объединенный
институт
ядерных
исследований
дубна

E11-89-442

M. Greguš, B.N. Khoromsky, G.E. Mazurkevich,
E.P. Zhidkov

ON APPROXIMATION OF NONLINEAR BOUNDARY INTEGRAL EQUATIONS FOR THE COMBINED METHOD

Submitted to XI International Conference
on Boundary Element Methods in Engineering, 1989
Cambridge, USA

1989

Грегуш М. и др. E11-89-442

Об аппроксимации нелинейных граничных интегральных уравнений для комбинированного метода

Рассматриваются нелинейные ГИУ, возникающие при решении нелинейных задач магнитостатики в комбинированной постановке для неограниченной области. На основе метода Галеркина изучаются аппроксимации возникающих операторных уравнений. Рассматриваемые граничные операторы обладают свойством сильной монотонности, Липшитц-непрерывности, потенциальности и имеют симметричную производную Гато. На основе этих свойств получены оценки погрешности галеркинских приближений в пространствах Соболева дробного порядка на соответствующих поверхностях. Рассмотрены двумерный и трехмерный случаи. Изучены вопросы сходимости итерационных процессов решения возникающих дискретизированных систем уравнений.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1989

Gregus̆ M. et al. E11-89-442

On Approximation of Nonlinear Boundary Integral Equations for the Combined Method

The nonlinear boundary integral equations that arise in research of nonlinear magnetostatic problems are investigated in combined formulation on an unbounded domain. Approximations of the derived operator equations are studied based on the Galerkin method. The investigated boundary operators are strongly monotone, Lipschitz-continuous, potential and have a symmetrical Gateaux derivative. The error estimates of the Galerkin's approximation in Sobolev spaces of fractional powers are obtained using the abovementioned properties of the operators, too. The problem has been studied on surfaces in two and three-dimensional spaces. We answer also some questions on convergence connected with the discretized systems of equations.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1989

1. INTRODUCTION

This paper is concerned with some problems that appear in the process of solving quasilinear elliptic equations in an unbounded domain with a bounded domain of nonlinearity. One of the problems is how to approximate solutions taking into account the boundary conditions at infinity. Different methods were devised to solve this problem [1-12]. One of the most general approaches consists in the coupling of the boundary element method and the finite element method [1,2,3]. Though, a number of variants exist in the frame of this concept.

We will discuss the questions of approximation of nonlinear operator equations for the trace of the unknown function on an auxiliary boundary (enveloping the domain of nonlinearity) by the Galerkin method. The equations are formed using a special class of the Poincare-Steklov operators [9]. We also mention the iterative methods of solving the discretized equations.The rate of convergence of the given iterative processes for the mentioned class of equations does not depend on the discretization step. At the end we give error estimates of the Galerkin approximations for some spline spaces that are defined on the selected auxiliary surface in R^n, n = 2,3.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

We suppose that $\Omega_1 \subset R^3$ is a bounded domain with Lipschitz boundary Γ_1 (it corresponds to the nonlinearity region), Ω is an auxiliary domain with Lipschitz boundary Γ and $\Omega_1 \subseteq \Omega$.

The function $\mu(x, t)$ is given and fulfills some or all of the following conditions (with $x \in \Omega_1$, $t, \tau \in [0, \infty)$):

$$\mu(x, t)t - \mu(x, \tau)\tau \geq m(t - \tau), \quad t \geq \tau, \quad m > 0, \tag{1}$$

$$|\mu(x, t)t - \mu(x, \tau)\tau| \leq M|t - \tau|, \tag{2}$$

$$\left|\frac{\partial}{\partial t}\mu(x, t)t\right| \leq M. \tag{3}$$

Let g_0 denote the Robin potential on the boundary Γ. We use the following function spaces: $W^1_{2, g_0}(\Omega) \equiv V$ is the subspace of functions $u \in W_2^1(\Omega)$ with the property that on Γ is their trace orthogonal to g_0, i.e. $(\gamma_0 u, g_0) = 0$, $W^{1/2}_{2,g_0}(\Gamma) \equiv X$ is the subspace of functions $u \in W_2^{1/2}(\Gamma)$ orthogonal to g_0 and $W^{-1/2}_{2,1}(\Gamma)$ is the subspace of functions $u \in W_2^{-1/2}(\Gamma)$ orthogonal to unity.

