объединенный
институт
ядерных
исследований
дубна

E11-89-811

B.N.Khoromskij, G.E.Mazurkevich

ONE FAST SOLVER FOR DISCRETE LAPLACE EQUATION ON RECTANGLE REGIONS

Submitted to "Журнал вычислительной математики и математической физики"

1989

I. We consider the standard five-point difference approximation of the Laplace equation on a rectangle using a uniform mesh $0 < i < N_1$, $0 < j < N_2$:

$$\frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h_1^2} + \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h_2^2} = 0 \qquad (1)$$

with boundary conditions

$$\begin{aligned} &u_{0j} = u_{N_1 j} = 0 , && 1 \le j \le N_2 - 1 \\ &u_{i0} = \varphi_i , \quad u_{iN_2} = \psi_i , && 1 \le i \le N_1 - 1 . \end{aligned} \qquad (2)$$

In what follows we shall designate as U, u the exact solution of (1),(2), as V, v its approximate solution.

Suppose that we can find the values of the grid function on mesh lines j_0, j_0+1, i.e. the vectors $u(\ell) = [u_{1\ell}, \dots, u_{N_1-1\,\ell}]$, $\ell = j_0, j_0+1$ are known. Then we shall evaluate the solution using explicit scheme: for increasing subscripts j -

$$\begin{aligned} &V(j_0+s) = B\,V(j_0+s-1) , \quad s = 1 \div n_1 \\ &V(j_0) = U(j_0) \end{aligned} \qquad (3)$$

for decreasing ones -

$$\begin{aligned} &V'(j_0-s) = B\,V'(j_0-s+1) , \quad s = 1 \div n_2 \\ &V'(j_0) = U'(j_0) , \end{aligned} \qquad (4)$$

where $V(j_0+s) = [v(j_0+s+1), v(j_0+s)]^T$,

$V'(j_0-s) = [v(j_0-s), v(j_0-s+1)]^T$, $s = 0, 1, \dots$

$$B = \begin{bmatrix} 2E - h_2^2 A & -E \\ E & 0 \end{bmatrix} - 2(N_1-1) \times 2(N_1-1) \text{ matrix;}$$

A is tridiagonal matrix corresponding to one dimensional Laplace operator; E, unity matrix.

Such formulation is equivalent to the Cauchy problem for Laplace equation which is / 1,2 / conditionaly correct in the sence of Tikhonov, and that leads to numerical instability of the processes (3) and (4) / 3 /. When evaluating solution according to (3), (4) on computer with floating-point arithmetics, vector $V(j_0)$ is known with computer rounding-off error $\delta_c = 2^{-t}$, i.e.

$$\|\delta_0\|_2 \cdot \|U(j_0)\|_2^{-1} \sim \delta_c \quad , \qquad \delta_0 = U(j_0) - V(j_0) \ .$$

Therefore the question arises: what number of steps n can be done in the processes (3),(4) after which the relative error of the approximate solution does not exceed ε as compared with the exact solution of (1),(2), i.e.

$$\|\delta_n\|_2 \cdot \|U(j_0 \pm n)\|_2^{-1} \le \varepsilon \quad , \qquad \delta_n = U(j_0 \pm n) - V(j_0 \pm n) \ .$$

In oder to do that it is necessary to estimate the maximum eigenvalue (ev) of the matrix B . One can easily obtain the following connection between ev λ of matrix B and ev μ of matrix A :

$$\lambda + \frac{1}{\lambda} - 2 = \mu \ ,$$

from which it follows

$$\lambda_{max} = \frac{1}{2}\left(2 + \mu_{max} + \left((2+\mu_{max})^2 - 4\right)^{1/2}\right), \ \mu_{max} = 4\frac{h_2^2}{h_1^2}\cos^2\frac{\pi}{2N_1} \qquad (5)$$

and since $\|\delta_n\|_2 \le \lambda_{max}^n \|\delta_0\|_2$ then the condition for n is

$$n \le n^* \quad ; \qquad \varepsilon = \lambda_{max}^{n^*} \cdot \delta_c \ . \qquad (6)$$

