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ALGEBRAIC-NUMERIC CALCULATIONS OF PROTON TRAJECTORIES IN BENDING MAGNETS OF SYNCHROTRON ACCELERATOR

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1. INTRODUCTION

The computational problems, arising from the theory of particle motion inside an accelerator are very complicated because of their nonlinear character. These problems are related to big requirements of accuracy of computation and a complicated structure of magnetic fields in a real installations.One of the most actual problems in this field is the problem of investigating beam charged particle trajectories in cyclic accelerators. The perspective of solving this problem is related to developing and using the highly precise mathematical and computational methods, and the realization of these methods on the powerful computers, including special parallel processors [1].

Nowadays there is a rapid growth of developing computer algebra methods and systems [2,3]. The computer algebra is a powerful method of computing with cumbersome algebraic manipulations and there are new possibilities of solving the complicated applied problems of the beam dynamics in accelerators. For example, in [4] the REDUCE realization of the Bogolubov-Krylov method of averaging is given.In this paper the precise investigation of influence of nonlinear resonances on the stability of charged particle motion is also proposed. We want to mention that this investigation is impossible when we use other methods. Another useful application of computer algebra REDUCE is given in [5]. In this paper the so called "betatron" integrals which appear in the perturbation theory of a particle motion are calculated. The authors also showed that the total computer accuracy is in a much shorter time by using analytical formulae than by resorting to a numerical integration.

Certainly, the analytical analysis of the motion of charged particles in cyclic accelerators is possible only in terms of perturbation theory. In these methods the computation of corrections of the linear approximation is possible. We also want to mention the practical usage of these formulas is related to the numeric computational problems. This means that we need the analytic-numerical interface.

In the present paper we study the problem of modeling a beam charged particle in the bending magnetic field of accelerator. There are many works devoted to this complicated problem (for example, see

[1,4-17]).In the section we give the mathematical formulation of our model.This model describes the betatron oscillations of a particle in a median plane of accelerator. We obtain and exactly solve the nonlinear ordinary differential equation which describes this motion of charged particle. Next we give the method of solution which is based on our proposition of existing the Laurent series. The realization of our algorithm on the REDUCE 3.2 computer algebra uses the known computer algebra procedures proposed by [13].The series which we obtain was investigated by using standard Fortran code. We also take into account that the problem is exactly solvable to estimate the numerical errors in the computation. The comparison with standard numerical methods is given.

2. THE MATHEMATICAL PRELIMINARIES

As a starting point we consider the motion of a single particle with impulse $\vec{p}=m\vec{v}$ and charge in a Lorentz field $\vec{F}=q/c\,[\vec{v}\times\vec{B}]$

$$\frac{d(m\vec{v})}{dt} = \vec{F}. \qquad (1)$$

In a Cartesian coordinate system $(x,y,s\equiv z)$, when we study the horizontal plane of particle motion s oscillations of a particle in a median plane the initial equation (1) may be given in the following form (nonlinear ordinary differential equation)

$$x''_{ss} = \frac{1}{B\rho}[-(1+x'^2_s)^{3/2}]\, B_y(x,y,s) \qquad (2)$$

with the initial conditions

$$x(s_o) = x_o, \qquad x'(s_o) = x'_o, \qquad (3)$$

where $B\rho$ is the particle rigidity ($B\rho = \frac{1}{qm}$), and

$$B_y(x,y,s) = B_1(1 + b_3 x^2 + b_5 x^4 + b_7 x^6 + \dots). \qquad (4)$$

In the last expression of the variables b_n (the relative amplitudes) n-th field oscillation related to the base dipole field B_1. Further we use the following relation

$$\frac{B_1}{B\rho} = \frac{1}{\rho}, \qquad (5)$$

where ρ is the radius of a bend of a particle in the homogeneous magnetic field $|(b_n=0$ for each $n>1)$.

The particle motion in the median plane (x,s) of the accelerator is described as follows. The single (alone) particle input in a magnetic aperture with angle θ_1 to the axes s, turn out on the curve with radius ρ (for all $b_n=0$) and output with angle θ_2. The particle dynamics is the union of the all possible trajectories of initial phase set $\{x_o,x_0'\}$. The problem of finding these sets $\{x_k,x_k'\}$ in the direction s is the basic goal of the present paper. The solution of equation (2) gives us the solution of problem stated above.

