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ITERATIVE METHODS OF DOMAIN DECOMPOSITION WITH CROSS-POINTS FOR THE SOLUTION OF DISCRETE ELLIPTIC PROBLEMS

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In the process of solving elliptic boundary value problems by domain decomposition one can distinguish two main stages /17/:

i. solution of independent problems in subdomains (that can be done in parallel);and

ii. solution of a problem on the separator lines(surfaces),which arises from the conditions for the behaviour of unknown function and its conormal derivatives on the boundaries of subdomains(the latter,in its discrete variant,is called sometimes *capacitance matrix equation* /6,7,17/). The second stage is the most difficult one and is accomplished by iterative methods,usually by the Preconditioned Conjugate Gradient (PCG) method. The problem of the construction of preconditioners in case of box-decomposition (the domain is partitioned by lines or surfaces with cross-points into the great number of subdomains) and finite element approximation of second order elliptic equations have been discussed in /5,6,7,9,10,17/ and see also literature cited there.

We shall consider the problem of the construction of effective preconditioners in the case of finite difference approximation of elliptic operators in the model boundary value problem:a rectan - gular region in $\mathbf{R}^n$,n=2,3,is partitioned by vertical and horizontal lines into $\sim m^2$ (in three-dimensional problem $\sim m^3$) subdomains. In each subdomain the value of elliptic operator coefficients are constants,which can differ from each other by several orders for different subdomains. To formulate the problem for unknowns φ on the boundaries of subdomains (*capacitance matrix problem*)

$$\mathbb{A}\ \varphi = \psi \tag{0.1}$$

and construct preconditioner $\mathbb{B}$ for matrix $\mathbb{A}$ we use discrete analogues of Poincaré-Steklov operators/12/. Poincaré-Steklov operators have been used in the analysis of convergence properties of the domain decomposition iterative methods when region is partitioned into strips in /1,13-15,11/. The discrete analogs of Poincare-Steklov operators and their applications have been studied in /2,3,10,15,18/. Some multigrid methods with Poincare-Steklov operator for the discrete solution of elliptic problems is discussed in /12/.

The main result of this work is given in Theorems 4 and 5 where the condition number $\mathbb{K}(\mathbb{B}^{-1}\mathbb{A})$ dependence on elliptic problem parameters is discussed, and can be summarized as follows: the convergence properties of iterative methods for the solution of (0.1) with discussed preconditioners are determined by $(\mathbb{N}/m)$, where $\mathbb{N}$ is the mean number of unknowns in one direction, and convergence properties are independent on jumps of elliptic operator coefficients as long as these jumps only occur across the subdomain boundaries. For the condition number $\mathbb{K}$ of matrix $\mathbb{B}^{-1}\mathbb{A}$ for two-dimensional problem there is an estimate

$$\mathbb{K} \le \mathbb{C}(1+\ln(\mathbb{N}/m))^2 ,$$

for three-dimensional problem -

$$\mathbb{K} \le \mathbb{C}(\mathbb{N}/m)(1+\ln(\mathbb{N}/m))^2 .$$

The discussed preconditioners $\mathbb{B}$ can also be used for the solution of elliptic problems when matrix $\mathbb{A}$ from (0.1) corresponds to the elliptic operator with variable coefficients in subdomains. To do this it is necessary that the following condition holds true

$$c_1\tilde{\mathbb{A}} \le \mathbb{A} \le c_2\tilde{\mathbb{A}} ,$$

here $c_1>0$ and $c_2<\infty$, $\bar{\mathbb{A}}$ corresponds to the elliptic operator with constant coefficients in subdomains.

1. FORMULATION OF THE PROBLEM AND SOME PRELIMINARY DEFINITIONS

Let us consider on plane rectangle Π with boundary $\partial\Pi$, which is partitioned by (m_1-1) vertical and by (m_2-1) horizontal lines into $p=m_1 m_2$ subdomains Ω_{ij} which are rectangles with sides a_1^i, a_2^j. These lines form internal boundaries G of subdomains Ω_{ij}, $i=1\div m_1$, $j=1\div m_2$.

We shall consider the solution of the finite difference analogue of the following problem :

$$\begin{aligned}
&-\mu_{ij}\Delta\tilde{W} = 0 && x\in\Omega_{ij},\ i=1\div m_1,\ j=1\div m_2 \\
&[\tilde{W}]=0,\ [\mu\partial\tilde{W}/\partial n]=\tilde{\psi}, && x\in G \qquad\qquad (1.1)\\
&\tilde{W}=0 && x\in\partial\Pi
\end{aligned}$$

As $[\cdot]$ we denote the jumps of the unknown function and its conor - mal derivatives. Suppose, that μ_{ij}=const>0 in Ω_{ij}, $i=1\div m_1$, $j=1\div m_2$.

To approximate differential equations in (1.1) we use a stan - dard five-point centered difference scheme on rectangular grid

"with displacement on h/2"/15,16/.(In each subdomain Ω_{ij} we use a uniform mesh with a grid size $h_1^i=a_1^i/N_1^i$, $h_2^j=a_2^j/N_2^j$ with N_1^i, N_2^j internal grid points in x- and y-directions respectively, and with boundary nodes displaced on $h_1^i/2$ or on $h_2^j/2$ relative the subdomain boundary $\partial\Omega_{ij}=\bigcup_{k=1}^{4} G_{ij}^k$, here G_{ij}^k are sides of rectangle Ω_{ij}). As Ω_{ij}^h we denote the union of internal node set $\mathring{\Omega}_{ij}^h$ and nodes which go out of subdomain boundary on h/2;
as $\partial\Omega_{ij}^h=\bigcup_{k=1}^{4}\Gamma_{ij}^k$ we denote the union of the points on $\partial\Omega_{ij}$ which are in the middle of corresponding nodes. Respectiveiy as $\partial\Pi^h$ and Γ we denote a mesh on external $\partial\Pi$ and internal G boundaries. As γW we denote a trace of a gridfunction W on Γ or on $\partial\Pi^h$ or on $\partial\Omega_{ij}^h$ respectively - it is an arithmetic mean value of two nodal layers $W_{\Gamma+h/2}$ and $W_{\Gamma-h/2}$ between which boundaries are situated:

$$\gamma W=(W_{\Gamma+h/2}+W_{\Gamma-h/2})/2 \;;$$

as $\Delta W/\Delta n$ we denote an outward normal derivative of gridfunction W:

$$\Delta W/\Delta n=(W_{\Gamma+h/2}-W_{\Gamma-h/2})/h \;.$$

Then we approximate (1.1) by the system :

$$\begin{aligned} &-\mu_{ij}\Delta_h W=0 && \text{on } \mathring{\Omega}_{ij}^h,\ i=1\div m_1,\ j=1\div m_2 \\ &[\gamma W]=0,\quad [\mu\ \Delta W/\Delta n]=\psi, && \text{on } \Gamma \\ &\gamma W=0 && \text{on } \partial\Pi^h \end{aligned} \qquad (1.2)$$

Here Δ_h corresponds to the discrete Laplacian, ψ is the projec - tion of the given function $\tilde{\psi}$ on the set of nodes on Γ.

We shall also consider the problem (1.1) in the case of three dimensions - in parallelepiped Π with boundary $\partial\Pi$ partitioned into $p=m_1 m_2 m_3$ subdomains, and its discrete analog on the rectangular grid "with displacement on h/2" (in each subdomain we use a uniform mesh Ω_{ijr}^h with grid size $h_1^i=a_1^i/N_1^i$, $h_2^j=a_2^j/N_2^j$, $h_3^r=a_3^r/N_3^r$)

It is convenient to analyse methods for the solution of (1.2) with the help of Poincaré-Steklov inverse operators /1,12÷15,11/. We shall briefly describe the discrete analogs of those operators as have been done in /15/.

