_l|^2 dx .$$

The Lemma is proved.

Remark 5.3. The estimate of the type (5.16) for the spaces of piecewise linear elements in the two-dimensional case is obtained in /35,18/. Lemma 7 is a generalization of Lemma 8 from /1,p.II/ for the three-dimensional case, which results in appearence of auxiliary coefficient $1/h_0$ (Lemma 8 /1/ as it was formulated in /1/ is true only for the two-dimensional case).

Denote by $\kappa_k = \kappa\ (\mathbb{B}_k^{-1}\mathbb{A}_0)$ the condition number of the operator $\mathbb{B}_k^{-1}\mathbb{A}_0$. Using Theorems 2,3 , Lemmas 4,6,7 and the Remark 5.2 it is easy to obtain

Corollary 1. For the space $\mathbb{Y}_0 = \mathbb{X}_1 \oplus \mathbb{X}_{2,h} \oplus \mathbb{X}_{3,h}$ the following estimates are true

$$\kappa_1 \le c\cdot(1 + N\cdot \ln N)\cdot \left[(1 + \ln^2 N)\cdot N\right] \qquad (5.20)$$

$$\kappa_i \le c\cdot(1 + N\cdot \ln N)\cdot(1 + \ln N_3)\ , \quad i=2,3 \qquad (5.21)$$

$$\kappa_4 \le c\cdot(1 + N\cdot \ln N) \qquad (5.22)$$

$$\kappa_5 \le c\cdot(1 + \ln N\cdot \ln N_2)\cdot N_2 \qquad (5.23)$$

§6 About operators spectrally equivalent to $\mathbb{A}_o$.

The most laborious part of calculations in solving the equation (2.4), for example, by the PCG method as well as in using the operator $\mathbb{A}_o$ as a preconditioner for solving equations of a more general form is computation of the residual vector $\mathbb{A}_o v$. Therefore operators $\overline{\mathbb{A}}_o$ spectrally equivalent to $\mathbb{A}_o$ in two- and three-dimensional cases are of interest for which calculation of the residual vector $\overline{\mathbb{A}}_o v$ can be made by faster methods. Examples of using such operators $\overline{\mathbb{A}}_o$ are given in §7.

Consider one class of such operators $\overline{\mathbb{A}}_o$. Let for the two-dimensional case all Ω_i be the quadrangles with the edge 1. Consider a set of circles G_i with the radius $R=1/2$ inscribed in Ω_i. Every circle $\partial\Omega_i$ is divided into four equal arcs (requirement of equality for arcs is not necessary) with the middle points of the arcs coinciding with the tangent points of adjacent circles. We call adjacent arcs the pairs of tangent arcs having common points. Let for all i $\mu_i>0$ are given. Consider the following

Problem ($\overline{\mathbb{A}}_o$): Find harmonic functions u_i in the subdomains G_i (for all i) such that the trace functions u_i and u_j (at the boundaries of each pair of tangent circles G_i and G_j) coincide at the pair of adjacent arcs and their normal derivatives at these arcs are bound by the relation

$$\mu_i \frac{\partial u_i}{\partial n}(\xi) + \mu_j \frac{\partial u_j}{\partial n}(\bar{\xi}) = \Psi(\xi), \quad \xi \in \partial G_i,\ \bar{\xi} \in \partial G_j\ ,$$

where $\bar{\xi}$ is the point symmetric to ξ with respect to the common tangent. In addition, for the arcs corresponding to the external boundaries of the quadrangles $u_i=0$ is true.

The problem $(\bar{\mathbb{A}}_o)$, apparently, has no physical analogues (the domain of definition of functions u_i is not simply connected), though it's a good mathematical model for constructing easily invertible preconditioners for more complex problems (where subdomains Ω_i are not rectangles, there are variable coefficients in Ω_i, quasi-linear operators in Ω_i etc.) It is convenient to consider the generalized (weak) formulation of the problem $(\bar{\mathbb{A}}_o)$ after its transformation to the boundary equation (of the type (1.4) with the operator $\mathbb{A}$) of the domain decomposition method, which for the two-dimensional case is thoroughly considered in /1,2/. We denote by $\mathbb{A}_{cir}$ the corresponding boundary operator defined on some direct sum of spaces $\mathbb{H}^{1/2}(\partial G_i)$. Note that the condition $R_i=1/2$ is not necessary, values of R_i can be chosen according to some optimization considerations.

Consider one-to-one bi-Lipschitz mapping $\gamma : \partial\Omega_i \Rightarrow \partial\mathbb{G}_i$ (for example, by means of the projection from the centre of the circles G_i) which associates every function $u(\eta) \in \mathbb{H}^{1/2}(\partial\Omega_i)$, $\eta \in \partial\Omega_i$ with the function $u_i(\xi) = u(\eta)$, $\xi = \gamma(\eta) \in \partial G_i$, $u_i(\xi) \in \mathbb{H}^{1/2}(\partial G_i)$.