Figure 1a presents dependence of $\lg\varepsilon$ on n when $\delta_c = 10^{-16}$ which corresponds to the double precision data (REAL*8) of EC - 1037, EC - 1061 computers. The dependences marked 1, 2, 3 are evaluated according to (5),(6) when $N_1 = 32$ and $h_1 = h_2$, $h_2 = 2h_1$, $h_2 = h_1/2$, respectively. The dependences 1', 2', 3' are obtained in the process (3) when $\varepsilon_s = \|U(j_0+s) - V(j_0+s)\|_2 \cdot \|U(j_0+s)\|_2^{-1}$, $s = 1 \div n$.

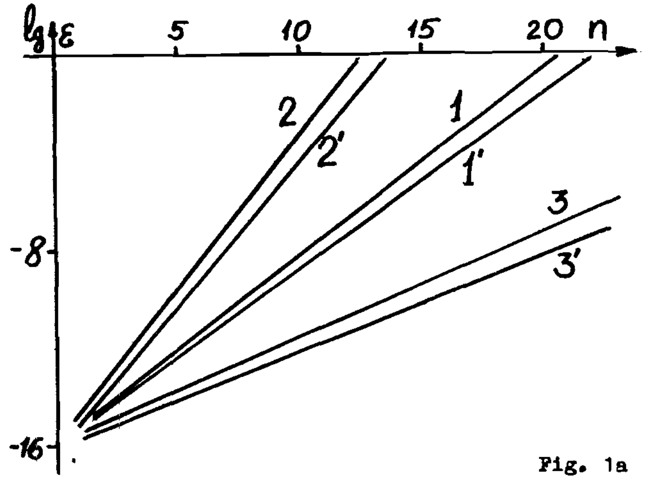

Fig. 1a

The values of $\sigma(\ell)$, $\ell = j_0, j_0+1$ are determined by boundary data φ and ψ as / 6,7,8,/ :

$$\sigma(\ell) = S_{N_2-\ell}\,\varphi + S_\ell\,\psi \ ,$$
$$S_\alpha = F D_\alpha F \quad , \quad \alpha = \ell, N_2 - \ell \ ; \quad \ell = j_0, j_0+1 \ , \qquad (7)$$

where $F \equiv F(N_1) - (N_1-1)\times(N_1-1)$ is matrix of discrete sine Fourier transform; D_α is a diagonal matrix

$$D_\alpha = \mathrm{diag}\left\{ d_m(\alpha) = \frac{\beta_m^{\alpha} - \beta_m^{-\alpha}}{\beta_m^{N_2} - \beta_m^{-N_2}} \ , \ m = 1 \div N_1 - 1 \right\} \qquad (8)$$
$$\beta_m = 1 + 2x + 2\left(x + x^2\right)^{1/2} \quad , \qquad x = \frac{h_2^2}{h_1^2}\sin^2\frac{\pi m}{2N_1} \ .$$

Evaluations according to (7) do not insert the additional error as compared with the computer rounding-off error δ_c of the boundary data φ and ψ , since for the maximum eigen values $\zeta_{max}(\alpha)$ of the matrices S_α , $\zeta_{max}(\alpha) = \max_m d_m(\alpha)$, holds true

$$\max_\alpha \zeta_{max}(\alpha) \le C \ ,$$

where $C \sim 1$ and is independent of N_1 and N_2 . (See also Remark 1)

