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NUMERICAL ANALYSIS OF BLOW-UP AND DEGENERACY OF A SEMILINEAR HEAT EQUATION

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1. Introduction

The purpose of this paper is the numerical study of the blow-up of solutions of the semilinear heat equation

(1.1) $$u_t = \Delta u + (1+u)\ln^{\beta}(1+u),$$

where $u = u(x,t)$ is nonnegative and $\beta>1$ is a fixed constant. This equation was first introduced in [10]. Different results concerning the asymptotic behaviour of solutions to (1.1) near finite blow-up time $t = T_0$ are given in [4],[8],[10],[12], [14],[19],[20],[22,p.272],[23]. In particular, results of numerical investigation for one space dimension are presented in [10],[14]. There is a good qualitative understanding of unusual asymptotic behaviour of solutions as $t \to T_0$, but many principle mathematical problems of blowing-up behaviour are open now.

Our main aim is to characterize by numerical experiment the asymptotic behaviour of $u(x,t)$ near $t = T_0$ in two and three space dimensions. We show that the qualitative results [10,23] hold for radial symmetric solutions $u(r,t)$, $r = |x|$ in many-dimensional case, and the asymptotic behaviour of $u(r,t)$ as $t \to T_0$ is described by the self-similar solution $v(r,t)$ (see [12],[22]) of the first order nonlinear equation of Hamilton-Jacobi type

(1.2) $$v_t = \frac{|\nabla v|^2}{1+v} + (1+v)\ln^{\beta}(1+v) .$$

Note that (1.1) doesn't admit any nontrivial self-similar or invariant solution with blowing-up properties for fixed $\beta>1$ [9]. But nonlinear Hamilton-Jacobi equation (1.2) has the following simple self-similar solution for arbitrary $\beta>1$:

(1.3) $$v(r,t) = \exp\{(T_0 - t)^{-1/(\beta-1)}\theta_a(\xi)\} - 1,$$

(1.4) $$\xi = r\,(T_0 - t)^{-m}, \quad m = \frac{\beta-2}{2(\beta-1)} ,$$

where T_0 is finite blow-up time and function $\theta_a \ge 0$ satisfies nonlinear ordinary differential equation

(1.5) $$(\theta_a')^2 - m\theta_a'\xi - \frac{1}{\beta-1}\,\theta_a + \theta_a^{\beta} = 0.$$

The function (1.3) satisfying (1.2) is not exact solution of the initial equation (1.1), but it describes the asymptotic properties of $u(r,t)$. Therefore $v(r,t)$ is said to be approximate self-similar solution (*a.s.-s.s.*) of (1.1). One can see from (1.4) that the main asymptotic properties of v depend on the sign of the parameter m.

There exist three types of blow-up behaviour:

(i) $1<\beta<2$ $(m<0)$ - total blow-up (HS-evolution);
(ii) $\beta = 2$ $(m=0)$ - regional blow-up (S-evolution);
(iii) $\beta > 2$ $(m>0)$ - single point blow-up (LS-evolution),

see [22, p. 274]. We show by numerical experiment that the same classification of blow-up behaviour holds for equation (1.1) for $N=2$ and $N=3$. We obtain "numerical proof" of convergence in specific norm of $u(r,t)$ to a.s.-s.s. (1.3) as $t\to T_0$. It is important to note that the asymptotic behaviour near $t=T_0$ doesn't depend on N, see equation (1.5) which doesn't contain space dimension. In general, the problem of convergence of $u(r,t)$ to a.s.-s.s. $v(r,t)$ as $t\to T_0$, i.e. the specific degeneracy of nonlinear parabolic equation (1.1) into Hamilton-Jacobi equation (1.2) near a finite blow-up time, is open now. An exception is the case $N = 1$ and $\beta = 2$, when equation (1.1) has the following explicit noninvariant solution [8],[16]:

(1.6) $$u_*(x,t) = \exp\{\varphi(t)[\psi(t) + \cos x]\} -1,$$

Functions $\varphi(t)$, $\psi(t)$ satisfy the system of nonlinear ordinary differential equations

$$\varphi' = -\varphi + 2\varphi^2\psi, \quad \psi' = \psi + \varphi - \varphi\psi^2, \quad t>0.$$

It was proved in [8] that blow-up solution (1.6) describes convergence to a.s.-s.s. (1.3). For $N\geq1$ and $\beta>2$ an exact upper estimate of $u(r,t)$ near $t = T_0$ with space-time structure of the a.s.-s.s. $v(r,t)$ was obtained in [14]. Different results on degeneracy of quasilinear parabolic equations into first order equations of Hamilton-Jacobi type were proved for equations without source under blowing-up boundary functions [22,p.346] and for global solutions growing to infinity as $t\to\infty$ [12].

Note that degeneracy of parabolic equation into equation of first order near finite blow-up time is common property of semilinear heat equations with sources of different kinds. The degeneracy mentioned above is the first example. Degeneracy of other kind exists for semilinear equation with power nonlinearity

(1.7) $$u_t = \Delta u + u^\beta, \quad \beta>1.$$

In this case Hamilton-Jacobi equation has the form

$$v_t + (\nabla v\cdot x)\{2(T_0 - t)|\ln(T_0 - t)|\}^{-1} = v^\beta,$$

and its self-similar solution

$$v(x,t) = (T_0 - t)^{-1/(\beta-1)}\{\beta-1 + [(\beta-1)^2/(4\beta)]\eta^2\}^{-1/(\beta-1)},$$

$$\eta = |x|((T_0 - t)|\ln(T_0 - t)|)^{-1/2},$$

describes the behaviour of $u(x,t)$ as $t\to T_0$, see different qualitative and numerical results in [1],[13],[14],[17],[18]. The equation with exponential nonlinearity

(1.8) $$u_t = \Delta u + e^u$$

degenerates as $t\to T_0$ into Hamilton-Jacobi equation (see [2],[14])

$$v_t + (\nabla v\cdot x)\{2(T_0 - t)|\ln(T_0 - t)|\}^{-1} = e^v,$$

which has blow-up self-similar solution

$$v(x,t) = -\ln(T_0 - t) - \ln(1 + \eta^2/4),$$

where η is defined above.

As for quasilinear equations

(1.9) $$u_t = \Delta u^{\sigma+1} + u^\beta \quad \text{and} \quad u_t = \nabla\cdot(|\nabla u|^\sigma\nabla u) + u^\beta,$$

where $\sigma>0$, $\beta>1$ are fixed constants, in many cases the asymptotic evolution of blow-up solutions is described by nontrivial self-similar solutions, see different results [5],[6],[9],[15], [22] for the first equation (1.9) and [7],[11] for the second one.