In analogy with the Dirichlet operator γ_0 /11/ the trace operator with the domain of definition V, $\gamma_{0,g_0}: V \to X$ is a linear continuous operator, too. Let us suppose that a linear operator $G_1 \in (X \to X^*)$ is selfadjoint and positively definite, i.e. $\forall u, v \in X$:

$$(G_1 u, v) = (u, G_1 v), \quad (G\ u, u) \geq m_{G_1} \|u\|_X^2, \quad m_{G_1} > 0. \tag{4}$$

In the boundary problem that we will investigate in general form, it is necessary to find a function $u \in \tilde{V}$, satisfying the integral identity:

$$\int_\Omega \sum_{i=1}^{3} a_i(x, w) \frac{\partial \eta}{\partial x_i} d\Omega + \alpha(G_1 \gamma_0 u, \gamma_0 \eta) = \int_{\Gamma_1} \psi \gamma_0 \eta ds + \beta \int_\Gamma q(s)\, \gamma_0 \eta(s) ds, \tag{5}$$

for arbitrary $\eta \in \tilde{V}$. The functions $\psi \in W^{-1/2}_{2,1}(\Gamma_1)$ and $q \in W^{-1/2}_{2,1}(\Gamma)$, and also the numbers $\alpha \geq 0$, $\beta \geq 0$ and coefficients $a_i(x, w)$, $w = \text{grad } u$, $i = 1, 2, 3$ are given. By $\tilde{V}$ we denote the space V, or the space $\dot{H}^1(\Omega)$.

If we set $\alpha = 1$, $\beta = 0$, $\tilde{V} = V$ and $G_1 = L^{-1}(E + K)$, with integral operators L, K defined by the formulas

$$Ku = \frac{1}{2\pi} \int_\Gamma \frac{\cos(r_{PM}, n_P)}{|r_{PM}|^2} u(P)\, d\sigma_P, \tag{6}$$

$$Lv = \frac{1}{2\pi} \int_\Gamma |r_{PM}|^{-1} v(P)\, d\sigma_P, \tag{7}$$

$P, M \in \Gamma$, and $|r_{PM}|$ is the length of the vector r_{PM} defined by the segment PM, then we obtain combined magnetostatic equations in general form. The solution of this problem can be harmonically extended onto all $\dot{R}^3$ under the condition that $|u(x)| \leq 0\left(\frac{1}{|x|}\right)$, $|x| \to \infty$. If we set $\alpha = 0$, $\beta = 1$, $\tilde{V} = V$, we get the Neumann problem, and for $\tilde{V} = \dot{H}^1(\Omega)$ we get the homogeneous Dirichlet problem. There holds the following theorem.

Theorem 2.1. Assume that $\mu(x, t)$ fulfills the conditions (1), (2). Then the boundary value problem (5) of the Dirichlet, Neumann, or combined ($\alpha = 1$, $\beta = 0$, $\tilde{V} = V$) type has a unique solution $u \in \tilde{V}$.

The nonlinear Poincare-Steklov operator $S: X^* \to X$ is defined by the relation /8,10/

$$(Sq, \eta) = (\gamma_{0,g_0} u, \eta) \quad \forall \eta \in X^*, \tag{8}$$

where γ_{0,g_0} is the trace of the solution of the Neumann problem (5) on Γ.

Theorem 2.2. Suppose there hold the conditions (1), (2). Then the operator $S^{-1}: X \to X^*$ is potential, Lipschitz-continuous and strongly monotone. If the condition (3) is fulfilled, the operator S^{-1} is Gateaus differentiable and there holds the estimate

$$((S^{-1})'(z) u, u) \leq M \|u\|_X^2, \tag{9}$$

and if the function $t \to \frac{\partial}{\partial t}(\mu(x, t)\, t)$ is continuous for almost all $x \in \Omega$, then the operator $(S^{-1})'$ is symmetric and positively definite.