Consider the Dirichlet problem in one of subdomains (for the simplicity - $h_1^i=h_2^j=1/N$); see figure 1:

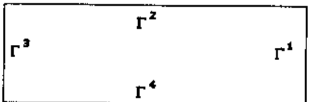

Fig.1

$$\begin{aligned} &-\Delta_h W=0 \qquad \text{in } \mathring{\Omega}^h \\ &\gamma W=[\varphi^1,\varphi^2,\varphi^3,\varphi^4]^T\equiv\varphi \ \text{ on } \partial\Omega^h=\bigcup_{k=1}^{4}\Gamma^k \end{aligned} \qquad (1.3)$$

Let gridfunction W be the solution of the problem (1.3). Let us find $V \equiv \Delta W/\Delta n = [\Delta W/\Delta n_1, \Delta W/\Delta n_2, \Delta W/\Delta n_3, \Delta W/\Delta n_4]^T$ and define operator S^{-1} by :

$$V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{vmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{vmatrix} \begin{bmatrix} \varphi^1 \\ \varphi^2 \\ \varphi^3 \\ \varphi^4 \end{bmatrix} \equiv S^{-1}\varphi . \qquad (1.4)$$

Matrix S^{-1} is the discrete analog of Poincaré-Steklov inverse operator. In this case it is easy to obtain formulas for evalua - tion of the elements of the matrices P_{ij} which form S^{-1}, but below we shall need only the elements of the diagonal blocks P_{ii}. They can be found by the solution of the problem (1.3), for example, with $\varphi = [\varphi^1, 0, 0, 0]^T$. Diagonal matrices P_{ii} have the following representation /15/:

$$P_{ii} = U_N^T \Lambda U_N , \qquad U_N = \left(u_{kl} = \sqrt{2/N}\, \sin\frac{\pi k(l-1/2)}{N} ;\ k,l = 1 \div N\right)$$
$$\Lambda = \mathrm{diag}\left\{ \lambda_k = 2N \frac{\beta_k - 1}{\beta_k + 1}\, \frac{\beta_k^N + \beta_k^{-N}}{\beta_k^N - \beta_k^{-N}} ;\ k = 1 \div N \right\} \qquad (1.5)$$
$$\beta_k = 1 + 2\alpha + 2\sqrt{\alpha + \alpha^2}, \qquad \alpha = \sin^2 \frac{\pi k}{2N}.$$

Here U_N is a matrix of Fast Fourier Transform (FFT). In the case if subdomain Ω is parallelepiped, block dimension of S^{-1} in (1.4) is equal to six and the diagonal blocks have representation:

$$P_{ii} = U^T \Lambda U, \qquad U = U_N * U_N$$
$$\Lambda = \mathrm{diag}\left\{ \lambda_{kl} = 2N \frac{\beta_{kl} - 1}{\beta_{kl} + 1}\, \frac{\beta_{kl}^N + \beta_{kl}^{-N}}{\beta_{kl}^N - \beta_{kl}^{-N}}, \quad k,l = 1 \div N \right\} \qquad (1.5')$$
$$\beta_{kl} = 1 + 2\alpha + 2\sqrt{\alpha + \alpha^2}, \qquad \alpha = \sin^2 \frac{\pi k}{2N} + \sin^2 \frac{\pi l}{2N}.$$

In (1.5') '*' designates tensor multiplication of the matrices, U_N is defined in (1.5).

Properties of the operator S^{-1} result from its functional de - finition /15/:

$$(S^{-1}\gamma W, \gamma V) = D(W, V), \qquad (1.6)$$

where $D(\cdot,\cdot)$ is quadratic form (discrete analog of the Dirichlet form) which is given by, see /15/,

$$D(W,V)=\sum_{k=1}^{N}\sum_{l=1}^{N}\left(\frac{W_{k+1l}-W_{kl}}{h_1}\frac{V_{k+1l}-V_{kl}}{h_1}+\frac{W_{kl+1}-W_{kl}}{h_2}\frac{V_{kl+1}-V_{kl}}{h_2}\right)h_1 h_2+ \tag{1.7}$$

$$+\frac{1}{2}\left(\frac{\Delta W}{\Delta n},\frac{\Delta V}{\Delta n}\right).$$

From properties of $D(\cdot,\cdot)$ it follows that operator S^{-1} is symmetric, non-negative definite in $L_2^h(\partial\Omega)$ and

$$\mathrm{Ker}S^{-1}=\{\gamma W=\mathrm{const}\ \mathrm{on}\ \partial\Omega\}.$$

The form $D(\cdot,\cdot)$ for two- and three-dimensional problems has one easily verified property which will be useful below:

$$D(W_1+W_2,W_1+W_2)\le C(D(W_1,W_1)+D(W_2,W_2)) \tag{1.8}$$

here C is independent of h_1,h_2,h_3.

Now consider "black and white" partitioning of the initial domain $\Gamma=\Omega_B\cup\Omega_W$ in $\mathbf{R}^n$, $n=2,3$, where

$$\Omega_B=\bigcup_{i+j-\mathrm{even}}\Omega_{ij},\quad \Omega_W=\bigcup_{i+j-\mathrm{odd}}\Omega_{ij}\quad \text{in } \mathbf{R}^2$$

$$\Omega_B=\bigcup_{i+j+r-\mathrm{even}}\Omega_{ijr},\quad \Omega_W=\bigcup_{i+j+r-\mathrm{odd}}\Omega_{ijr}\quad \text{in } \mathbf{R}^3.$$

Further we introduce one-dimensional subscription and as $\mathcal{I}_B$ denote the set of subscripts 'l' for which $\Omega_l\in\Omega_B$, in the same manner we define the set $\mathcal{I}_W$.

In such subscription the grid Γ on internal boundary G has representation:

$$\Gamma=\bigcup_{l\in\mathcal{I}_B}\Gamma_l \qquad \text{or} \qquad \Gamma=\bigcup_{l\in\mathcal{I}_W}\Gamma_l, \tag{1.9}$$

where $\Gamma_l=\bigcup_{k=1}^{q}\Gamma_l^k$, Γ_l^k is the net domain on the side of rectangle Ω_l in two-dimensional case, or Γ_l^k is the net domain on the side of parallelepiped Ω_l in three-dimensional case; $q=4$ in two-dimensional case ($q=6$ in three-dimensional) if Ω_l is "internal" subdomain, i.e. there are no common points among the boundary $\partial\Omega_l$ of Ω_l and the boundary $\partial\Pi$ of initial domain Π; $q=3$ or $q=2$ ($q=5$ or $q=4$ or $q=3$) if Ω_l is "boundary" subdomain, i.e. some sides of Ω_l are on the boundary $\partial\Pi$.

A direct sum of a finite-dimensional spaces $\mathcal{V}_k$ we denote as $\mathcal{V}=\sum_k\oplus\mathcal{V}_k$, a vector φ which belongs to that sum - $\varphi=\sum_k\oplus\varphi_k$, $\varphi_k\in\mathcal{V}_k$, $\|\varphi\|_{\mathcal{V}}=\sum_k\|\varphi_k\|_{\mathcal{V}_k}$.