II. The approximate solution of the problem (1),(2) in mesh points $1 \le i \le N_1-1$, $1 \le j \le N_2-1$ is obtained in the following way. Let natural numbers $\mathcal{Z}$, n_1^k, n_2^k, $p_k = n_1^k + n_2^k$, $k = 1 \div \mathcal{Z}$ be given. They satisfy some conditions which will be given below. Represent the set of subscripts $\mathcal{J} = \{1 \le j \le N_2 - 1\}$ as a union of two groups:

$$\mathcal{J}_0 = \begin{cases} j_1 = 1 & \\ j_q = j_{q-1} + p_{q/2} + 1 & , \ q\text{- even} \\ j_q = j_{q-1} + 1 & , \ q\text{- odd} \end{cases} , \quad q = 2 \div 2\mathcal{Z}$$

which consists of $\mathcal{K}_0 = 2\mathcal{Z}$ elements, and

$$\mathcal{J}_1 = \mathcal{J} \setminus \mathcal{J}_0 = \bigcup_{k=1}^{\mathcal{Z}} \mathcal{J}_1^k ,$$

$$\mathcal{J}_1^k = \begin{cases} j_q = \sum_{i=1}^{k-1} p_i + 2k - 1 + q , \quad q = 1 \div p_k & , \ p_k \ne 0 \\ \varnothing & , \ p_k = 0 \end{cases}$$

which consists of

$$\mathcal{K}_1 = \sum_{k=1}^{\mathcal{Z}} p_k = N_2 - 1 - 2\mathcal{Z} \tag{9}$$

elements.

Determine n^* according to (6) by given ε and δ_c and demand that conditions

$$n_i^k \le n^* , \quad i = 1,2 ; \quad k = 1 \div \mathcal{Z} \tag{10}$$

hold true. Conditions (9),(10) lead to the restrictions on the choice of $\mathcal{Z}$:

$$\frac{N_2 - 1}{2(n^* + 1)} \le \mathcal{Z} \le \frac{N_2 - 1}{2} .$$

Having chosen $\mathcal{Z}$, then choose n_1^k, n_2^k, $k = 1 \div \mathcal{Z}$ from (9),(10). Now, on mesh lines with subscripts $j \in \mathcal{J}_0$ we find the solution according to (7), and then, on mesh lines with subscripts $j \in \mathcal{J}_1^k$, evaluate it by (3) with $n_1 \equiv n_1^k$ and by (4) with $n_2 \equiv n_2^k$.

The number of arithmetical operations q necessary for the solution of the problem (1),(2) using the method described above (we shall call it FAES - Fourier Analysis + Explicit Scheme) consists of expenditures for evaluations in (7), which are

$$q_0 = 2 \cdot M_{FFT}(N_1) + 2\mathcal{Z}\left(M_{FFT}(N_1) + 3(N_1 - 1)\right)$$

operations (suppose that the elements of D_d matrices are known and that for multiplication of $F(N_1)$ matrices on vector the FFT algorithm with the number of operations $M_{FFT}(N_1)$ is used), and of expenditures for evaluations in (3),(4) which are

$$q_1 = \mathcal{K}_1 \cdot 6(N_1 - 1) = 6(N_2 - 1)(N_1 - 1) - 12\mathcal{Z}(N_1 - 1)$$

operations, so that $q = q_0 + q_1$.

Compare this quantity with the number of operations of one of the most effective algorithms - FACR algorithm / 4,5 /. In oder to solve the problem (1) with nonhomogeneous boundary conditions on all sides of rectangle and with $N_1 = N_2 = N = 2^m$ the FACR algorithm needs / 4, p.204 / (see also / 20, 21/)

$$Q = 2N^2 m + 6N^2 + O(mN) \tag{11}$$

arithmetical operations, when $M_{FFT}(N) = 2mN - 3N - m + 3$.

The FAES method needs

$$Q = \frac{4N^2}{n+1} m + 12N^2\left(1 - \frac{1}{n+1}\right) + O(mN) \tag{12}$$

operations, where n depends on accuracy of the problem ε and computer accuracy δ_c . So, when $n = 1$ ($\varepsilon \sim 10^{-15}$, when $\delta_c \sim 10^{-16}$) these two methods coincide in the number of operations and accuracy.