The linear case (all $b_n = 0$).

In this case, when the magnetic field in aperture of homogeneous field (in (4) all $b_n=0$), the equation (2), using (4) and (5) has the form

$$x''_{ss} = -\frac{1}{\rho}(1 + x'^2_s)^{3/2}. \qquad (6)$$

It is well known that (6) has the following solution

$$x(s) = \sqrt{\rho^2+(s-c_1)^2} + c_2 \qquad (7)$$

$$x'(s) = -\frac{s-c_1}{\sqrt{\rho^2+(s-c_1)^2}}$$

$$c_1 = \frac{x'_o\,\rho}{\sqrt{1+x'^2_o}}$$

$$c_2 = x_o - \frac{\rho}{\sqrt{1+x'^2_o}}.$$

3. THE METHOD OF SOLUTION

Let us introduce the standard substitution $\sigma(x)=x'$, which allows one to solve the problem (2) in a general case

$$B_y(x)=B_1(1+\sum_{j=1}^{\infty} b_{2j+1} x^{2j}) \qquad (8)$$

of the form

$$\rho \frac{\sigma\, d\sigma}{(1+\sigma^2)^{3/2}} = -[1+\sum_{j=1}^{\infty} b_{2j+1} x^{2j}]dx . \qquad (9)$$

Integrating (9) we obtain the standard problem of inversion

$$S + \int \frac{f}{[1-f^2]^{1/2}} dx - C_1 = 0, \qquad (10)$$

where

$$f= \frac{C_2}{\rho} - \frac{1}{\rho}(x+\sum_{j=1}^{\infty} \frac{b_{2j+1}}{2j+1} x^{2j+1}), \qquad (11)$$

C_1 and C_2 are constants of integration which are fixed by the initial conditions (3):

$$C_2 = [x_o+\sum_{j=1}^{\infty} \frac{b_{2j+1}}{2j+1} x_o^{2j+1}] - \frac{\rho}{\sqrt{1+x_o'^2}} \qquad (12)$$

and C_1 we will be obtained next (see (19)). Next we want to investigate the problem (10) $j \le 3$ which has the clear physical meaning. The reason is the following: the magnets which we study are optically pure enough. This fact is in a good comparison with experimental data.

As a solution of (10) we find S as a function of x. But our problem is to find the betatron oscillations i.e. we try to find x as a function of S

$$x(S) = \sum_{n=0}^{\infty} a_n S^n . \qquad (13)$$

Next we obtain the series (13) step by step. Expanding the expression (10) in a series in f^{-1} we have

$$\frac{1}{\frac{C_2}{\rho} - \frac{1}{\rho}[x+\sum_{j=1}^{\infty} \frac{b_{2j+1}}{2j+1} x^{2j+1}]} \equiv \sum_{n=0}^{N} A_n x^n . \qquad (14)$$

In the right side of (14) in general we have $N=\infty$ but in the practical use we have the concrete N fixed by the accuracy which we needed. The next step of our algorithm is to get the square of the series (14), i.e.

$$\frac{1}{f^2} = (\sum_{n=0}^{N} A_n x^n)^2 = \sum_{n=0}^{2N} A_{1,n} x^n \qquad (15)$$

next we extract 1 from the series (15) , i.e.

$$\frac{1}{f^2} - 1 = \sum_{n=0}^{2N} A_{2,n} x^n \equiv (A_{1,1}-1)+ \sum_{n=1}^{2N} A_{1,n} x^n . \qquad (16)$$

Finally we have

$$\frac{f}{\sqrt{1-f^2}} \equiv (f^{-2}-1)^{-1/2}=\sum_{n=0}^{M} A_{3,n} x^n , \qquad (17)$$

where $M \le 2N$.