In each subdomain Ω_l we introduce the space $\tilde{\mathcal{V}}(\Omega_l^h)$ of h-harmonic functions V_l, i.e. $\Delta_h V_l=0$ in Ω_l^h. We shall say that some grid -

function(vector) $V\in\mathcal{V}(\Pi^h)$ if $V=\sum_{l=1}^{p}\oplus V_l$, $V_l\in\mathcal{V}(\Omega_l^h)$, $\mathcal{V}(\Omega_l^h)\subset\bar{\mathcal{V}}(\Omega_l^h)$ and $[\gamma V]=0$ on Γ and $\gamma V=0$ on $\partial\Pi^h$. The set of traces γV on Γ of functions from $\mathcal{V}(\Pi^h)$ with $L_2^h(\Gamma)$ inner product we denote $\mathcal{X}(\Gamma)$.

Each element $V_l\in\mathcal{V}(\Omega_l^h)$ can be represented as

$$V_l=\sum_{k=1}^{q} V_l^k, \qquad \text{where} \qquad \gamma V_l^k=\begin{cases}\varphi_l^k & \text{on } \Gamma_l^k\\ 0 & \text{on } \Gamma_l^i,\ i\neq k\ .\end{cases} \tag{1.10}$$

Then, $\mathcal{X}(\Gamma)=\sum_{l\in\mathcal{I}_B}\oplus\mathcal{X}(\Gamma_l)$, $\mathcal{X}(\Gamma_l)=\sum_{k=1}^{q}\oplus\mathcal{X}(\Gamma_l^k)$; $\mathcal{X}(\Gamma_l)$ consists of the elements $\varphi_l=\gamma V_l$, $V_l\in\mathcal{V}(\Omega_l^h)$; $\mathcal{X}(\Gamma_l^k)$ consists of non-zero components φ_l^k of the trace γV_l^k, and respectively each element $\varphi\in\mathcal{X}(\Gamma)$ is $\varphi=\sum_{l\in\mathcal{I}_B}\oplus\varphi_l$, $\varphi_l=\sum_{k=1}^{q}\oplus\varphi_l^k$. In accordance with (1.9) we introduce operator of permutations $\mathbb{T}$ such that $\mathbb{T}^T\mathbb{T}=\mathbb{E}$, $\mathcal{X}(\Gamma)\xrightarrow{\mathbb{T}}\mathcal{X}'(\Gamma)$, where $\mathcal{X}'(\Gamma)=\sum_{l\in\mathcal{I}_V}\oplus\mathcal{X}(\Gamma_l)$.

Let us introduce operators $\mathbb{S}^{-1}=\sum_{l\in\mathcal{I}_B}\oplus\mu_l\mathbb{S}_l^{-1}$ and $\mathbb{R}^{-1}=\sum_{l\in\mathcal{I}_V}\oplus\mu_l\mathbb{S}_l^{-1}$, here $\mathbb{S}_l^{-1}$ is defined in (1.4). It must be mentioned that matrices $\mathbb{S}_l^{-1}$ have block dimension q in accordance with definition of internal boundary Γ_l, see above. Consider the system of algebraic equations from which the unknown vector $\varphi\in\mathcal{X}(\Gamma)$ must be found:

$$\mathbb{A}\ \varphi\ \equiv\ \mathbb{S}^{-1}\varphi\ +\ \mathbb{T}^T\mathbb{R}^{-1}\mathbb{T}\varphi\ =\ \psi\ , \tag{1.11}$$

where ψ is taken from (1.2). Let us determine properties of $\mathbb{A}$ from (1.11), to do this we shall follow /2,15/.

Lemma 1. Matrix $\mathbb{A}$ is symmetrical and positive definite in $\mathcal{X}(\Gamma)$.

The proof of symmetry of $\mathbb{A}$ is based on the properties of $D(\cdot,\cdot)$ in (1.6),(1.7):
for each $U,V\in\mathcal{V}(\Pi^h)$, $\gamma U,\gamma V\in\mathcal{X}(\Gamma)$ we have

$$\begin{aligned}(\mathbb{A}\gamma U,\gamma V)&=\sum_{l\in\mathcal{I}_B}\mu_l\,(\mathbb{S}_l^{-1}\gamma U_l\,,\gamma V_l\,)+\sum_{l\in\mathcal{I}_V}\mu_l\,(\mathbb{S}_l^{-1}\gamma U_l\,,\gamma V_l\,)=\\&=\sum_{l\in\mathcal{I}_B}\mu_l\,D_l\,(U_l\,,V_l\,)\ +\sum_{l\in\mathcal{I}_V}\mu_l\,D_l\,(U_l\,,V_l\,)\ =\ (\gamma U,\mathbb{A}\gamma V)\ .\end{aligned} \tag{1.12}$$

Positive definiteness follows from inequality

$$\min_{l=1\div p}\mu_l\cdot(\mathbb{A}_\Delta\gamma U,\gamma U)\ \le\ (\mathbb{A}\gamma U,\gamma U)\ \le\ \max_{l=1\div p}\mu_l\cdot(\mathbb{A}_\Delta\gamma U,\gamma U)\ ,$$

where $\mathbb{A}_\Delta$ is operator from (1.11) under the condition that $\mu_l=1$, $l=1 \div p$, with easily verifying properties

$$\mathbb{A}_\Delta=\mathbb{A}_\Delta^*;\ \mathrm{Ker}\mathbb{A}_\Delta=0;\ \mathbb{A}_\Delta\geq\alpha\mathbb{E},\ \alpha>0.$$

Now let us assume that in (1.2) function ψ is given in such a way that system (1.2) is solvable, i.e. for each $V\in\mathcal{V}(\Pi^h)$ holds true

$$(\psi,\gamma V)=\sum_{l=1}^{p}\mu_l D_l(W_l,V_l)\ . \qquad (1.13)$$

Theorem 1. The solutions of (1.2) and (1.11) are equivalent, i.e. if $W\in\mathcal{V}(\Pi^h)$ is the solution of (1.2), then $\varphi=\gamma W\in\mathcal{X}(\Gamma)$ is the solution of (1.11) and vice versa, if $\varphi\in\mathcal{X}(\Gamma)$ is the solution of (1.11), then there exists $W\in\mathcal{V}(\Pi^h)$ solution of (1.2) such that $\gamma W=\varphi$ on Γ.

The proof in one direction is obvious because the system (1.11) is non other than different record of the conditions on Γ from (1.2).

Let $\varphi\in\mathcal{X}(\Gamma)$ be solution of (1.11). Solving Dirichlet problems in each subdomain Ω_l with φ_l as boundary condition on $\partial\Omega_l$ we find gridfunctions W_l such that $\gamma W_l=\varphi_l$ and for each $U\in\mathcal{V}(\Pi^h)$ holds true

$$(\gamma U_l,\mu_l\frac{\Delta W_l}{\Delta n})=\mu_l D_l(U_l,W_l),\ l=1\div p.$$

Summing these expressions we obtain:

$$\sum_{l\in\mathcal{I}_B}(\gamma U_l,\mu_l\frac{\Delta W_l}{\Delta n})+\sum_{l\in\mathcal{I}_W}(\gamma U_l,\mu_l\frac{\Delta W_l}{\Delta n})=\sum_{l\in\mathcal{I}_B}(\gamma U_l,[\mu\frac{\Delta W}{\Delta n}]_l)=$$

$$=\sum_{l=1}^{p}\mu_l D_l(U_l,W_l).$$

On the other hand, from (1.11) $(\mathbb{A}\varphi,\gamma U)=\sum_{l=1}^{p}\mu_l D_l(U_l,W_l)=(\psi,\gamma U)$.