Proposition 1. Operators $\mathbb{A}$ (of the type (1.4)) and $\mathbb{A}_{cir}$ are spectrally equivalent in the following sense

$$c_2\Big(\mathbb{A}_{cir} u_i(\gamma(\eta)),\ u_i(\gamma(\eta))\Big) \le \Big(\mathbb{A}\ u(\eta),\ u(\eta)\Big) \le$$
$$\le c_1 \cdot \Big(\mathbb{A}_{cir} u_i(\gamma(\eta)),\ u_i(\gamma(\eta))\Big)$$

for all $u(\eta) \in \mathbb{H}^{1/2}(\partial\Omega_i)$, where constants c_1, $c_2 > 0$ depend only on the ratio $R_i/1$.

Proposition 1 is a consequence of results in /36/ concerning the stability of Sobolev's spaces on the Lipschitz surfaces at smooth replacement of variables.

Exactly analogous construction of operators spectrally equivalent to operator $\mathbb{A}$ (of the type(1.4)) can be performed in the three-dimensional case if one chooses as G_l inscribed spheres with the radius $R_l=l/2$, where l is an edge of the cube (domain Ω_l). Denote these operators by $\mathbb{A}_{sph}$. One can use as G_l also cylinders with axes parallel to one of coordinate directions (operator $\mathbb{A}_{cyl}$). In turn, these operators $\mathbb{A}_{cir}$, $\mathbb{A}_{sph}$, $\mathbb{A}_{cyl}$ will be spectrally equivalent to operators $\mathbb{A}$ (of the type (1.4)) constructed for such subdomains Ω_l as rectangles, parallelepipeds, convex quadrangles or prisms. We call the proposed method of constructing operators $\mathbb{A}_{cir}$, $\mathbb{A}_{sph}$, $\mathbb{A}_{cyl}$ the method of variating substructures (VS-method). More detailed consideration of these operators will be performed in a separate paper.

§7 Discussion of results

Let us estimate the computational work (for example, for the PCG method) necessary for solving the problem (2.4) for different combinations of preconditioners $\mathbb{B}_k$, $k = 1,\dots,5$, as well as for similar constructions in the two-dimensional case /1,2/.

According to /2/ for the two-dimensional problems (with rectangular Ω_l) the condition number of the operator $\mathbb{B}^{-1}\mathbb{A}$ is $O(\ln^2 N)$, where $N=O(d/h)$ is the maximum (over all Ω_l) number of unknowns for one variable, and $p^2 = p_x \cdot p_y$ is the overall number of subdomains Ω_l. For the three-dimensional problem the number N is defined in a similar way. Suppose that $N = 2^m$ and the necessary accuracy of computations is $\varepsilon = N^{-\nu}$, $\nu>0$. If we use the algorithm proposed in /37/ (for $N\le32$ the method proposed in /38/ is one of the most effective) for solving the partial

Dirichlet problem in subdomains, then the solution of the two-dimensional problem by the PCG method will require

$$Q_{2,c} = O\ (\ p^2N\ [log_2N]^4\ ) \tag{7.1}$$

operations for defining the solution on the cutting lines. Computing of the solution at the inner points of Ω_i will require auxiliary $O(N_o^2)$ operations , where $N_o^2 = N^2 \cdot p^2$ is the total number of variables. The estimate of the type (7.1) is also true for the similar algorithms in the case of finite-difference schemes /3/. For the operator $\tilde{\mathbb{A}}$ with variable (but "weakly" changing inside Ω_i) coefficients in subdomains (Ω_i may be convex quadrangles) the quantity $Q_{2,v}$ for the preconditioner $\mathbb{B}$ /2/ will require already

$$Q_{2,v} = O\ (\ p^2N^2[log_2N]^3\ ) \tag{7.2}$$

operations.

Estimates of the condition number for the operators $\mathbb{B}_n^{-1}\mathbb{A}_o$ (see Corollary 1 § 5) for the subspaces $\mathbb{X}_{2,h}$ and $\mathbb{X}_{3,h}$ show that the most cost-effective version of solving the equation (2.4) is the two-level iteration process PCG, on the first level of which the preconditioner $\mathbb{B}_2$ is used (with the condition number $O(N\ lnN\ lnN_3)$), while for solving the equation in the subspace $\mathbb{X}_1 \oplus \mathbb{X}_{2,h}$ - the preconditioner $\mathbb{B}_4$ ($\kappa_4 = O\ (\ N\ lnN\ )$). Assuming $N_2 = N_3 = N$ we obtain that solution of (2.4) at the inner boundaries (with accuracy $\varepsilon = N^{-\nu}$) will require

$$Q_{3,c} = O(N ln^{3/2}N \cdot Q_{12}(N,p) + N^{1/2} lnN\ (p^3N^2 ln^2N)) lnN \tag{7.3}$$

operations, where p^3 is the overall number of substructures and $Q_{12}(N,p)$ is labour required for one step of iterations in solving the problem on $\mathbb{X}_1 \oplus \mathbb{X}_{2,h}$ which is estimated as $Q_{12} \le O\ (p^{7/2} + 3p^3 N lnN)$. Computing the solution at the inner points of Ω_i will require auxiliary $O(N^3p^3)$ operations.