The effect of degeneracy and convergence to a.s.-s.s. is well defined when studying the Cauchy problem to (1.1). But for numerical calculations we consider initial boundary value problem in $B_R\times(0,T_0)$, where $B_R = \{|x|<R\}$ is a ball in $\mathbb{R}^N$ of radius $R>0$, with Dirichlet or Neumann boundary conditions. The choice of boundary condition is not essential (see Sections 3 and 4). In radial symmetric case this problem has the form

(1.10) $$u_t = \frac{1}{r^{N-1}}(r^{N-1}u_r)_r + (1+u)\ln^\beta(1+u) \quad \text{in } (0,R)\times(0,T_0),$$

(1.11) $$u_r(0,t) = 0 \quad \text{for } t\in[0,T_0),$$

(1.12) $$u(R,t) = 0 \quad \text{or} \quad u_r(R,t) = 0 \quad \text{for } t\in[0,T_0),$$

(1.13) $$u(r,0) = u_0(r)\geq0 \quad \text{in } [0,R], \ u_0\in C([0,R]).$$

Now a.s.-s.s. $v(r,t)$ satisfies nonlinear Hamilton-Jacobi equation

(1.14) $$v_t = \frac{(v_r)^2}{1+v} + (1+v)\ln^\beta(1+v).$$

The qualitative theory of a.s.-s.s. is given in Section 2. Section 3 is devoted to numerical method used in this paper. In next sections numerical results for the cases $\beta = 2$ (Section 4.1), $1<\beta<2$ (Section 4.2) and $\beta>2$ (Section 4.3) are given.

2. Qualitative Method of Construction of Approximate Self-Similar Solution

Let $\ln(1+u(r,t)) = U(r,t)$. After this transformation we get the following semilinear parabolic equation

(2.1) $$U_t = \frac{1}{r^{N-1}}(r^{N-1}U_r)_r + (U_r)^2 + U^\beta \quad \text{in } (0,R)x(0,T_0).$$

Function U satisfies the same boundary conditions

(2.2) $$U_r(0,t) = 0 \quad \text{for } t\in[0,T_0),$$

(2.3) $$U(R,t) = 0 \text{ or } U_r(R,t) = 0 \quad \text{for} \quad t\in[0,T_0),$$

and initial condition

(2.4) $$U(r,0) = U_0(r) \equiv \ln(1+u_0(r)) \quad \text{in } [0,R].$$

Equation (2.1) with power nonlinearities is more convenient for studying and numerical calculations.

First we note that (2.1) has no nontrivial blowing-up self-similar solution. Some preliminary results from [4,10,22,23] concerning the construction of blow-up lower solutions to (2.1) show that the first term (Laplacian) in the right of (2.1) is smaller in comparison with two others as $U\longrightarrow\infty$. The problem of "degeneracy" of Laplacian for (2.1) near a blow-up point is open now.

Full results of such kind were obtained for equation without source $U_t=U_{xx}+(U_x)^2$ in $\{x>0\}x(0,T)$ with blow-up boundary function $U(0,t) = (T-t)^n$ for $t\in[0,T)$, where $n<0$ is constant, i.e. $U(0,t)\longrightarrow\infty$ as $t\longrightarrow T^-<\infty$. It is proved [22,p.348] that in this case there exists convergence of $U(x,t)$ to the exact self-similar solution of Hamilton-Jacobi equation $V_t = (V_x)^2$.

Thus we consider now Hamilton-Jacobi equation

(2.5) $$V_t = (V_r)^2 + V^\beta.$$

For any fixed $\beta>1$ it has blowing-up self-similar solution

(2.6) $$V(r,t) = (T_0-t)^{-1/(\beta-1)}\theta_a(\xi),$$

where ξ is defined in (1.4), θ_a satisfies ordinary differential equation (1.5) and symmetry condition

(2.7) $$\theta_a'(0) = 0.$$

One can obtain that if the blow-up solution goes to infinity according with the space-time structure of a.s.-s.s. (2.6), then

$$\left|\frac{1}{r^{N-1}}(r^{N-1}U_r)_r\right|/(U_r)^2 \sim (T_0-t)^{1/(\beta-1)}\longrightarrow 0 \quad \text{as} \quad t\longrightarrow T_0,$$

i.e. Laplacian in (2.1) is negligible with respect to the first-order term $(U_r)^2$ as $t\longrightarrow T_0$. The existence of the solution of the problem (1.5),(2.7) satisfying additional condition $\theta_a(\infty) = 0$ is given below.

LEMMA (see [22,p.275],[14]). *For any fixed $\beta>1$ there exists solution $\theta_a\geq 0$ of (1.5), (2.7) such that*

(2.8) $$\theta_a(\xi) = \theta_0 - \frac{1}{4(\beta-1)}\xi^2(1+o(1)) \text{ as } \xi\longrightarrow 0,$$

where $\theta_0 = (\beta-1)^{-1/(\beta-1)}$ and

(i) for $1<\beta<2$ this function is unique solution of (1.5), (2.7) and $\theta_a=0$ for some $\xi=\xi_0>0$; we let $\theta_a\equiv 0$ for $\xi>\xi_0$; the following estimates of ξ_0 hold

$$2\left(\frac{2-\beta}{\beta-1}\right)^{(2-\beta)/(2(\beta-1))} < \xi_0 < \pi^{1/2}(\beta-1)^{-\frac{\beta}{2(\beta-1)}}\frac{\Gamma(\frac{1}{2(\beta-1)})}{\Gamma(\frac{\beta}{2(\beta-1)})},$$

where $\Gamma(p)$ is Euler's Gamma-function;

(ii) for $\beta=2$ the unique solution has the form

(2.9) $$\theta_a(\xi) = \cos^2(\xi/2) \text{ for } 0\leq\xi\leq\pi, \qquad \theta_a(\xi) = 0 \quad \text{for } \xi>\pi;$$

(iii) for $\beta>2$ the solution with asymptotic behaviour (2.8) is strictly *positive and*

(2.10) $$\theta_a(\xi) = C\xi^{-2/(\beta-2)}(1+o(1)) \quad \text{as } \xi\longrightarrow+\infty,$$

where $C = C(\beta)>0$ is a constant; there exists one-parametric set of positive solutions with another behaviour near $\xi=0$:

(2.11) $$\theta_a(\xi) = \theta_0 - c\xi^{\frac{2(\beta-1)}{\beta-2}}(1+o(1)) \text{ as } \xi\longrightarrow 0,$$

here $c>0$ is arbitrary constant.