Then the expansion of (10) in the series in x included the following operations: getting inverse of series (in the series which we obtain we get N terms), then we square the result and extract from them 1 and finally we get the minus square root of the result. After that we obtain the coefficients $A_{3,n}$ in (17), we integrate the result

$$\int \frac{f}{\sqrt{1-f}} dx \simeq \int [\sum_{n=0}^{M} A_{3,n} x^n]dx = \sum_{n=0}^{M+1} \tilde{A}_n x^n , \qquad (18)$$

where $\tilde{A}_{n+1}=-\frac{1}{n+1}A_{3,n}$ for n>0 and $\tilde{A}=C_1$. The last expression is fixed by S=0

$$C_1=\sum_{n=1}^{M+1} \tilde{A}_n\ x_o^n\ . \qquad (19)$$

Then the equation (10) has the form

$$S-C_1 = -\sum_{n=1}^{M+1} \tilde{A}_n\ x^n\ . \qquad (20)$$

The last step is the getting of inverse of (20) in terms of the generalized variable $\tilde{S} = S-C_1$:

$$x=\sum_{n=0}^{\infty} \tilde{\tilde{A}}_n\ \tilde{S}^n\ , \qquad (21)$$

Note, that from the last series (13) is easy to obtain.

Next we formulate the criteria of errors in the processes of calculation. Differentiating the series (13) in the variable s we obtain the exact result

$$x'=\sum_{n=1}^{\infty} n\ a_n\ S^{n-1}\ , \qquad (22)$$

which after the substitution in the left side of the expression of x', we have

$$x'= \frac{\sqrt{1-f}}{f}\ . \qquad (23)$$

In the right side of (23) we substitute the expression of x from (21). Then we have

$$\left|\ x' - \frac{\sqrt{1-f}}{f}\ \right| < \varepsilon\ . \qquad (24)$$

Note that in the process of calculation we get such a number of terms which corresponds to the fixed error in x. We also want to note that in comparison with standard numerical methods, for example with Runge-Kutta such calculations are impossible. Then we have a good reason of using the mixed methods i.e. analytic-numerical methods in the study of complicated problems. In section 5 we shall give the explicit comparison.

4. THE COMPUTER ALGEBRA PACKAGE

Our goal in this section is to describe the REDUCE 3.2/3.3 package and FORTRAN code of numerical investigation of proton beam dynamics.

In our calculations we use the CERN package [13] for working with generalized series.

Next we describe the computer algebra algorithm of solution of problem (2). We have two different cases - linear and nonlinear.

NONLINEAR CASE

Introduce the following notations

```
      AS:= f;     (see (11), when  j=3)
      AS1:= AS**(-2);
      AS2:= AS1-1;
      AS3:= AS2**(-1/2);
      SS1:= S-C1;      (SS1 ≡ -S+C1 = ∫ AS3 dx , C1 see (19)).
```

INPUT. SERIES

```
          AS:= CM1-AA1*X-AA3*X**3-AA5*X**5-AA7*X**7;
```

where CM1,AA1,AA3,AA5 and AA7 are constants.

OUTPUT. Inverse series.

```
          XS:= f(s)
```

as a function of s. This series we compute in the explicit analytical form. Next using the Fortran code we obtain the explicit numerical results.

The REDUCE 3.2 code we give in the Appendix.

LINEAR CASE

INPUT. `XL:= -(RO**2+(S-C1)**2)**(1/2)+C2;` ,

where C1 and C2 are defined in (7), and the constant R0 is given in (6).

OUTPUT. The expression XL is the Tailor series which we obtain using REDUCE procedure of getting Tailor expansion.

Next using the Fortran codes we calculate the nonlinear and linear cases.

THE DESCRIPTION OF THE COMPUTER ALGEBRA PACKAGE

The REDUCE procedures of operation on the generalized series (lines 1-164 of our program) are taken from [13].

The main procedures are given in the Appendix (line 165 - 239).

LINE	
165-176	the procedure ELRO1 give the realization of linear case of Taylor series of solution (7) i.e. we obtain the solution of the form (13).
177-206	procedure ELRO2 realizes the nonlinear case.
180	(see the expression (11)).
181-194	the useful operations on the series (11).
204	print of coefficients of the series (13).
207-239	the generation of FORTRAN code.
215-227	the generation of input numerical constants.
228-232	print of coefficients and numerical expressions of these (see (13)).
233-235	print of final results.

5. THE RESULTS

The main numerical results are given in the figure 1 where we show the function

$$\Delta(x_o) = x_K^N - x_K^L$$

of the initial coordinate x_o. Further all x'_o are const = $tg\frac{\theta}{2}$ and $\theta=7.5^o$, x_K^L is the solution x(s) in the linear case at the point $s = s_k = 43.16$ cm. (i.e. at the end of the magnet) ; x_k^N is the solution x(s) for the nonlinear case in the same point $s=s_k$. In Table 1 we show the function $\Delta(x_o)$ only in these cases when one of b_n is different from zero and the others are zeros.