Comparing these expressions we obtain $\sum_{l\in\mathcal{I}_B}\oplus[\mu\frac{\Delta W}{\Delta n}]_l=\psi$. That proves theorem 1.

2. THE CONSTRUCTION OF PRECONDITIONERS

For the approximate solution of the system (1.11) let us consider an iterative scheme:

$$\mathbb{B}\,\frac{\varphi_{n+1}-\varphi_n}{\tau_{n+1}}+\mathbb{A}\,\varphi_n=\psi\ .$$

In our case the choice of a particular iterative method which is defined by the choice of iterative parameters τ_n is not essential. For the purpose of this exposition we may think of PCG method /8/.

The importance of making a "good" choice for preconditioner $\mathbb{B}$ is well known. $\mathbb{B}$ should have two properties:

a) operator $\mathbb{B}$ should be easily invertable, i.e. expenditures to evaluate $\mathbb{B}^{-1}\psi$ should be much smaller than those to evaluate $\mathbb{A}^{-1}\psi$;

b) operator $\mathbb{B}$ should be spectrally close to $\mathbb{A}$ in the sense that condition number $\mathbb{K}$ of $\mathbb{B}^{-1}\mathbb{A}$ should not be large. Clearly, $\mathbb{K}\le\frac{\alpha_2}{\alpha_1}$, where α_1 and α_2 are constants such that

$$\alpha_1(\mathbb{B}\varphi,\varphi) \le (\mathbb{A}\varphi,\varphi) \le \alpha_2(\mathbb{B}\varphi,\varphi) \quad \text{for all } \varphi\in\mathcal{X}(\Gamma). \qquad (2.1)$$

These two properties will guarantee that the work per iterative step in applying preconditioned method will be small, and that the number of steps to reduce the error to a given size will be also small.

To construct such preconditioner $\mathbb{B}$ we decompose $\mathcal{X}(\Gamma)$ on $\mathcal{X}_L(\Gamma)$ and $\mathcal{X}_0(\Gamma)$ so that each function $\varphi\in\mathcal{X}(\Gamma)$ can be uniquely represented as $\varphi=\varphi_0+\varphi_L$, where $\varphi_0\in\mathcal{X}_0(\Gamma)$, $\varphi_L\in\mathcal{X}_L(\Gamma)$. The expediency of such de - composition will be obvious from the below exposition when the examples of the choice of $\mathcal{X}_L(\Gamma)$ and $\mathcal{X}_0(\Gamma)$ will be given.

For all $\varphi,\upsilon\in\mathcal{X}(\Gamma)$ holds true

$$(\mathbb{A}\varphi,\upsilon)=(\mathbb{A}\varphi_0,\upsilon_0)+(\mathbb{A}\varphi_L,\upsilon_L)+2(\mathbb{A}\varphi_0,\upsilon_L)$$

and as preconditioner let us define operator $\mathbb{B}$ such that

$$(\mathbb{B}\varphi,\upsilon)=(\mathbb{B}_0\varphi_0,\upsilon_0)+(\mathbb{A}\varphi_L,\upsilon_L), \qquad (2.2)$$

where $\mathbb{B}_0$ is block-diagonal matrix

$$\mathbb{B}_0=\sum_{l\in\mathcal{I}_B}\oplus\mu_l\,\mathrm{diag}S_l^{-1} + \mathbb{T}^T\Big(\sum_{l\in\mathcal{I}_V}\oplus\mu_l\,\mathrm{diag}S_l^{-1}\Big)\mathbb{T}$$

$$\mathrm{diag}S_l^{-1}=\{P_{ii},\ i=1\div q\},\ \text{see (1.4)} \qquad (2.3)$$

and for all $\gamma W,\gamma V\in\mathcal{X}(\Gamma)$ holds true

$$(\mathbb{B}_0\gamma W,\gamma V)=\sum_{l=1}^{p}\mu_l\,(\mathrm{diag}S_l^{-1}\gamma W_l,\gamma V_l)=\sum_{l=1}^{p}\mu_l\sum_{k=1}^{q}D_l\,(W_l^k,V_l^k) \qquad (2.4)$$

W_l^k, V_l^k have been defined in (1.10).

The process of inversion of $\mathbb{B}$ consists of two stages:
I. The solution of the problem

$$(\mathbb{A}\varphi_L, v_L)=(\psi, v_L) \qquad \text{for all } v_L \in \mathcal{X}_L(\Gamma). \qquad (2.5)$$

Below estimates of the work for the solution of (2.5) will be given for the concrete choice of $\mathcal{X}_L(\Gamma)$. Usually $\mathcal{X}_L(\Gamma)$ is chosen in such a way that realization of the first stage is not difficult.

II. The solution of the problem

$$\mathbb{B}_0\varphi_0=f, \quad f\equiv\psi-\mathbb{A}\varphi_L. \qquad (2.6)$$

From (2.3) we have that evaluation of vector $\varphi_0=\sum_{l\in\mathcal{J}_B}\oplus[\sum_{k=1}^{q}\oplus(\varphi_0)_l^k]$ in two-dimensional case can be done by solving $[(m_1-1)m_2+(m_2-1)m_1]$ problems on common interface Γ_l^k of each two subdomains (fig.2):

$$\mathbb{F}u \equiv \mu_l P^l_{kk}u + \mu_{l_1} P^{l_1}_{k_1k_1}u = f^l_k, \quad l\in\mathcal{J}_B, \; l_1\in\mathcal{J}_V. \qquad (2.7)$$

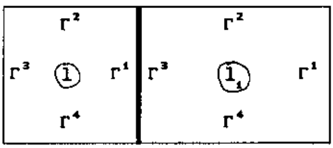

Fig.2

Here we denote $u\equiv(\varphi_0)_l^k$. On fig.2 $k=1$, $k_1=3$, P^l_{kk} and $P^{l_1}_{k_1k_1}$ are given by (1.5). Operator $\mathbb{F}^{-1}$ have represen - tation:

$$\mathbb{F}^{-1} = U_N^T\Phi U_N, \quad \Phi=\mathrm{diag}(\phi_i=(\mu_l\lambda_i^l+\mu_{l_1}\lambda_i^{l_1})^{-1}); \; i=1 \div N) \qquad (2.8)$$

λ_i^l, $\lambda_i^{l_1}$, U_N are given in (1.5).

So, for the solution of (2.6) in two-dimensional case Fast Fourier Transform (FFT) can be used and the work for inversion of $\mathbb{B}_0$ is estimated by

$$Q=C(m_2-1)\sum_{i=1}^{m_1}N_1^i\ln N_1^i+C(m_1-1)\sum_{j=1}^{m_2}N_2^j\ln N_2^j. \qquad (2.9)$$

If we consider the solution of (2.6) in three-dimensional case, then for the evaluation of φ_0 it is necessary to solve $[(m_1-1)m_2m_3+(m_2-1)m_1m_3+(m_3-1)m_1m_2]$ problems (2.7) on common in - terface Γ_l^k of each two subdomains (now it will be a rectangle), where P^{l}_{kk} and $P^{l_1}_{k_1k_1}$ are given by (1.5'). Also FFT can be applied and the work required for inversion of $\mathbb{B}_o$ in three-dimensional case is estimated by

$$Q=C(m_3-1)\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} N_1^i N_2^j \ln N_1^i N_2^j + C(m_2-1)\sum_{i=1}^{m_1}\sum_{k=1}^{m_3} N_1^i N_3^k \ln N_1^i N_3^k +$$
$$+C(m_1-1)\sum_{j=1}^{m_2}\sum_{k=1}^{m_3} N_2^j N_3^k \ln N_2^j N_3^k . \tag{2.10}$$

The estimates for α_1 and α_2 from (2.1) depend on the choice of $\mathcal{X}_L(\Gamma)$ and $\mathcal{X}_o(\Gamma)$ and will be obtained for the concrete examples. Now we shall formulate some general assertions. In what follows, $\mathbb{C}$ without subscripts will denote positive costant which is inde - pendent on mesh size h_l^j and of μ_l.