For the first two cases $1<\beta<2$ and $\beta=2$ we have unique function θ_a and therefore unique a.s.-s.s. (2.6). For $\beta>2$ the problem (1.5), (2.7) has infinitely many solutions satisfying $\theta_a(+\infty)=0$, and there exist many functions (2.6) which may be a.s.-s.s. of (2.1). We show by numerical calculations that the asymptotic behaviour of blowing-up solution $U(r,t)$ is described by a.s.-s.s (2.6) with θ_a satisfying (2.8) (for N=1 and $\beta>2$ such result was obtained in [14], where the exact upper estimate of $U(r,t)$ near $t = T_0$ with structure corresponding to (2.8) was proved). Hence we expect that function (2.6) with θ_a satisfying nonanalytic expansion (2.11) doesn't appear on asymptotic stage of blowing-up process.

Finally we state method of rescaling of solution $U(r,t)$ in order to show the convergence to a.s.-s.s. $V(r,t)$ as $t\longrightarrow T_0$. By usual approach the rescaled function has the form

(2.12) $$\theta(\xi,t) = (T_0 - t)^{\frac{1}{\beta-1}}U(\xi(T_0-t)^m,t),$$

which is defined by the space-time structure of a.s.-s.s. (2.6). The asymptotic stability of a.s.-s.s. is equivalent to condition

(2.13) $$\theta(\xi,t)\longrightarrow\theta_a(\xi) \text{ as } t\longrightarrow T_0,$$

where θ_a is defined in the Lemma.

But for numerical calculations we use another method of rescaling. Let $\gamma(t) = \sup_r U(r,t)/\theta_0$ and consider the function

(2.14) $\theta(\xi,t) = U(\xi(\gamma(t))^{(2-\beta)/2},t)/\gamma(t).$

In comparison with (2.12) T_0 doesn't occur here. It is important, since T_0 is defined after finishing numerical calculations. One can see that (2.12) and (2.14) are equivalent if (2.13) holds.

3. Numerical method

We solved numerically both - the original problem (1.10)- (1.13) and the reduced one - (2.1)-(2.4). In spite of the fact, that the first one has a selfadjoint elliptic operator, and hence, it has many advantages in the algorithmic realization of the numerical method, we chose the second. The reason is that the new unknown function $U(r,t)=ln(1+u(r,t))$ grows slowly than $u(r,t)$when $t\longrightarrow T_0$, so we can approach closer the blow-up time T_0. Thus, we describe below the numerical method for solving the initial boundary value problem:

(3.1) $U_t = AU \equiv \frac{1}{r^{N-1}}(r^{N-1}U_r)_r + (U_r)^2 + U^\beta$ in $(0,R)x(0,T_0)$,

(3.2) $U_r(0,t) = 0$ for $t \in [0,T_0)$,

(3.3) $U(R,t) = 0$ or $U_r(R,t) = 0$ for $t \in [0,T_0)$,

(3.4) $U(r,0) = U_0(r) = ln\ (1+u_0(r))$ in $[0,R]$.

We choose the radius R so as to avoid the influence of the boundary condition (3.3) over the solution, i.e. in fact, we solve the Cauchy problem.

We use the lumped mass finite element method (FEM) [24] with quadratures.

The discretization is made on the basis of the problem (3.1)-(3.4) in weak form:

(3.5) $(U_t,\chi) = A(t;U,\chi),\quad \forall\chi\in H^1_\alpha(0,R),\quad 0<t<T_0,$

(3.6) $U(0,\cdot) = U_0,$

where $(\chi,\phi) = \int_0^R r^{N-1}\chi(r)\phi(r)dr,$

(3.7) $A(t;\chi,\phi) = \int_0^R(-\chi_r\phi_r + \chi_r^2\phi + \chi^\beta\phi)r^{N-1}dr,$

$H^1_\alpha(0,R) = \{\chi:\chi,\ r^{(N-1)/2}\chi'\in L^2(0,R),\ (1-\alpha)\chi(R)=0\},$

$\alpha=0$ corresponds to the condition $U(R,t)=0$, $\alpha=1$ - to the condition $U_r(R,t) = 0$.

For the spatial discretization of (3.5),(3.6) we consider standard piecewise polynomial Lagrangian finite element spaces. Let

$\{0=r_1<r_2<\ldots<r_m=R,\ r_{i+1}-r_i\le h\}$

be a partition of the interval $[0,R]$ into elements $e_i=[r_i,r_{i+1}]$. Thus, we denote by $S_{\alpha,h}$ the space of continuous functions on $[0,R]$ that reduce to polynomials of degree $\le$ $k-1$ on each element $e_i, i=1,2,\ldots,m-1$:

$S_{\alpha,h}=\{W(r)\in C([0,R]);\ W_{(r_i,r_{i+1})}\in P_{k-1};\ (1-\alpha)W(R)=0\}.$

The approximation properties of $S_{\alpha,h}$ are well known [24]:

$\|I_hW-W\|_{L^2(0,R)} + h\|\nabla I_hW-\nabla W\|_{L^2(0,R)} \le Ch^k\|W\|_{H^k},$

$\|I_hW-W\|_{L^\infty[0,R]} \le Ch^k\|W\|_{W^2_\infty[0,R]}.$

Here I_h is the interpolation operator:
$I_h\ :\ C([0,R])\longrightarrow S_{\alpha,h},\ (I_hW)(\eta_j) = W(\eta_j)$ for each of the nodes η_j, $j=1,2,\ldots,M$, that define the degrees of freedom of $S_{\alpha,h}$.

Let $U_h(r,t)$ denote the approximate solution in $S_{\alpha,h}$. We pose the semidiscrete problem:

To find $U_h\in S_{\alpha,h}$ for each t, such that

(3.8) $(U_{h,t},W) = A_h(t;U_h,W)$ for all $W\in S_{\alpha,h}$,

(3.9) $U_h(0) = U_{0h}.$

Let $\{\varphi_i\}_{i=1}^M$ be the standard Lagrangian nodal basis of $S_{\alpha,h}$. Representing $U_h(r,t)$ in the form

$$U_h(r,t) = \sum_{i=1}^M U_i(t)\varphi_i(r)\in S_{\alpha,h},$$

and using the lumped mass method our semidiscrete problem (3.8),(3.9) may be written in matrix form:

(3.10) $\tilde{\mathbf{M}}\dot{\mathbf{U}}=\mathbf{K}(\mathbf{U})\mathbf{U},$

(3.11) $\mathbf{U}(0)=\mathbf{U}_0.$

Here $\mathbf{U}=\mathbf{U}(t)=(U_1(t),U_2(t),\ldots,U_M(t))^T$, $\tilde{\mathbf{M}}$ is the lumped mass matrix,

$$\tilde{\mathbf{M}} = \mathrm{diag}(\tilde{m}_{ii}),\quad \tilde{m}_{ii} = \sum_{j=1}^M m_{ij},\quad m_{ij} = \int_0^R r^{N-1}\varphi_i\varphi_j dr,\ i,j=1,\ldots,M,$$

$$\mathbf{K}(\mathbf{U}) = \sum_e \mathbf{k}_e = \sum_e(\mathbf{k}_e^{(1)}+\mathbf{k}_e^{(2)}+\mathbf{k}_e^{(3)}),\quad \mathbf{k}_e^{(l)}=\{k_{ij}^{(l)}\},\quad l=1,2,3,$$

(3.12) $k_{ij}^{(1)} = -\int_e r^{N-1}\psi_i'\psi_j'dr,\quad k_{ij}^{(2)} = \int_e r^{N-1}a(U)\psi_i\psi_j'dr,$

(3.13) $k_{ij}^{(3)} = \int_e r^{N-1}b(U)\psi_i\psi_j dr,$

$$a(U) = \sum_{i=1}^{k} U_i \psi_i', \qquad b(U) = \Big(\sum_{i=1}^{k} U_i \psi_i\Big)^{\beta-1},$$

ψ_i, $i=1,\dots,k$ are the form functions of the element e.

Let us note, that the matrix **K** is nonsymmetric. When solving the system of ODE (3.10),(3.11), we don't calculate matrix **K** in explicit form - we calculate only the product **K**$(U)U$, accumulating it by means of the element matrices $\mathbf{k}_e$.

To solve the system (3.10),(3.11) of ODE we use a modification of the explicit Runge-Kutta method, which has second order of accuracy and an extended region of stability [21]. Moreover, the time-step τ is chosen automatically so as to guarantee stability and a desired accuracy ε at the end of the time-interval.

In computations we use linear (for N=1,2,3) and quadratic (for N=1) finite elements on uniform and nonuniform grid. To approximate the integrals in (3.12),(3.13) we use the trapezoidal rule (N=1) or the two-points Gauss rule (N=2,3) in the case of linear elements, and the three-points Gauss rule in the case of quadratic elements.

Optimal order error estimates of the standard Galerkin FEM for linear singular initial boundary value problem are proved in [3]. The effect of the lumped mass matrix for nonlinear nonsingular parabolic problems is studied in [24],[25]. Our numerical experiments show that the convergence rate of the procedure described above for solving the singular nonlinear problem (3.1)-(3.4) probably remains optimal. We are preparing a next paper dealing with the error analysis and the algorithm of mesh refinement in the case of a single point blow-up of the solution ($\beta>2$).

To analyze numerically the accuracy of the realized method, we observed the behaviour of the most sensitive characteristic of the process described by the equation (3.1) - the blow-up time T_0. The results for the case N=3, $\beta=2$, $U_0(r)=2\cos^2(\frac{r}{2})$ for $0\le r\le\pi$, $U_0(r)=0$ for $\pi\le r\le 6$, $U(6,t)\equiv 0$ are shown in the table:

h	M	ε	T_0
0.2	31	10^{-3}	0.8877025
0.1	61	10^{-3}	0.8832341
0.1	61	10^{-5}	0.8825855
0.05	121	10^{-3}	0.8817950
0.05	121	10^{-5}	0.8815855.

Note that in the case $\beta\le 2$ the realized method gives a possibility to compute sufficiently exactly the solution $U(r,t)$ when its amplitude is on the order of $\approx 10^{11}$ on uniform grid and $\tau_{min}=10^{-11}$. To reach such amplitudes of U and such accuracy in the case of a single point blow-up (LS-evolution), we use $\tau_{min}=10^{-16}$ and special mesh refinement.

4. Numerical results and interpretation

As it was told, the aim of the numerical experiments was:

- to analyze the space-time structure of the unbounded solutions of the problem (3.1)-(3.4);
- to confirm the degeneracy of the parabolic equation when $t\to T_0$ by showing convergence of its solution $U(r,t)$ to the a.s.-s.s. $V(r,t)$ in the sense of (2.13),(2.14):

(4.1) $\qquad \theta(\xi,t)\to\theta_a(\xi)$ as $\|U(t)\|_{C_r} \equiv \sup_r U(r,t)\to\infty$;

- to show that for $\beta>2$ in the asymptotic stage $t\to T_0$ only the function $\theta_a(\xi)$ with analytic expansion (2.8) is realizable;
- to investigate the interaction of structures, determined by the solutions $U(r,t)$ for $\beta\ge 2$.

To confirm (4.1), we compute the solution $\theta_a(\xi)$ of the problem (1.5) with initial condition from (2.8): $\theta_a(0)=\theta_0=(\beta-1)^{-1/(\beta-1)}$. We use the same method as for the solution of the system of ODE (3.10)-(3.11). Note that under such initial condition for $\beta>2$ we get not only the solution, satisfying (2.8), but also different solutions from the family (2.11). In this case we use the expansion

$$\theta_a(\xi) = \theta_0 - (4(\beta-1))^{-1}\xi^2 + \{\beta/[32(\beta+1)(\beta-1)^{(\beta-2)/(\beta-1)}]\}\xi^4 + O(\xi^6)$$

in order to find some additional values of $\theta_a(\xi)$ near $\xi=0$, so to choose the desired solution, satisfying (2.8). The graph of $\theta_a(\xi)$ is signed with ■ on Figures 1b-5b, 9b-12b.

We used the one-dimensional case and the exact results for $\beta=2$, N=1 to test the numerical method and its program realization. So, we explain here only the results for N=2,3 and $\beta=2$, $1<\beta<2$, $\beta>2$. They show, there exist three types of blow-up solutions for equation (3.1) (as well as for (1.10)). We'll consider every one separately, but first we determine the main characteristic of them - the effective semi-width $r_{ef}=r_{ef}(t)$ (it is signed with "x" on Figures 1a-15a) of the solution. For monotone in r solution $U(r,t)$ with unique maximum point at the origin $r=0$ the semi-width r_{ef} is defined from the equation:

(4.2) $\qquad U(r_{ef},t) = U(0,t)/2.$

The semi-width has simple geometrical interpretation for arbitrary non-monotone in r function $U(r,t)$. As it is known [22, Chapter IV], $r_{ef}\sim\text{const}>0$ as $t\to T_0$ characterizes S-evolution, $r_{ef}\to\infty$ - HS-evolution and $r_{ef}\to 0$ - LS-evolution.