The results which we obtain are in a good agreement with [14-16] where the Runge-Kutta (RK) method is used. The precision of calculation of the function x(s) in term of RK method is 10^{-12}. The precision of $x_k(s)$ in our method strongly depends on many mathematical and physical parameters for example the length of magnet,the length of the series (21)and so on. In our study (see the Fig. 2) the precision ε depends on the length of the series (21).

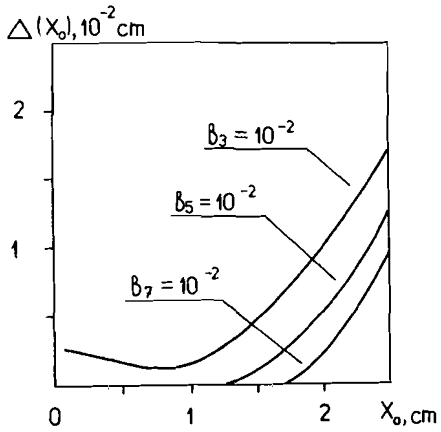

Fig.1

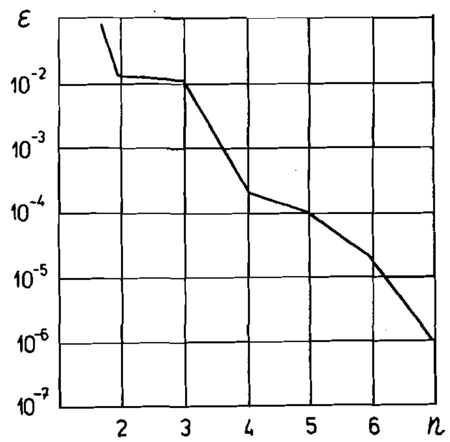

Fig.2

6. CONCLUSIONS

In our study of the charged particle dynamics in accelerators we show that analytic-numerical interface is very useful. In these calculations we have a profit in the comparison with the standard one. This is very remarkable in the case of the multyturn accelerations where the errors have a tendency of accumulation. Our results can be used in the concrete calculations of the beam dynamics of the nuclotron accelerator.The clear comparison of the methods of the solution of the problem which we stated in our paper we give in the future publication.