3. THE DISCRETIZATION METHODS

The properties of the operator $G = (E + K)^{-1} L$ describes

Lemma 3.1. The norm $\|v\|_G^2 = (Gv, v)$, $v \in X^*$ is equivalent to the norm of the space X^*, the norm $\|u\|^2_{G^{-1}} = (G^{-1}u, u)$, $y \in X$ is equivalent to the norm $\|\cdot\|_X$, and the operator is symmetric.

For $\alpha = 1$, $\beta = 0$, $\tilde{V} = V$ the equation (5) is equivalent to the operator equation

$$\Phi u \equiv S^{-1} u + G^{-1} u = 0, \quad u \in X. \tag{10}$$

According to Theorem 2.2 Φ is Lipschitz-continuous and strongly monotone and therefore the equation (10) has a unique solution $u^* \in X$.

We will study a finite dimensional approximation of the operator equation (10), thus creating new equations, and formulate a theorem on the convergence of the iterative processes that solve the created equations.

Let $X_n \subset X$ be a linear subspace in X with the induced norm, and $h_1, \dots, h_n$ complete, linearly independent system of base functions in X_n. The operator $I_n \in L(X_n \to X)$ is the inclusion operator and it's adjoint is the operator $I_n^* \in L(X^* \to X_n^*)$. We will study a system of equations with Galerkin type solution $u_n \in X_n$:

$$(S^{-1} u_n, h_i) + (G^{-1} u_n, h_i) = 0, \quad i = 1, \dots, n, \tag{11}$$

that, following [9], can be written as an operator equation in X_n:

$$\Phi_n u_n = 0, \quad \Phi_n = I_n^* \Phi I_n, \quad \Phi_n : X_n \to X_n^*. \tag{12}$$

The identity $\|I_n u_n\| = \|u_n\|$ implies [9] that the properties of the operator Φ are transferred to Φ_n. It is not difficult to prove [7] the following assertion on the error estimate of the solution u_n:

Lemma 3.2. Assuming that the conditions (1), (2) are fulfilled, the equation (12) has a unique solution, such that there holds an estimate

$$\|u_n - u^*\|_X \le 3 \frac{M}{m} \inf_{v \in X_n} \|v - u^*\|_X . \tag{13}$$

Next we outline the iterative process to solve equation (12). Equivalent norms in X_n and X_n^* are defined [7] via the operator $I_n^* G^{-1} I_n = \mathcal{J}_n$.

Theorem 3.1. Suppose the conditions (1), (2) are fulfilled. Then for $\tau \in (0, 2M_\Phi^{-1})$ the iterational process

$$\mathcal{J}_n \left[\frac{u_{n,i} - u_{n,i-1}}{\tau} \right] = -\Phi_n u_{n,i-1}, \quad i = 1, 2, \dots \tag{14}$$

converges to the solution $u_n \in X$ of equation (12) at the rate

$$\|u_{n,i} - u_n\| \le \frac{\tau q^i}{1-q} \|\Phi u_{n,0}\|_{X^*},$$

where $q = \max\{1 - m_\Phi \tau, 1 - M_\Phi \tau\}$, for arbitrary initial approximation $u_{n,0} \in X_n$.

It is easy to see that for $\tau = 2(M_\Phi + m_\Phi)^{-1}$ we get $q(\tau) = (M_\Phi - m_\Phi)(M_\Phi + m_\Phi)^{-1}$. Here m_Φ and M_Φ are constants of strong monotonicity and Lipschitz-continuity of the operator Φ.

Remark 3.1. Since the operator S^{-1} is potential, equation (10) can be solved also by gradient methods (like the method of steepest descent of the method of conjugate gradients).

If we set $\mathcal{J}_n = \Phi_n'(u_{n,i_0})$, we get the modified Newton-Kantorovich method, and for $\tau = 1$, $\mathcal{J}_n = \Phi_n'(u_{n,i-1})$ we get the

Newton method. Local convergence of these methods follows from the properties of the operator Φ. Nonlocal convergence of newtonian processes is given by the next theorem [13].

Theorem 3.2. Suppose that the function $t \to \frac{\partial}{\partial t}(\mu(x,t)t)$, $t \in [0, \infty)$, is differentiable in t for almost all $x \in \Omega_i$. Then the continuous Newton method

$$\frac{\partial u}{\partial \tau} = -[\Phi_n'(u(\tau))]^{-1} \Phi_n(u(\tau)), \quad u(0) = u_0 \in X_n, \quad u(\tau) \in X_n, \tag{15}$$

converges to u^* for arbitrary initial approximation $u_0 \in X_n$.