4.1. Regional blow-up (S-evolution)

It corresponds to $\beta=2$. All our computations confirm (4.1) with

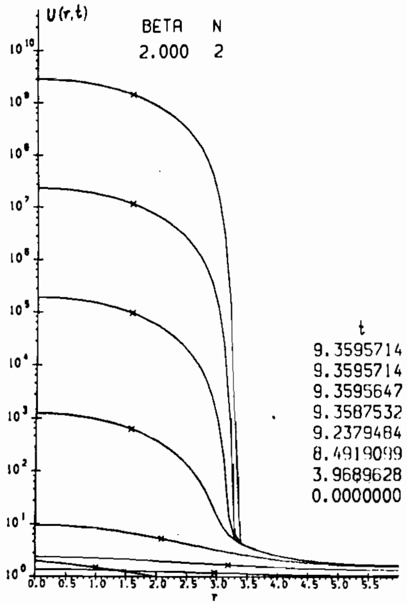

Fig. 1a

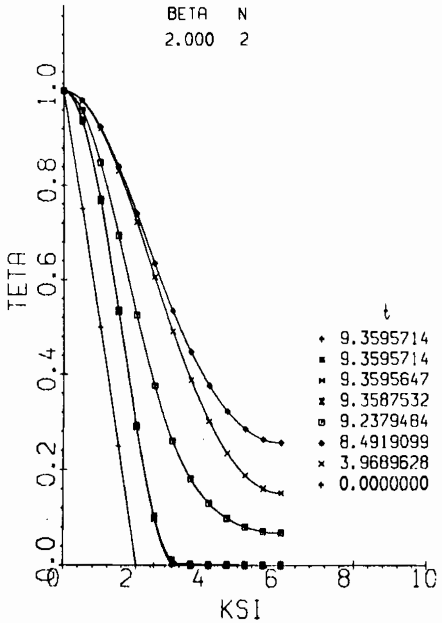

Fig. 1b

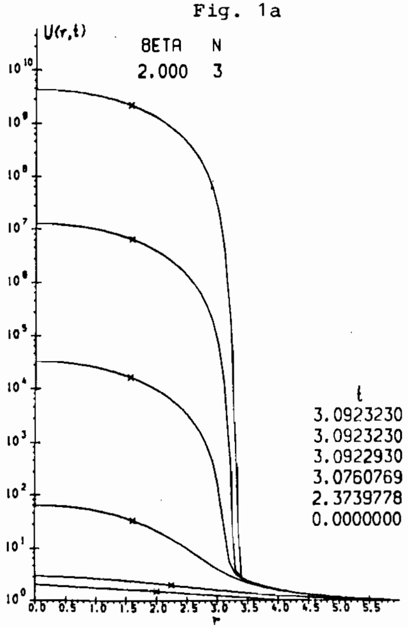

Fig. 2a

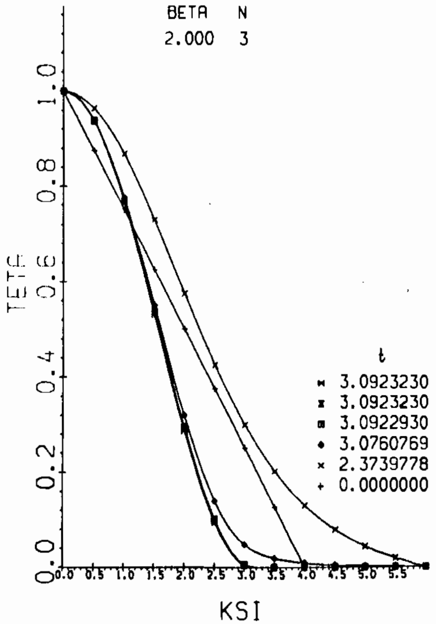

Fig. 2b

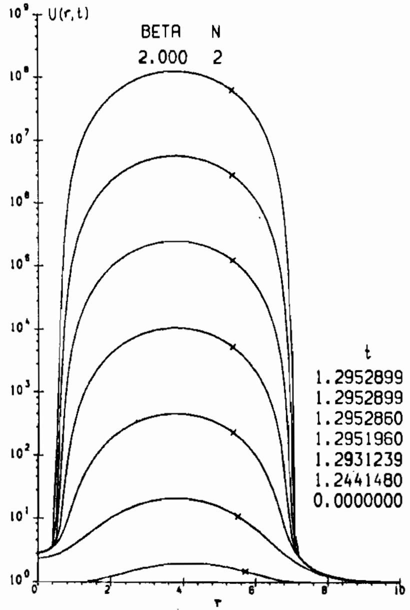

Fig. 3a

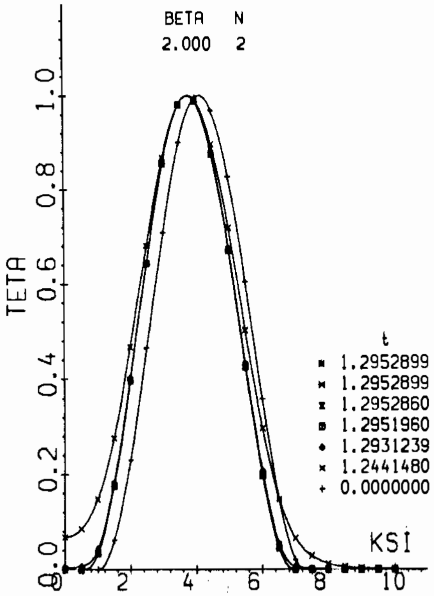

Fig. 3b

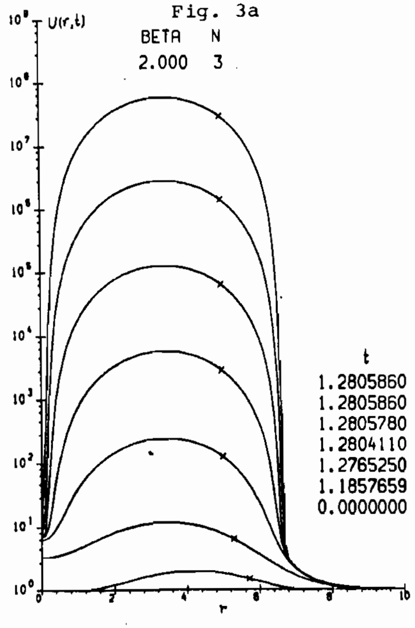

Fig. 4a

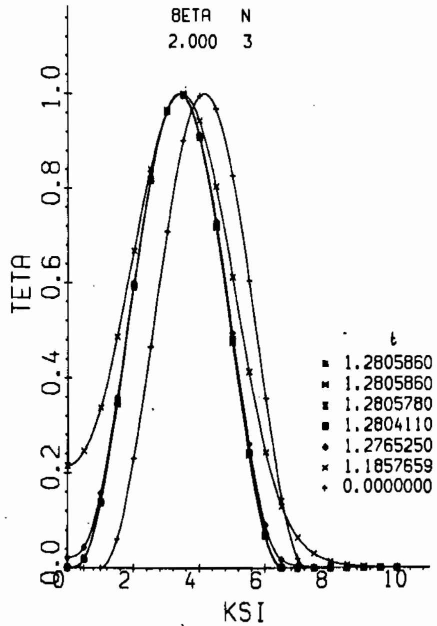

Fig. 4b

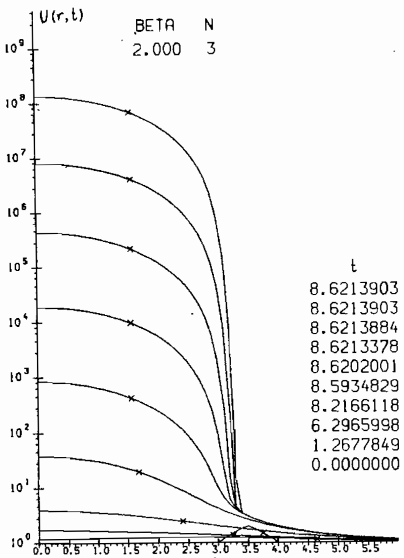

Fig. 5a

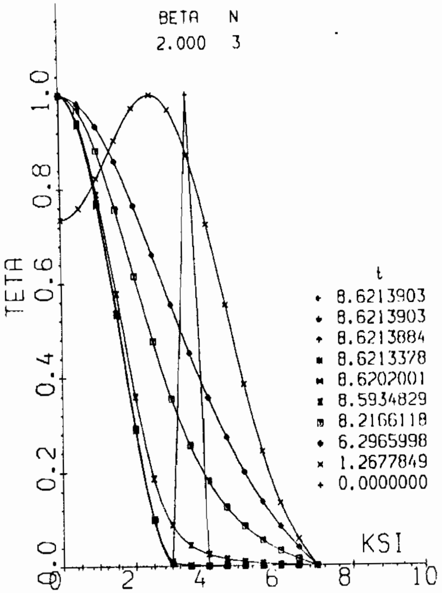

Fig. 5b

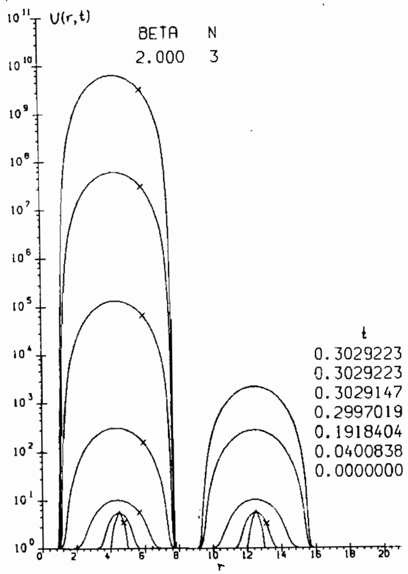

Fig. 6

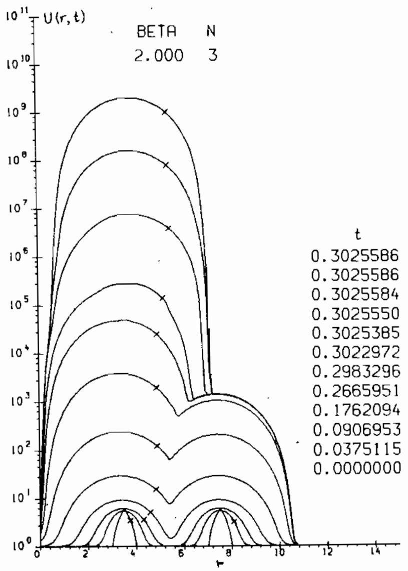

Fig. 7

$\theta_a(\xi)$, given by (2.9), for arbitrary $U_0(r)$ when N=1, and for $U_0(r)$ with sufficiently "large" energy $\|U_0\|_1$ when N=2,3 (see [14],[22]). It can be easily seen from Figures 1b-5b, where the rescaled function $\theta(\xi,t)$ is shown. In all the cases the last three or four profiles of $\theta(\xi,t)$ and the graph of $\theta_a(\xi)$ (shifted or not on the ξ-axes) coincide to within plotting resolution. It takes place not only for central initial functions $U_0(r)$ (Figures 1,2), but for any noncentral as well (Figures 3-5). So, function (2.9) represents an universal (for every N) function of the medium, described by the equation (3.1) for β=2. In addition

$$r_{ef} \longrightarrow \pi/2 \quad \text{as} \quad t \longrightarrow T_0.$$

Note, that the measure of the support of U_0 on Figure 1 (Figure 2) is smaller (greater) than 2π, but the solution $U(r,t)$ is effectively localized in a region of diameter 2π.