When n increases, i.e. when accuracy of the solution decreases or number λ_{max} from (5) decreases, FAES method needs $(n+1)/2$ -times less operations than FACR method. This conclusion is true of course only for the sufficiently great N. The Table columns (11) and (12)) gives the real correlation between two methods. It must also be noted that when $m < n \leq n^*$ we have algorithm with $Q = O(N^2)$ operations.

Table

m	(11)	$n=7$, $\varepsilon=10^{-10}$ (12)	(13)	$n=15$, $\varepsilon=10^{-5}$ (12)	(13)
5	15,535	13,368	3,702	12,880	4,574
6	71,761	55,884	11,679	52,884	11,691
7	323,219	231,504	41,064	215,312	32,968
8	1,431,829	956,420	158,425	874,980	105,465
9	6,269,463	3,948,216	649,578	3,556,152	379,962

III. The problem of partial solution of (1),(2) when it is necessary to find grid function only on mesh line $i = i_0$ / 8,9,10 / often arises in practice. FAES method easily permits one to do that in the following way: all the elements v_{ij} of the vectors $V(j_0 + S)$ in the processes (3),(4) must be put equal to zero except those for which $i_0 - n_\ell^k + S \leq i \leq i_0 + n_\ell^k - S$, $\ell = 1,2$, holds true. Then of course the calculations in (3),(4) must be carried out only for the elements with such subscripts i. So, $q \approx q_0 + 6\cdot\sum_{k=1}^{k}(n_1^k)^2 + (n_2^k)^2$ arithmetical operations are necessary for the solutions of the partial problem. When $N_1 = N_2 = N = 2^m$ this quantity is estimated by

$$q = \frac{2N^2}{n+1}m + 6Nn + O(mN) \qquad (13)$$

and it is possible to minimize this value when $n \sim \sqrt{N} \leq n^*$, so $q = O(N^{3/2}m)$. The column marked (13) in the Table shows the number of operations necessary for the partial solutions of (1) under nonhomogeneous boundary conditions on all sides of rectangle.

IV. The FAES algorithm allows one to solve discrete Laplace equation in parallelepiped with nonhomogeneous boundary conditions on mesh planes $j=0$ and $j=N_2$. In that case the maximum eigenvalue of transition operator from mesh plane (i, j_0, k) to mesh plane (i, j_0+1, k), $1 \leq i \leq N_1-1$, $1 \leq k \leq N_3-1$ is connected with maximum eigen value of two-dimensional grid Laplace operator according to (5) when

$$\mu_{max} = 4\left(\frac{h_2^2}{h_1^2}\cos^2\frac{\pi}{2N_1} + \frac{h_2^2}{h_3^2}\cos^2\frac{\pi}{2N_3}\right).$$

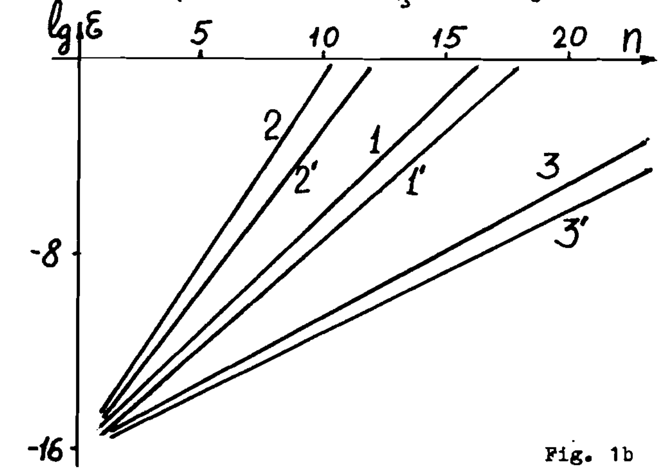

Fig. 1b

The number of steps n which can be done in processes (3),(4) is defined by (6). Figure 1b shows dependences of $\lg\varepsilon$ on n for three-dimensional problem analogous to those in two-dimensional case (see Fig. 1a).