Note that the use of equation (11) is connected with certain inconveniences in practice. Each part is therefore modified in such manner, that we obtain a constructive way of computing coefficients of the algebraic system (12). Pertaining to the first addend, it is possible to use finite-element analogue of the Poincare-Steklov operator S^{-1} for suitable triangulation of the domain Ω [18]. Operator G^{-1} is usually approximated by the collocation method and then the corresponding inner and outher equations are "glued" together in a certain set of points on the boundary Γ. However, the questions of convergence of approximations of this type for the combined problem still do not have strict theoretical foundations [2].

4. ERROR ESTIMATES

Here, specific error estimates of the Galerkin method for equation (11) are given assuming that the boundary Γ is a smooth surface. Suppose that for $n = 2$, or 3 the boundary Γ is a simply connected surface (manifold) of class C^∞, and $H^\sigma(\Gamma)$, $\sigma \ge 0$ is Sobolev space of fractional degree [11] σ.

The family of subspaces $X_n = S_h^{k,r}$, $n = 3$ (R^3) is defined [2] in accordance with the choice of regular family $S_h^{k,r} \subset H^r(\Gamma)$ of boundary finite-element spaces [20]. Letter h is a parameter of triangulation, and r denotes the smoothness of piece-wise polynomial elements of order $k-1$.

Note that the solution u^* of equation (11), $n = 2,3$, is an element of $H^\sigma(\Gamma)$, $\sigma \ge 1/2$. Approximation properties [2,20] of the system $S_h^{k,r}$, $n = 3$ imply

Theorem 4.1. For arbitrary $\ell, \sigma \in R$, such that $\ell \le r \le k$, $\ell \le \sigma \le k$ for $u^* \in H^\sigma$, $\sigma \ge 1/2$, there holds the estimate

$$\|u_n - u^*\|_{H^\ell(\Gamma)} \le c h^{\sigma - 1} \|u^*\|_{H^\sigma(\Gamma)}, \tag{16}$$

where u_n is a solution of equation (11), $n = 3$.

In the case $n = 2$, we use /14/ the family of spaces $X_n = S_h^d(\Omega_h)$ of boundary elements on Γ, corresponding to the space of 1-periodic, (d-1)-continuously differentiable splines of degree d on Γ, where the grid-region $\Omega_h = \{t_i \mid i = 0, \dots, N\}$ satisfies on Γ the condition $t_0 = t_N$ with a regular step $h = |t_{i+1} - t_i|$, $i = 0, \dots, N$. The approximation properties /14/ of the system $S_h^d(\Omega_h)$ imply the following:

Theorem 4.2. For arbitrary $t, \sigma \in R$ such that $\sigma \le d + 1$, $t \le \sigma$, $t \le d + 1/2$ there holds the estimate

$$\| u_n - u^* \|_{H^t} \le c h^{\sigma - t} \| u^* \|_{H^\sigma}, \tag{17}$$

where $u^* \in H^\sigma(\Gamma)$, and u_n is a solution of the equation (11).

Notice that in the case of piece-wise linear elements, i.e. $k = d + 1 = 2$, $r = 1$, for $u^* = H^2(\Gamma)$, i.e. $\sigma = 2$ there holds an estimate

$$\| u_n - u^* \|_{H^t} \le c h^{2-t} \| u^* \|_{H^2}, \tag{18}$$

where $0 \le t \le 3/2$ for $n = 2$ and $0 \le t \le 1$ for $n = 3$.