Figures 3-7 show the evolution of noncentral initial data. Some of them (Figures 3,4) are self-similar for the degenerate equation (2.5) and they have sufficiently "large" energy. It is seen, that they grow almost standing still, in difference with the nonself-similar data with small energy in Figure 5. So, the almost resonance excitation of the medium depends on the self-similarity of the initial data and on their energy (see Figure 6, where the initial data are not self-similar, but they have large energy). The interaction of different initial data in the process of their evolution is illustrated in Figures 6,7. It is seen that the interaction of the structures depends on the distance d between their maxima: if this distance remains greater than or equal to 2π, the structures grow independently (Figure 6), while one of them "die" in comparison with the other (with smaller T_0). On the contrary, if d is (or becomes) less then 2π, the structures merge and degenerate into one (Figure 7).

4.2. Total blow-up (*HS*-evolution)

It corresponds to $1<\beta<2$. Here

$$r_{ef} \longrightarrow \infty \quad \text{as} \quad t \longrightarrow T_0.$$

Thus, all initial perturbations merge in the process of evolution (Figure 8) and degenerate into one wave, which propagates through the whole space as $t \longrightarrow T_0$ (Figures 9,10). In all cases (4.1) holds where $\theta_a(\xi)$ is the unique solution of (1.5) (Figures 9b,10b). When $t \longrightarrow T_0$, an effective wave front r_f ($U(r_f,t) \approx 0$) [22, Chapter IV] occurs and moves in consistence with the self-similar low:

$$r_f \approx \xi_0(\gamma(t))^{(2-\beta)/2} \quad \text{as} \quad t \longrightarrow T_0.$$