Lemma 2. Suppose that C_o, C_L, C_1, C_2 from inequalities

$$C_o(\mathbb{A}\varphi_o,\varphi_o)+C_L(\mathbb{A}\varphi_L,\varphi_L) \le (\mathbb{A}\varphi,\varphi)$$
$$C_1(\mathbb{B}_o\varphi_o,\varphi_o) \le (\mathbb{A}\varphi_o,\varphi_o) \le C_2(\mathbb{B}_o\varphi_o,\varphi_o)$$

for all $\varphi_o\in\mathcal{X}_o(\Gamma)$, $\varphi_L\in\mathcal{X}_L(\Gamma)$, $\varphi=\varphi_o+\varphi_L\in\mathcal{X}(\Gamma)$ are known.
Then, α_1 and α_2 in (2.1) are defined by $\alpha_1=\min(C_o,C_L)\min(C_1,1)$, $\alpha_2=\max(C_2,1)$.

Lemma 3. C_2 is independent on mesh size h_l^j and on μ_l;
$C_o=\mathbb{C}(1+C_L^{-1})^{-1}$.

The first statement of *Lemma* 3 follows from property (1.8) of Dirichlet form and from (2.4): for all $\varphi_o=\gamma W\in\mathcal{X}_o(\Gamma)$ holds true

$$(\mathbb{A}\varphi_o,\varphi_o)=\sum_{l=1}^{p}\mu_l D_l(W_l,W_l)=\sum_{l=1}^{p}\mu_l D_l\Big(\sum_{k=1}^{q}W_l^k,\sum_{k=1}^{q}W_l^k\Big)\le$$
$$\le\mathbb{C}\sum_{l=1}^{p}\mu_l\sum_{k=1}^{q}D_l(W_l^k,W_l^k)=\mathbb{C}(\mathbb{B}_o\varphi_o,\varphi_o) .$$

Suppose that we know C_L such that holds $(\mathbb{A}\varphi_L,\varphi_L)\le\frac{1}{C_L}(\mathbb{A}\varphi,\varphi)$. Then for all $\varphi=\varphi_o+\varphi_L\in\mathcal{X}(\Gamma)$

$$(\mathbb{A}\varphi_o,\varphi_o)=(\mathbb{A}(\varphi_o+\varphi_L-\varphi_L),\varphi_o+\varphi_L-\varphi_L)\le\mathbb{C}[(\mathbb{A}\varphi,\varphi)+(\mathbb{A}\varphi_L,\varphi_L)]\le\mathbb{C}(1+C_L^{-1})(\mathbb{A}\varphi,\varphi)$$

That proves *Lemma* 3.

From *Lemma* 2 and *Lemma* 3 follows

Theorem 2. Let C_1 and C_L are known from inequalities (2.11):

$$\left.\begin{aligned} C_L(\mathbb{A}\varphi_L,\varphi_L) &\le (\mathbb{A}\varphi,\varphi)\\ C_1(\mathbb{B}_o\varphi_o,\varphi_o) &\le (\mathbb{A}\varphi_o,\varphi_o)\end{aligned}\right\}\ \text{for all } \varphi=\varphi_o+\varphi_L\in\mathcal{X}(\Gamma) \tag{2.11}$$

Then, for the condition number $\mathbb{K}=\alpha_2/\alpha_1$ from (2.1) holds true

$$\mathbb{K}\le\mathbb{C}[C_L\min(C_1,1)]^{-1}.$$

3.THE STUDY OF SOME PRECONDITIONERS

Decomposition $\mathcal{X}(\Gamma)=\mathcal{X}_L(\Gamma)+\mathcal{X}_o(\Gamma)$ which defines preconditioner $\mathbb{B}$ is based on the idea that estimates for C_L and C_1 in (2.11) for operators $\mathbb{A}$ and $\mathbb{B}_o$ corresponding to the whole domain should be obtained by means of estimates for the operators corresponding to subdomain or a group of subdomains. In practice this condition gives that convergence properties of iterative methods for the solution of (1.11) depend on one parameter of subdomains - N_i^j, and are independent on the number of subdomains into which initial domain is partitioned.

We shall consider in detail two examples of preconditioners $\mathbb{B}$ for two-dimensional problems and one for three-dimensional. It is clear that the set of possible preconditioners is not limited by those examples.

Each gridfunction $u\in\mathcal{X}(\Gamma_l^k)$ in two-dimensional case can be uni - quely represented as $u=u_o+u_L$, where gridfunction $u_o=0$ at edge nodes ζ_1 and ζ_2 of mesh subdomain Γ_l^k and u_L is linear function along Γ_l^k with the same values as u at edge nodes: $u_L(\zeta_i)=u(\zeta_i)$, i=1,2.

So, we define decomposition $\mathcal{X}(\Gamma_l^k)=\mathcal{X}_L(\Gamma_l^k)+\mathcal{X}_o(\Gamma_l^k)$, where $\mathcal{X}_o(\Gamma_l^k)$ consists of u_o elements, $\mathcal{X}_L(\Gamma_l^k)$ - of u_L.

Then, $\mathcal{X}_o(\Gamma)=\sum_{l\in\mathcal{J}_B}\oplus[\sum_{k=1}^{q}\oplus\mathcal{X}_o(\Gamma_l^k)]$, $\mathcal{X}_L(\Gamma)=\sum_{l\in\mathcal{J}_B}\oplus[\sum_{k=1}^{q}\oplus\mathcal{X}_L(\Gamma_l^k)]$. Precondi - tioner with such choice of subspaces we denote as **P1L2**.

In second example for two-dimensional case we choose u_L=const as $\mathcal{X}_L(\Gamma_l^k)$ so that for the gridfunction $u_o=u-u_L$ holds true $(u_o,1)=0$ $u_o\in\mathcal{X}_o(\Gamma_l^k)$. This preconditioner we denote as **P1C2**.

Preconditioner for three-dimensional problem with a choice of $\mathcal{X}_L(\Gamma)$ and $\mathcal{X}_o(\Gamma)$ as in **P1C2**-case we denote **P1C3**.