Let us consider the problem of the type (2.4) with the operator $\tilde{\mathbb{A}}$ of a more complicated form, but "closer" in spectrum to operator $\mathbb{A}_o$. For example, $\tilde{\mathbb{A}}$ corresponds to the equation with variable coefficients inside Ω_i or corresponds to partitioning to subdomains (topologically equivalent to the initial subdivision) in which Ω_i are convex quadranles (for R^2) or quadrangular prisms etc. The solution of the problem is accomplished with the help of "boundary", as well as with "inner" unknowns in the subdomains. In this case computation of the residual vector $\tilde{\mathbb{A}}v$ assumes computation of the solution in all inner points of the subdomains and accounts not less than $O\,(N^3p^3 lnN)$ operations. Therefore for solving the equation with the operator $\tilde{\mathbb{A}}$ it is advisable to use at once the preconditioner $\mathbb{A}_o$ or the operator of the type $\mathbb{A}_{vs}=(\mathbb{A}_{cir}, \mathbb{A}_{sph}, \mathbb{A}_{cyl})$, considered in § 6. For operators of the type $\mathbb{A}_{vs}$ one can use fast methods of computing the residual operator $\mathbb{A}_{vs}v$ (the Poincare-Steklov operator) on the circle, sphere or on the cylinder surface. For example, in the two-dimensional case for converting the operator $\mathbb{A}_{cir}$ ($4N$ unknowns on every circle ∂G_i) only a double FFT transformation is needed (i.e. $C_F 8Nlog_2 4N$ operations, where $C_F\in[2,3]$) to compute the residual operator on the boundary of one subdomain ∂G_i by direct method, whereas for the similar grid quadrangle $C_s 8Nlog_2^2 N$ operations are required (by the method proposed in /37/) with sufficiently large constant C_s. Apparently, the operator $\mathbb{A}_{cir}$ is the most effective preconditioner in solving the equations by the domain decomposition method with the checkerboard subdivision for not very "stretched" subdomains.

Note that for conversion of the operators $\mathbb{A}_{vs}$ the preconditioners $\mathbb{B}_{k,vs}$ must be used in which the block-diagonal part of the operator $\mathbb{A}_{vs}$ is substituted by the spectral equivalent one (as in /18/) of the type $(\alpha_i\mu_i+\beta_j\mu_j)(-\Delta^{1/2})$ on every adjacent pair of arcs of the subdomains G_i and G_j, where Δ is the Laplace operator /33/ defined on the part Σ of the surface ∂G_i (or of the circle) in spaces of the type $\mathring{H}^{1/2}(\Sigma)$.

Now let us consider some particular cases of partitioning. For subdivisions of the "strips" type, i.e. at $m_y = m_z = 1$ the condition number $\kappa(\mathbb{B}_k^{-1}\mathbb{A}_0)$, $k=1,\dots,4$ doesn't depend on the quantity d/h (due to disconnectedness of the components $\mathbb{X}_{3,i}$ of the space $\mathbb{X}_3$) and is defined only by the form of subdomains Ω_i. Here $\mathbb{X}_1 = \mathbb{X}_2 = \varnothing$. For the "quasi-two-dimensional" subdivision, i.e. at $m_z = 1$ we have $\mathbb{X}_1 = \varnothing$, while $\kappa(\mathbb{B}_5^{-1}\mathbb{A}_0)$ doesn't depend on d/h. Apparently for this type of decomposition the operator of the type $\mathbb{A}_{cyl}$ will be the most effective preconditioner. If we have $d=O(h)$, then assuming $\mathbb{X}_2 = \mathbb{X}_3 = \varnothing$ we come to finite-difference system of equations for the family of the "serendipity type" elements (with constant coefficients μ_i inside elements Ω_i). The problem is reduced to a single solution of the equation (2.4) using, for example, iteration methods for difference elliptic equations with highly varying coefficients /11,40-42/.

The greater flexibility of the considered family of algorithms may be achieved using the technique of the local grid refinement /28-31/. Besides the partitioning into substructures Ω_i by a "rough" grid can be used not only for the global data transform, but also for approximating the nonlinearity in the framework of the incomplete-nonlinear formulation for the quasi-linear elliptic equations/1,2/.

In some problems we can put $\mathbb{X}_2 = \varnothing$ with all consequent simplifications. This case takes place, for example, for finite-difference schemes with the shift $h/2$.

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