On mesh planes j_0, j_0+1 grid functions can be found according to (7),(8) when $F = F(N_1) * F(N_3)$; $m = (i-1)\cdot(N_3-1) + k$;

$$x = \left(\frac{h_2}{h_1}\sin\frac{\pi i}{2N_1}\right)^2 + \left(\frac{h_2}{h_3}\sin\frac{\pi k}{2N_3}\right)^2, \quad i = 1 \div (N_1-1),\ k = 1 \div (N_3-1).$$

So, the number of arithmetical operations necessary for approximate solution of three-dimensional problem is defined by $q = q_0 + q_1$, where

$$q_0 = 2\cdot M^{\bar{w}}_{FFT} + 2z\left(M^{\bar{w}}_{FFT} + 3(N_1-1)(N_3-1)\right),$$

$$q_1 = 10(N_1-1)(N_2-1)(N_3-1) - 20z(N_1-1)(N_3-1),$$

$$M^{\bar{w}}_{FFT} = (N_1-1)\cdot M_{FFT}(N_3) + (N_3-1)\cdot M_{FFT}(N_1).$$

Solution of the partial problem about determination of the grid function on mesh planes $i = i_0$ or $k = k_0$ needs

$$q = q_0 + 10N_\nu \sum_{r=1}^{z} (n_1^r)^2 + (n_2^r)^2, \quad \nu = 1 \text{ or } \nu = 3,$$

operations. When $N_1 = N_2 = N_3 = N = 2^m$ this value is

$$q = \frac{4N^3}{n+1}m + 10N^2 n + O(N^2 m)$$

and so as in two-dimensional case, it is possible to minimize this quantity when $n \sim \sqrt{N} \leq n^*$.

The FAES method of partial and total solution of discrete Laplace equation is very effective when used jointly with domain decomposition method (DD-method) for the numerical solution of elliptic equations with discontinuous coefficients in domains composed from a great number of rectangles or parallelepipeds /11÷18/ and when in the process of iterations on subdomains it is possible to use partial solution in each subdomain only. After completion of global iterations the total solution in all subdomains is found. The weak dependence of DD-method convergence on the number of substructures / 13 ÷ 18 / allows one to use grids of small dimensions N in each subdomain, i.e. to use the $Q = O(N^{(2d-1)/2}\log_2 N)$, $Q = O(N^d)$, $d = 2,3$, complexity algorithms for the solution of partial and total problems in subdomains.

Remark 1. The lack of full independence of problem's dimension for the error of solution in (7) must be mentioned: the use of FFT for multiplication of matrix $F(N)$ on vector leads to the error / 19/:

$$\left(\frac{m}{3}\right)^{1/2}\delta_c \leq \|v - u\|_2 \cdot \|u\|_2^{-1} \leq m^{1/2}\delta_c,$$

when $N = 2^m$. In real scale of variations of N it practically does not influence on the accuracy of further calculations.

Remark 2. After slight modifications our method can be used for the solution of Laplace equation in cylindrical or spherical coordinates and for the solution of mixed buondary value problems.

Authors are gratefull to prof. Zhidkov E. P. for usefull discussions.

REFERENCES :

1. Lavrent'ev M.M. Some Ill-Posed problems of mathematical physics, Novosibirsk, Vychisl. Tsentr Sib. Otd. Akad. Nauk SSSR , 1962, (in Russian)
2. Tikhonov A.N., Arsenin V.Ya. Methods for Solution of Ill-Posed Problems, Moscow: Nauka, 1986 (in Russian)
3. Vabishchevich P.N. The Difference Methods in the Solution of Cauchy Problem for Elliptic Equations. Zh. Vychisl. Mat. & Mat. Phys., 1981, 21, No. 8, p.p. 509 - 511 (in Russian)
4. Samarsky A.A., Nicolaev E.S. Methods for Solution of Grid Problems, Moscow: Nauka, 1978 (in Russian)
5. Swarztrauber P.N. The Method of Cyclic Reduction, Fourier Analysis & FACR Algorithms for Discrete Solution of Poisson Equation on a Rectangle. SIAM Rev., 1977, v. 19, p.p. 490 - 501
6. Wasow W. Journ. Res. Nat. Bur. Stand., 1952, v. 48, p. 345