Now, for the solution of the problem (2.5) a method similar to Galerkin method can be applied: the unknown function $\varphi_L\in\mathcal{X}_L(\Gamma)$ is represented as

$$\varphi_L=\sum_{l\in\mathcal{J}_B}\oplus(\sum_{k=1}^{q}\oplus u_l^k),\quad u_l^k\in\mathcal{X}_L(\Gamma_l^k);$$

where $u_l^k=\eta_1 v_1+\eta_2 v_2$, $\eta_i=u_l^k(\zeta_i)$, i=1,2 for **P1L2** (3.1)

$u_l^k=\eta_l^k v_c$, for **P1C2**, **P1C3**

here η_1,η_2,η_l^k are the numbers; v_i - linear functions such that $v_i(\zeta_j)=\begin{cases}1, i=j\\0, i\neq j\end{cases}$, i,j=1,2; v_c - gridfunction which is equal to 1 in each node of Γ_l^k. Choosing $(v_i)_l^k=\begin{cases}v_i \text{ on}\Gamma_l^k\\0 \text{ on}\Gamma_s^r, r\neq k, s\neq l\end{cases}$ and

$(v_c)_l^k=\begin{cases} v_c \text{ on } \Gamma_l^k \\ 0 \text{ on } \Gamma_s^r, r\neq k, s\neq l \end{cases}$ as basis functions in $\mathcal{X}_L(\Gamma)$ and substituting (3.1) into (2.5), we obtain a system of algebraic equations

$$A_L \eta = \psi_L \qquad (3.2)$$

for the unknowns $\eta=(u_l^k(\xi_i), i=1,2;\ k=1\div q;\ l\in\mathcal{I}_s)$ for **P1L2**-case,

$\eta=(\eta_l^k,\ k=1\div q;\ l\in\mathcal{I}_s)$ for **P1C2**, **P1C3**.

Matrix A_L in (3.2) is symmetrical, positive definite and sparse: in **P1L2**-case there are 14 non-zero elements in one row(column) of matrix A_L; in **P1C2**-case - 7 non-zero elements; in **P1C3** - 11 non-zero elements. Dimension of A_L is independent of the dimension of the whole problem (1.11) and is equal to 2R for **P1L2**, R for **P1C2**, where $R=[(m_1-1)m_2+(m_2-1)m_1]$. In **P1C3**-case A_L has dimension $[(m_1-1)m_2m_3+(m_2-1)m_1m_3+(m_3-1)m_1m_2]$. For the solution of the problem (3.2) direct and iterative methods can be applied (for instance PCG method).

Now we shall make an estimate of condition number $K=\frac{\alpha_2}{\alpha_1}$, see *Theorem* 2, for the preconditioners introduced above. This estimate can be obtained by estimates of C_1^l and C_L^l for operators from (2.11) corresponding to each subdomain, because in all cases mentioned above, on the elements $\varphi_o\in\mathcal{X}_o(\Gamma_l)$ operator S_l^{-1} is positive definite and $\mathrm{Ker}S_l^{-1}=0$, therefore the following theorem can be verified directly by (1.12) and (2.4):

Theorem 3. Suppose that for each subdomain Ω_l, $l=1\div p$, we know C_L^l and C_1^l from inequalities

$$C_L^l(S_l^{-1}\varphi_L,\varphi_L) \le (S_l^{-1}\varphi,\varphi) \qquad (3.3)$$

$$C_1^l(\mathrm{diag}S_l^{-1}\varphi_o,\varphi_o) \le (S_l^{-1}\varphi_o,\varphi_o) \qquad (3.4)$$

for all $\varphi_o\in\mathcal{X}_o(\Gamma_l)$, $\varphi_L\in\mathcal{X}_L(\Gamma_l)$. $\varphi=\varphi_o+\varphi_L\in\mathcal{X}(\Gamma_l)$. Then in (2.11) $C_L=\min_{l=1\div p} C_L^l$, $C_1=\min_{l=1\div p} C_1^l$.

Remark 1. In (3.4) $C_1^l=\lambda_{min}^l$, λ_{min}^l is the minimal eigenvalue of the problem

$$(S_l^{-1}v,z) = \lambda^l(\mathrm{diag}S_l^{-1}v,z),\ v\in\mathcal{X}_o(\Gamma_l) \qquad (3.5)$$

for all $z\in\mathcal{X}_o(\Gamma_l)$.

For the estimate of the condition number K (2.1) two following theorems hold true

Theorem 4. For the preconditioners **P1L2**, **P1C2** we have:

$$K \le C(1+\ln N)\lambda^{-1},$$

where $\mathbb{N}=\max\limits_{i,j}(N_1^i,N_2^j)$, $\lambda=\min\limits_{l=1\div p}\lambda^l_{min}$, λ^l_{min} is a minimal eigenvalue of the problem (3.5) on $\mathcal{X}_0(\Gamma_l)$ corresponding to **P1L2**, **P1C2**.

Theorem 5. For the preconditioner **P1C3** we have:

$$\mathbb{K}\le C\mathbb{N}(1+\ln\mathbb{N})\lambda^{-1},$$

where $\mathbb{N}=\max\limits_{i,j,r}(N_1^i,N_2^j,N_3^r)$, $\lambda=\min\limits_{l=1\div p}\lambda^l_{min}$, λ^l_{min} is a minimal eigen - value of the problem (3.5) on $\mathcal{X}_0(\Gamma_l)$ corresponding to **P1C3**.

In order to prove theorems 4 and 5 estimates of C_L^l from (3.3) should be obtained. First we shall do that for two-dimensional case (*Theorem* 4). *Lemma 4* given below expresses C_L^l through para - meters (N_1^i,N_2^j) of mesh subdomain Ω_l^h for **P1L2**, **P1C2** cases. The proof of this lemma will be given for **P1L2**, in **P1C2**-case the proof is analogous to **P1L2**.

Vector $\varphi_L\in\mathcal{X}_L(\Gamma_l)$ has representation $\varphi_L=\sum_{k=1}^{q}\oplus\varphi_L^k$, where φ_L^k is projection of some linear functions $\tilde\varphi_L^k$ on the node set Γ_l^k. Gene - rally speaking, the values of $\tilde\varphi_L^k$ at the vertices of subdomain Ω_l can have discontinuities and let this difference be $2d_k$ at the k-vertice. (see fig. 3).

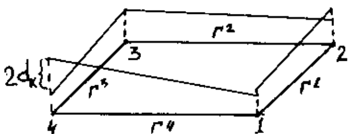

Fig. 3

Lemma 4. For C_L^l in (3.3) holds true:

$$C_L^l \ge C[(1+\ln N)(1+N(1+\ln N)\theta+O(1/N))]^{-1},$$

where $\theta=(\sum_{k=1}^{q}d_k^2)/\max|\varphi_L|^2$, $N=\max(N_1^i,N_2^j)$.

A proof of *Lemma* 4. For the simplicity we consider subdomain Ω_l^h as "internal" subdomain with mesh size $h_1^i=h_2^j=1/N$. Inequality (3.3) holds true for gridfunctions φ_L+const and φ+const . Let us choose a constant in such a way that grid functions $\dot\varphi_L=\varphi_L$+const, $\dot\varphi=\varphi$+const have zero values in one of the edge-nodes ζ_i of one of the mesh subdomains Γ_l^k, and let us prove *Lemma* 4 for such functi - ons. Let us give two auxiliary assertions combining which we immediately obtain *Lemma* 4.

*A*1. For all $\mathring{\varphi}_L\in\mathcal{X}_L(\Gamma_l)$ holds true

$$(S_l^{-1}\check{\varphi}_L,\mathring{\varphi}_L)\le C(1+N(1+\ln N)\theta+O(1/N))\max|\mathring{\varphi}_L|^2,$$

where θ is defined in the conditions of *Lemma* 4.