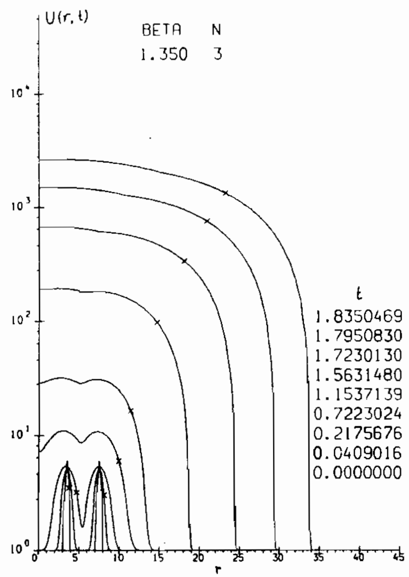

Fig. 8

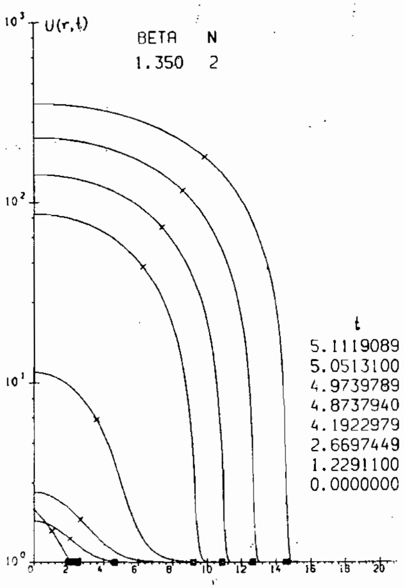

Fig. 9a

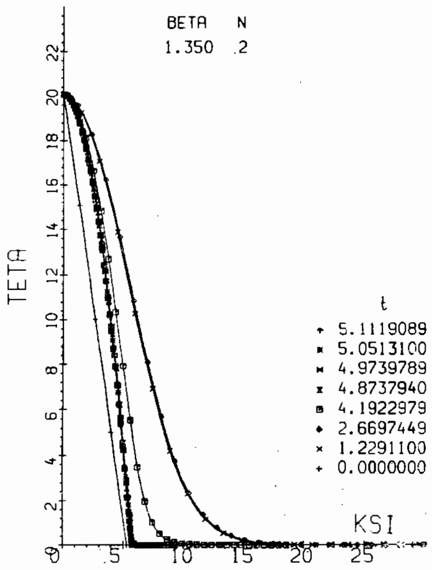

Fig. 9b

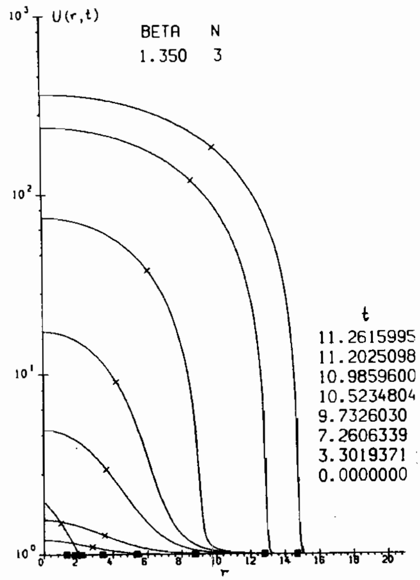

Fig. 10a

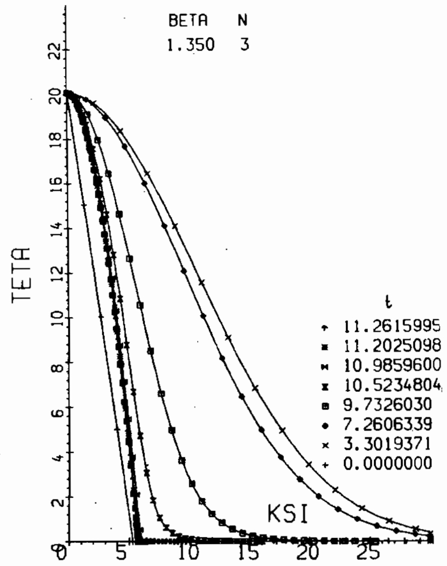

Fig. 10b

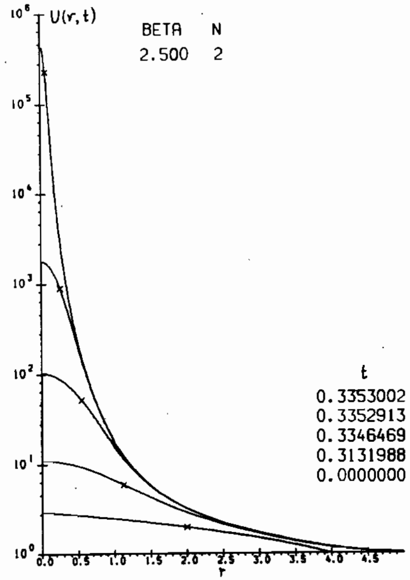

Fig. 11a

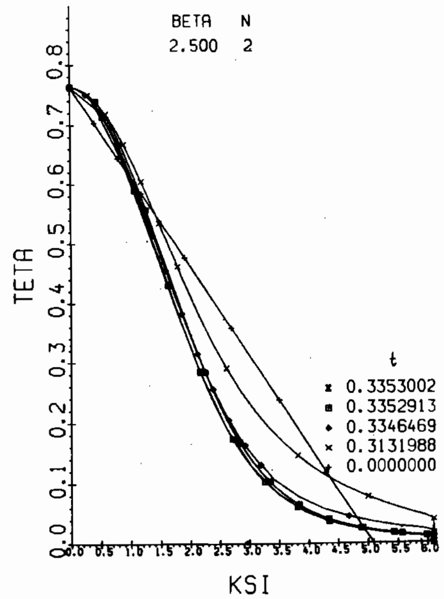

Fig. 11b

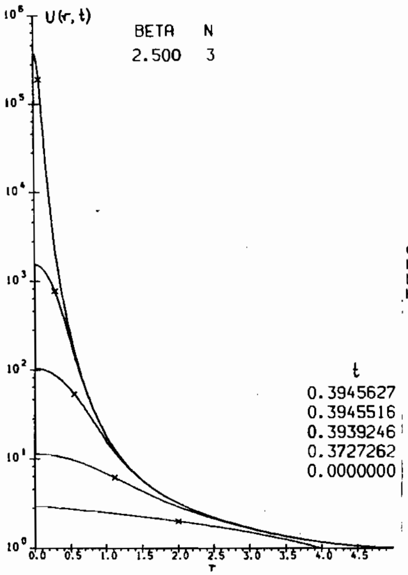

Fig. 12a

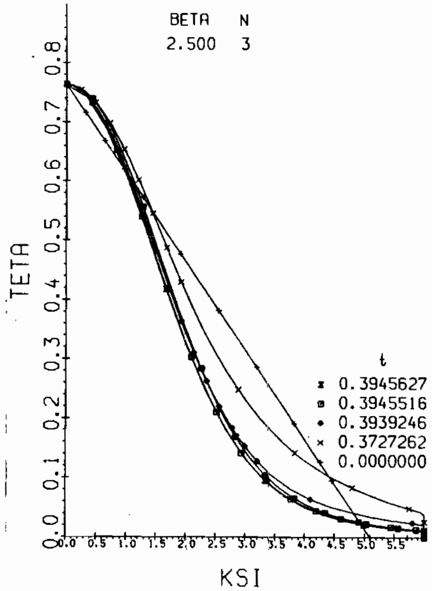

Fig. 12b

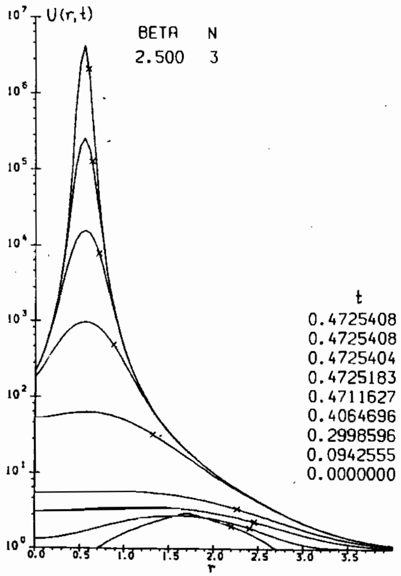

Fig. 13

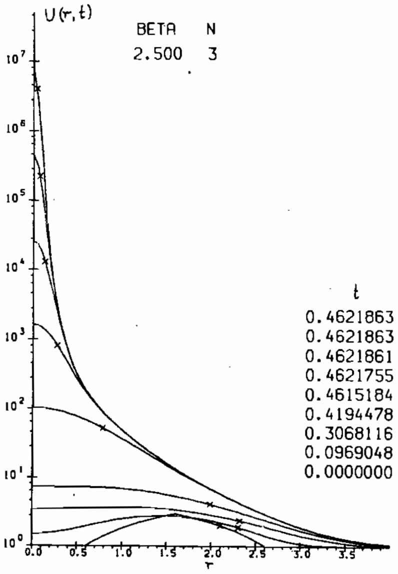

Fig. 14

4.3. Single point blow-up (*LS*-evolution)

It corresponds to $\beta>2$. Here

$$r_{ef}\longrightarrow 0 \quad \text{as} \quad t\longrightarrow T_0,$$

and there is an effective localization of the process [14],[22]. Our calculations for different values of β confirm (4.1) with $\theta_a(\xi)$ from (1.5),(2.7),(2.8) (Figures 11,12). Figures 13-15 show the evolution of some noncentral initial perturbations. Depending on their distance from the origin and on their energy they move to the origin or don't. Our efforts to find some conditions, under which the initial perturbation moves to the origin, fell. Figures 13,14 show one "limit" case - one and the same U_0 reaches the origin (Figure 14, d=1.6), and doesn't reach it (Figure 13, d =1.7). Figure 15 shows the interaction of two initial perturbations - they merge and degenerate into one central structure.

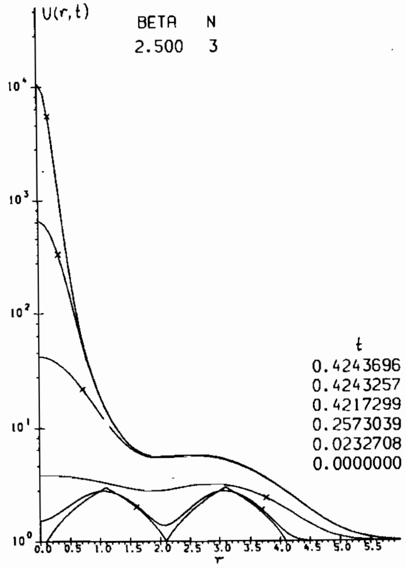

Fig. 15

Summary

The numerical work was undertaken in order to analyze the asymptotic behaviour near the finite blow-up time T_0 of the solutions of (1.10) for different N and $\beta>1$ and to confirm the degeneracy of (1.10) into (1.14) as $t\longrightarrow T_0$. It was shown that the asymptotic behaviour

doesn't depend on the initial data (if they have sufficiently large energy) and on the space dimension N. It depends on the parameter β only. The most exciting case is $\beta=2$, when the solution blows up in a region with diameter 2π.