APPENDIX

```
165.    PROCEDURE ELRO1(NN);
166.    BEGIN SCALAR P1,P2,I,H1,AA,H2;
167.    ON DIV,RATIONAL; OFF ALLFAC,MCD;
168.    XL:=-(RO**2+(SS-C1)**2)**(1/2)+C2;
169.    WRITE "XL:=",XL;
170.    XT:=TLOR(XL,SS,NN); WRITE "XT:=",XT;
171.    P1:=COEFF(XT,SS,MZ);
172.    XP:=DF(XT,SS);
173.    ON ALLFAC,MCD,GCD; OFF EXP,DIV,RATIONAL;
174.    FOR I:=0:P1 DO WRITE "MZ(",I,"):=",MZ(I);
175.    RETURN P1
176.    END;
```

```
PROCEDURE ELRO2(NN);
BEGIN SCALAR I,I1,M,Y1,R1,N1,P3;
ON RATIONAL; OFF ALLFAC,MCD;
AS:=CM1-AA1*X-AA3*X**3-AA5*X**5-AA7*X**7;
WRITE "AS:=",AS;
Y1:=COEFF(AS,X,BZ);
IF NN<Y1 THEN ML:=NN ELSE ML:=Y1;
AS1:=POTPOW(ML,BZ,0,1-2,X);
AS2:=AS1-1; R1:=COEFF(AS2,X,DZ);
ON MCD,ALLFAC,GCD; OFF RATIONAL;
FOR I:=0:R1 DO WRITE "DZ(",I,"):=",DZ(I);
OFF GCD,ALLFAC,MCD; ON DIV,RATIONAL;
IF NN<R1 THEN PP:=NN ELSE PP:=R1;
LET X**(PP+1)=0;
AS3:=POTPOW(PP,DZ,0,1,-1/2,X);
N:=COEFF(AS3,X,CZ);
ON MCL,ALLFAC,GCD; OFF RATIONAL;
FOR I:=0:N DO WRITE "CZ(",I,"):=",CZ(I);

OFF GCD,ALLFAC,MCD; ON RATIONAL;
SS1:=-FOR I:=1:(N+1) SUM CZ(I-1)*X**I;
N1:=COEFF(SS1/X,X,EZ);
IF NN<N1 THEN PL:=NN-1 ELSE PL:=N1;
XS:=REVPOW(PL,EZ,1,1,SS);
N2:=COEFF(XS,SS,PZ);
ON MCD,ALLFAC,GCD; OFF RATIONAL;
XX1:=DF(XS,SS);
WRITE "XX1:=",XX1;
FOR I:=0:N2 DO WRITE "PZ(",I,"):=",PZ(I);
RETURN N2
END;

ARRAY BZ(50),MZ(50),PZ(50),AF(50),DZ(50),CZ(50),EZ(50);
ARRAY AL(50),SM1(50),SM2(50),PZ(50),BP(50);
%EXAMPLE 1;
VL2:=ELRO2(5);
OFF ECHO; ON FORT; OUT FORFIL;
WRITE "      DIMENSION PZ(5),SM2(5)";
WRITE "      REAL PZ(5)";
WRITE "    1 FORMAT(G20,12)";
WRITE "      RO=3299536./10000.";

WRITE "      X0=0.";
WRITE "      XX0=656./10000.";
WRITE "      SK=4316./200.";
WRITE "      A3=1./100.";
WRITE "      A5=0.";
WRITE "      A7=0.";
WRITE "      AA1=1./RO";
WRITE "      AA3=A3/(3.*RO)";
WRITE "      AA5=A5/(5.*RO)";
WRITE "      AA7=A7/(7.*RO)";
```



```
WRITE "      CC1=AA1*X0+AA3*X0**3+AA5*X0**5+AA7*X0**7";
WRITE "      CM1=CC1-1./(1.+XX0**2)**(1/2)";
FOR I:=0:5 DO WRITE "PZ(",I,"):=",PZ(I);
WRITE "      SM2(I)=PZ(1)";
WRITE "      DO 7 I=1,5";
WRITE "      SM2(I)=SM2(I-1)+PZ(I)*SK**I";
WRITE "    7 CONTINUE";
WRITE "      DO 5 I=1,5";
WRITE "      PRINT 1,SM2(I)";
WRITE "    5 CONTINUE";
WRITE "      END";
SHUT FORFIL;
OFF FORT;
END;
```

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Гердт В.П. и др. E11-89-755
Численно-аналитические вычисления траекторий протонов в поворотных магнитах ускорителя-синхротрона

Исследовано решение нелинейного дифференциального уравнения второго порядка, описывающего траектории заряженных частиц в существенно неоднородном поле ускорителя-синхротрона. Приведена математическая постановка проблемы. Разработан и реализован на языке аналитических вычислений REDUCE 3.2 алгоритм решения поставленной задачи. В основу алгоритма положено предположение о существовании решения в виде степенного ряда. Использованы REDUCE-процедуры, реализующие операции деления ряда на ряд, возведения ряда в произвольную, в том числе и дробно-отрицательную степень, а также операцию обращения ряда. Полученный степенной ряд использован в программе на языке ФОРТРАН для последующего численного анализа, а также для исследования поставленной физической задачи. Приведены результаты моделирования поведения пучка протонов в поворотных магнитах ускорителя-синхротрона.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Gerdt V.P. et al. E11-89-755
Algebraic-Numeric Calculations of Proton Trajectories in Bending Magnets of Synchrotron Accelerator

We study a solution of nonlinear differential equation of the second degree which describes the trajectories of the charged particles in the fully inhomogeneous field of cyclic accelerator. We give the clear mathematical statement of the problem and algorithm of solving it. We realize this algorithm on the Computer Algebra System REDUCE 3.2. Our algorithm is based both on the existence of exact solution in terms of hyperelliptic integral and on the existence of power series solution of specific inversion problem. We use the known REDUCE procedures of operation on generalized power series. Using the FORTRAN code we give the numerical analysis of these series in the close relation to the concrete physical situation. We apply our results to the beam dynamics modeling of the protons in the bending magnets in synchrotron accelerator.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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