*A*2. For all $\dot\varphi_L\in\mathcal{X}_L(\Gamma_l)$, $\varphi_0\in\mathcal{X}_0(\Gamma_l)$, $\dot\varphi=\dot\varphi_L+\varphi_0\in\mathcal{X}(\Gamma_l)$ holds true

$$\max|\dot\varphi_L|^2\le C(1+\ln N)(S_l^{-1}\dot\varphi,\dot\varphi).$$

For a proof of A1 let us represent $\mathring{\varphi}_L$ as a trace of some h-harmonic function: $\mathring{\varphi}_L=\gamma W_L$, $W_L=V_L+\sum_{k=1}^{4}(W_1^k+W_2^k)$, where V_L is a pro-jection on mesh subdomain Ω_l^h of function $V_L=\alpha xy+\beta x+\varepsilon y+c$ with values ω_k, $k=1\div 4$, at the vertices of rectangle Ω_l which are equal to the mean values of $\tilde{\varphi}_L^k$ at these vertices. And the traces of W_1^k and W_2^k are $\gamma W_j^k=\begin{cases}\phi_j^k \text{ on } \Gamma_l^k\\ 0 \text{ on } \Gamma_l^i,\ i\neq k\end{cases}$, $j=1,2$; where ϕ_j^k are projections on Γ_l^k of linear functions $d_k\cdot x$ which vanish in one of the edge nodes of Γ_l^k. Using (1.8) we have:

$$(S_l^{-1}\mathring{\varphi}_L,\mathring{\varphi}_L)\ \le\ \mathbb{C}(S_l^{-1}\gamma V_L,\gamma V_L)+\mathbb{C}_k\sum_{k=1}^{4}\sum_{j=1}^{2}(P_{kk}\phi_j^k,\phi_j^k)\,. \tag{3.6}$$

It is easy to obtain that

$$(S_l^{-1}\gamma V_L,\gamma V_L)\le\mathbb{C}_k\sum_{k=1}^{4}\omega_k^2\le\mathbb{C}\Big(\sum_{k=1}^{4}\sum_{j=1}^{2}[\varphi_L^k(\zeta_j)]^2+\frac{1}{N}\sum_{k=1}^{4}[\varphi_L^k(\zeta_1)-\varphi_L^k(\zeta_2)]^2\Big), \tag{3.7}$$

where $\varphi_L^k(\zeta_j)$ are the values of $\mathring{\varphi}_L$ at the edge nodes ζ_j of Γ_l^k, $j=1,2$; $k=1\div4$. To estimate the second part of (3.6) we use (1.5), and, for instance for $j=1$, we have:

$$\delta_1^k\equiv(P_{kk}\phi_1^k,\phi_1^k)=(U_N^T\Lambda U_N\phi_1^k,\phi_1^k)=\sum_{i=1}^{N}\lambda_i b_i^2\ ;\ \ b_i=d_k\sqrt{\frac{2}{N}}\sum_{j=1}^{N}\Big(\frac{j-0.5}{N}\Big)\sin\frac{\pi i(j-1/2)}{N}\,.$$

After simple transformations we obtain $b_i^2\le\mathbb{C}d_k^2\ \frac{2}{N}\mathrm{cosec}^2(\frac{\pi i}{2N})$, and for δ_1^k:

$$\delta_1^k\ \le\ \mathbb{C}d_k^2\ \frac{2}{N}\sum_{i=1}^{N}\lambda_i\,\mathrm{cosec}^2\Big(\frac{\pi i}{2N}\Big)\,. \tag{3.8}$$

To estimate $\sigma_i=\lambda_i\,\mathrm{cosec}^2(\frac{\pi i}{2N})$, we use the form of λ_i given by (1.5). It can be easily shown that an equality

$$1+2\eta^2+2\eta(1+\eta^2)^{1/2}\ \ge\ \exp(\zeta\eta),\quad 0\le\eta\le1,\ \zeta=\ln(3+2\sqrt{2})$$

holds true. Since $\sin\frac{\pi i}{2N}\ge\frac{i}{N}$, $1\le i\le N$, so $\beta_i\le\exp(\zeta i/N)$, β_i are given by (1.5), and

$$\frac{\beta_i^N+\beta_i^{-N}}{\beta_i^N-\beta_i^{-N}}\ \le\ \frac{1+\exp(-2\zeta i)}{1-\exp(-2\zeta i)}\ \equiv\ \varkappa(i)\,.$$

Since $\varkappa(i)$ have maximum at $i=1$, so for σ_i we obtain

$$\sigma_i\ \le\ \mathbb{C}N\,\frac{\beta_i-1}{\beta_i+1}\mathrm{cosec}^2\Big(\frac{\pi i}{2N}\Big)\ \le\ \mathbb{C}N\Big(1+\mathrm{cosec}\Big(\frac{\pi i}{2N}\Big)\Big)\ \le\ \mathbb{C}N\Big(1+\frac{N}{I}\Big),$$

and respectively for δ_1^k (3.8):

$$\delta_1^k\ \le\ \mathbb{C}d_k^2N(1+\ln N)\,. \tag{3.9}$$

Now combining (3.9) and (3.7) we obtain first assertion.

Remark 1. Since in the initial conditions of the problem (1.1),(1.2) there is condition of continuity of unknown function across the boundaries of subdomains, so $d_k \sim C/N$, therefore we can consider

$$(S_l^{-1}\mathring{\varphi}_L,\mathring{\varphi}_L) \le C\max|\mathring{\varphi}_L|^2,$$

where $C=O(1)$.

A proof of *A*2. Since $|\mathring{\varphi}_L|$ attains its maximum at one of the edge nodes on one of the subdomains Γ_l^k where φ_0 vanishes, so an inequality holds true:

$$\max|\mathring{\varphi}_L|^2 \le \max|\mathring{\varphi}_L+\varphi_0|^2 = \max|\mathring{\varphi}|^2 .$$

Let maximum $|\mathring{\varphi}|^2$ is attained in node r of Γ_l, then

$$\max|\mathring{\varphi}|^2=|\mathring{\varphi}(r)|^2=|\sum_i a_i v_i(r)|^2 \le C(\sum_i|a_i|)^2, \qquad (3.10)$$

where a_i are Fourier coefficients of representation of $\mathring{\varphi}$ in basis $\{v_i\}$, v_i are normalized eigenfunctions of S_l^{-1}. We consider that for all v_i there exists M independent on N such that $\max_i|v_i|\le M$. This property follows from the same property of eigenfunctions of operator S_l^{-1} when S_l^{-1} corresponds to the cases in which the form of its eigenfunctions can be found (these are - when S_l^{-1} is given on a boundary of a circle, when S_l^{-1} is given on a part of a boundary of rectangle, see (1.5), (1.5')) and some topological considerations. Continuing the sequence of inequalities (3.10) and applying Hölder inequality, we obtain:

$$\max|\mathring{\varphi}|^2 \le C(\sum_i \frac{1}{e_i})\sum_i e_i a_i^2 = C(\sum_i \frac{1}{e_i})(S_l^{-1}\mathring{\varphi},\mathring{\varphi}),$$

where e_i are eigenvalues of operator S_l^{-1}. Since $\delta\equiv\sum_i\frac{1}{e_i} = \mathrm{Sp}S_l$, so $\delta=4\sum_{i=1}^{N}\frac{1}{\lambda_i}$, λ_i are given by (1.5). Function $(\beta_i^N-\beta_i^{-N})/(\beta_i^N+\beta_i^{-N})\le 1$, for all $i=1 \div N$, and since $\sin(\frac{\pi i}{2N})\ge\frac{i}{N}$, $i=1 \div N$, so

$$\delta \le C\,\frac{1}{N}\sum_{i=1}^{N}(1+\frac{N}{i}) \le C(1+\ln N). \qquad (3.11).$$

The second assertion is proved and hence *Lemma* 4.