The realized numerical method gives a possibility to analyze the semilinear equations mentioned above and we intend to do it.

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Димова С.Н., Галактионов В.А., Иванова Д.И. E11-89-785
Численный анализ режимов с обострением и вырождения для одного полулинейного уравнения теплопроводности

Вычислительным экспериментом исследовано асимптотическое поведение вблизи момента обострения T_0 решений уравнения

$$u_t = \frac{1}{r^{N-1}}(r^{N-1}u_r)_r + (1+u)\ln^{\beta}(1+u) \qquad (1)$$

для различных N и $\beta > 1$ и подтверждено вырождение уравнения (1) в нелинейное уравнение типа Гамильтона-Якоби

$$v_t = \frac{(v_r)^2}{1+v} + (1+v)\ln^{\beta}(1+v)$$

при $t \to T_0$. Показано, что асимптотическое поведение обостряющихся решений не зависит от начальных данных и от размерности пространства N. Оно зависит только от параметра β. Самый интересный случай - это $\beta = 2$, когда решение обостряется в области диаметра 2π. Для решения уравнения (1) использован метод конечных элементов.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1989

Dimova S.N., Galaktionov V.A., Ivanova D.I. E11-89-785
Numerical Analysis of Blow-up and Degeneracy of a Semilinear Heat Equation

Computational experiment was undertaken in order to analyze the asymptotic behaviour near the finite blow-up time T_0 of the solutions of

$$u_t = \frac{1}{r^{N-1}}(r^{N-1}u_r)_r + (1+u)\ln^{\beta}(1+u) \qquad (1)$$

for different N and $\beta > 1$ and to confirm the degeneracy of (1) into the nonlinear equation of Hamilton-Jacobi type

$$v_t = \frac{(v_r)^2}{1+v} + (1+v)\ln^{\beta}(1+v)$$

as $t \to T_0$. It was shown that the asymptotic behaviour of the blow-up solutions doesn't depend on the initial data and on the space dimension N. It depends on the parameter β only. The most exciting case is $\beta = 2$, when the solution blows up in a region with diameter 2π. The finite element method was used for solving equation (1).

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1989