7. Romanova S.E. The Cost-Effective Method for the Solution of Difference Laplace Equation on Rectangle Domains. Dokl. Akad. Nauk SSSR, 1980, 252, No. 1, p.p. 48 - 51 (in Russian)

8. Bakhvalov N.S., Orekhov M.Yu. On Fast Methods for the Solution of Poisson Equation. Zh. Vychisl. Mat. & Mat. Phys., 1982, 22, No. 6, p.p. 1386 - 1392 (in Russian)

9. Benegas A. Fast Poisson Solvers for Problems with Sparsity. Math. Comput., 1978, 32, No. 142, p.p. 441 - 446

10. Kuznetsov Yu.A., Matsokin A.M. On Partial Solution of Linear Systems of Algebraic Equations. In: "Computational Methods of Linear Algebra", Novosibirsk, 1978, p.p. 62 - 89 (in Russian)

11. Proceedings Of the First International Symposium of Domain Decomposition Methods for Partial Differential Equations, Eds Glowinski R. et al., Philadelphia, PA: SIAM, (ISBN 0-89871-220-3), 1988, 431 p.

12. Bramble J.H., Pasciak J.E., Schats A.H. The Construction of Preconditioners for Elliptic Problems by Substructuring I,II. Math. Comput., 1986, 47, No. 175, p.p. 103 - 134; 1987, 49, No. 179, p.p. 1 - 17

13. Kuznetsov Yu.A. Multilevel Methods of Domain Decomposition. In: "Numerical Analisis & Mathematical Modelling", Moscow, Otd. Vychisl. Mat. Akad. Nauk SSSR, 1988, p.p. 109 - 126 (in Russian)

14. Dryja M., Proskurowski W., Widlund O., A Method of Domain Decomposition with Cross-Points for Elliptie Finite Element Problems. In: "Proceedings of the Int. Symp. on Optimal Algoritms" Blagoevgrad, Bulgaria, April 21-25, 1986, p. 15

15. Widlund O. Iterative Methods of Substructuring. General Elliptic Case. In: "Numerical Processes and Systems", 6, Moscow, Nauka, 1988, p.p. 76 - 89 (in Russian)

16. Khoromsky B.N. Quasy-Linear Elliptic Equations in the Incomplete Nonlinear Formulation and Methods for their Preconditioning. Dubna, 1989, Preprint JINR E5-89-598, Dubna, 1989.

17. Zhidkov E.P., Mazurkevich G.E., Khoromsky B.N. Iterative Methods of Domain Decomposition with Cross-Points for the Solution of Discrete Elliptic Problems. Dubna, 1989, Preprint JINR E11-89-174, Dubna, 1989.

18. Zhidkov E.E, Mazurkevich G.E., Khoromsky B.N. Domain Decompo - sition Method for Magnetostatics nonlinear problems in Combined Formulation. Sov. J. Numer. Anal. Math. Model., 1990, vol. 5, No. 2 (to appear)

19. Kaneko T., Liu B. Accumulation of Round-Off Error in Fast Fourier Transforms. J. Assoc. Comput. Mach., 1970, 17, No. 4, p.p. 637 - 654

20. Kuznetsov Yu.A. Calculation Methods in Subspaces. In: Vychisl. Proc. & Syst., 2, Moscow, Nauka, 1985, p.p 265-350.

21. Kaporin I.E. Modificate Marching Algorithm for the Solution of Discrete Dirichlet Problem for Poisson equation in Rectangle. In: Difference Methods in Mathematical Physics, Moscow Univ., 1980, p.p. 11-21.

Received by Publishing Department
on December 6, 1989.