5. NUMERICAL EXPERIMENTS

As an illustration, results of a numerical experiment showing the convergence in h of the problem (10), defined on a boundary of a parallelepiped Π, are given /19/ using finite-differencies approximation of the operator S^{-1} and an approximation of K and L by the collocation method on piece-wise constant base functions, for $G^{-1} = L^{-1}(E + K)$. The boundary value problem in R^3, $X = (x, y, z) \in R^3$

$$\Delta u = \rho(X), \quad u(\infty) = 0, \quad u(X) = 0\left(\frac{1}{|X|^2}\right), \quad |X| \to \infty, \tag{19}$$

is transformed into the equation (10) defined on the boundary $\Gamma = \partial\Pi$. The function $\rho(X)$ is given by

$$\rho(X) = \begin{cases} C(X), & X \in \Pi_1 \subset \Pi = \{X \mid |x| \le 1.5,\ |y| \le 0.5,\ |z| \le 0.5\}, \\ 0, & X \in R^3 \setminus \Pi_1, \quad \int\limits_{\Pi_1} C(P)\, dP = 0 \end{cases}$$

and exact solution of (19) is $u^*(X) = \dfrac{1}{4\pi} \int \dfrac{C(P)}{|X - P|} dP$.

Numerical experiments were carried out on a sequence of three grids $h_1(8,8,8) \to h_2(16,16,16) \to h_3(32,32,32)$ on the boundary of a parallelepiped Π. The results of computations, given for $\Delta_{h_i} = \max\limits_{\partial\Pi} |u_{h_i} - u^*|$, $i = 1, 2, 3$ in the Table, clearly show the $O(h)$ approximation.

Table

h_k	8x8x8	16x16x16	32x32x32
Δ_{h_k}	0.0399	0.0195	0.0089

REFERENCES

1. Johnson C., Nedelec J.C. - Math.Comp., 1980, 35, No.152. p. 1063.
2. Wendland W.L. - On Asymptotic Error Estimates for Combined FEM and BEM. University of Stuttgart, Mathematisches Institut, A, preprint No.10, 1988.
3. Hsiao G.G. - In: Boundary Elements X, vol.1, ed.C.A.Brebbia, Springer-Verlag, 1988, p.431.
4. Troubridge C.W. - IEEE Transaction on Magn., 1982, Vol. Mag-18, No.1, p.293.
5. Il'in V.P. - Numerical Methods in Electrophysical Problems (in Russian). Moscow, Nauka, 1985.
6. Serdyukova S.I. - JINR, P5-84-718, Dubna, 1984, p.7.
7. Gajevski H., Gröger K., Zacharias K. - Nichtlineare operatorgleichungen und operatordifferentialgleichungen, Berlin, Academia-V, 1974.
8. Zhidkov E.P., Mazurkevich G.E., Khoromsky B.N. - JINR Communication P11-87-501, Dubna, 1987.
9. Agoshkov V.I., Lebedev V.I. - In: Numerical Processes and Systems, Vol.2, ed.G.I.Marchuk, Moscow, Nauka, 1985, p. 137.
10. Kuznetsov S.B. - Preprint of Comp.Centre of Acad.Sc.USSR, No.111, Novosibirsk, 1985.
11. Lions J.L., Magenes E. - Nonhomogeneous Boundary Value Problems and Applications. Moscow, Mir, 1971.
12. Zhidkov E.P., Khoromsky B.N. - Soviet J.Numer.Anal.M.M., 1987, vol.2, No.6, p.463.
13. Zhidkov E.P., Khoromsky B.N. - JINR Preprint P5-8244, Dubna, 1974.
14. Ruotsalainen K., Wendland W. - Numer.Math., 1988, Vol.53, p. 299.

15. Gregus M., Zhidkov E.P., Khoromsky B.N. - JINR Preprint E11-87-585, Dubna, 1987.
16. Gregus M. et al. - JINR Preprint E11-88-481, Dubna, 1988.
17. Zhidkov E.P., Khoromsky B.N. - JINR, P11-83-261, Dubna, 1983.
18. Khoromsky B.N. - JINR, P11-88-480, Dubna, 1988; JINR, P11-88-784, Dubna 1988.
19. Zhidkov E.P., Mazurkevich, Khoromsky B.N. - JINR Communication P11-86-333, Dubna 1986.
20. Babuska I., Aziz A.K. - In: The Mathematical Foundation of the Finite Element Method with Applications to Partial Differential Equations, ed.A.K.Aziz, New York, Academic Press, 1972.
21. Kral J. - Integral Operators in Potential Theory. Lecture Notes in Methematics, No.823, Berlin, Heidelberg, New York, Springer-Verlag, 1980.

Received by Publishing Department
on June 16, 1989.