A proof of the analog of *Lemma* 5 for three-dimensional problem (for the preconditioner P1C3) is accomplished in almost the same way as for two-dimensional. Therefore we shall not give its full proof, we shall only show the differences which in particular lead to the appearance of factor "N" in the expression for the condition number K.

Taking into account *Remark* 2, the analog of *Lemma* 4 in three-dimensional case is as follows:

Lemma 5. For **P1C3** case we have the estimate for C_L^l in (3.3):

$$C_L^l \gg \mathbb{C}[N(1+\ln N)]^{-1}, \quad N=\max(N_1^i, N_2^j, N_3^r).$$

In the proof of *Lemma* 5 the differences with two-dimensional case appear at the stage of estimate of the sum (3.11) of magni - tudes inverse to the eigenvalues of operator S_l^{-1} which corresponds to parallelepiped. In accordance with (1.5') for λ_{ij} we have:

$$\delta=6\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{\lambda_{ij}} \le \mathbb{C}\,\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}(1+\frac{N}{i+j}) \le \mathbb{C}N(1+\ln N).$$

So, combining *Theorems* 2 and 3, *Lemmas* 4 and 5, taking into account *Remarks* 1,2 we obtain assertions of *Theorems* 4 and 5.

Theorems 4 and 5 in the form of how they represent condition number **K** do not allow yet to speak about the dependence of the behavior of **K** on the parameters of discretisation, because for the time being the dependence of minimal eigenvalue of the problem (3.5) on those parameters is not defined.We have failed to obtain this dependence theoretically therefore we present hypothesis about the behavior of the minimal eigenvalue of the problem (3.5) which can be strictly justified in the case of finite-element approximation /5/. Below we shall illustrate our hypothesis by numerical experiments.

Hypothesis. For the minimal eigenvalue of the problem (3.5) holds true:

$$[\lambda^l_{min}]^{-1} \le \mathbb{C}(1+\ln N),$$

$N=\max(N_1^i, N_2^j)$ for **P1L2**, **P1C2**; $N=\max(N_1^i, N_2^j, N_3^r)$ for **P1C3**.

The results of numerical experiments which illustrate this hypothesis are given in table 1.

Numerical experiments have been carried out for the problem (3.5) with operators S_l^{-1} and $\mathrm{diag}S_l^{-1}$ corresponding to the unit square for **P1L2**, **P1C2** cases (to unit cube-for **P1C3**) with uniform mesh size h=1/N.

Table 1

log N	λ^{-1}_{min}		
	P1L2	**P1C2**	**P1C3**
2	1.9	5.1	5.0
3	3.1	7.8	7.5
4	4.5	10.3	10.5
5	6.3	13.2	12.9
6	8.1	16.3	15.3
7	10.6	19.7	—
8	12.4	23.8	—

Taking into account the suggested hypothesis about the dependence of the minimal eigenvalue of the problem (3.5) on the parameters N_k^i we have the following estimates for $K=\alpha_2/\alpha_1$ in (2.1) for preconditioners **P1L2**, **P1C2**:

$$K \le C(1+\ln\mathbb{N})^2, \quad \mathbb{N}=\max_{i,j}(N_1^i, N_2^j) \qquad (3.12)$$

for preconditioner **P1C3**:

$$K \le C\mathbb{N}(1+\ln\mathbb{N})^2, \quad \mathbb{N}=\max_{i,j,r}(N_1^i, N_2^j, N_3^r). \qquad (3.13)$$

To summarize shortly aforesaid let us note the basic properties of iterative methods for the solution (1.2) with preconditioners **P1L2**, **P1C2**, **P1C3**.

I. The convergence properties of iterative algorithms depend only on local characteristics of subdomains - the number of internal grid points N_1^i, N_2^j, N_3^r, and are independent of the number of subdomains into which initial domain is partitioned;

II. The convergence is independent on jumps of elliptic operator coefficients μ_l as long as these jumps only occur across the subdomain boundaries;

III. With a growth of the number of unknowns the convergence properties become worth and this deterioration is defined by the behavior of $K=\alpha_2/\alpha_1$ in (2.1) which is given, under the introduced suggestions, by (3.12), (3.13).

Let us estimate the work required for the solution of (1.11) by PCG method. For the simplicity we consider that the full number of grid points in initial domain is equal to M^2 (M^3 - in three dimensional case), the number of subdomains into which initial domain is partitioned is equal to m^2 (m^3 in three-dimensional case), the number N of nodes in each subdomain is $(M/m)^2$ ($(M/m)^3$). Applying for the solution of Dirichlet problems in each subdomain (evaluation of the vector $v=A\varphi$) the method suggested in /4/, we obtain that the work required for implementation of one iterative step of PCG with preconditioners **P1L2**, **P1C2** is estimated by

$$q_2=O[mM(\ln^2\frac{M}{m}+\ln\frac{M}{m})+(\varkappa m)^2(\frac{\mu_{max}}{\mu_{min}}m^2)^\alpha],$$

$\varkappa=1$ for **P1C2**, $\varkappa=2$ for **P1L2**.

In three-dimensional case with **P1C3** - by

$$q_3=O[mM^2(\ln^2\frac{M}{m}+\ln\frac{M}{m})+m^3(\frac{\mu_{max}}{\mu_{min}}m^2)^\alpha],$$

here α depends on the iterative method for the solution of the problem (3.2), for example, $\alpha=1/2$ for CG method; $\mu_{max}=\max_{l=1-p}\mu_l$,

$\mu_{min} = \min\limits_{l=1-p} \mu_l$. Then the work required to reduce initial $\mathbb{A}$-norm of the error $E_n = \varphi - \varphi_n$ by a factor of ε is estimated by

$$Q_2 = O(q_2 \ln\frac{M}{m} \ln\varepsilon^{-1})$$

for two-dimensional problems, and by

$$Q_3 = O(q_3 (M/m)^{1/2} \ln\frac{M}{m} \ln\varepsilon^{-1})$$

for three-dimensional problems.

It is also necessary to note that the main labor-intensive stages of iterative methods for the solution of (1.11) can be parallelized: evaluation of vectors $v = \mathbb{A}\varphi$ and $u = \mathbb{B}_0^{-1} v$, $\mathbb{B}_0$ is given by (2.3), is reduced to the solution of m^2 (m^3) independent Dirichlet problems in each subdomain and independent problems (2.7) on the common interface of each two subdomains. If there is computer with the corresponding number of processors then these problems can be solved in parallel. And as soon as the suggested above algorithms have mechanism of a global information transfer (the problem (2.5),(3.2)) then these algorithms can be parallelized in a wide range of the processors number variation /17/.

Remark 3. If in the conditions of the problem (1.1), (1.2) there is Neumann condition $\frac{\Delta W}{\Delta n}\Big|_{\partial\Pi_1} = \eta$ on a part $\partial\Pi_1$ (on one or on several sides of the domain Π) of the boundary $\partial\Pi$ instead of Dirichlet condition $\gamma W|_{\partial\Pi_1} = 0$ then preconditioner $\mathbb{B}$ is constructed in almost the same manner as discussed above. But in that case the trace $\varphi_1 = \gamma W|_{\partial\Pi_1}$ on $\partial\Pi_1$ must be considered as unknown function and preconditioner must be constructed for modificated system instead of (1.11):

$$\begin{bmatrix} \mathbb{A} & \mathbb{A}_{\varphi\varphi_1} \\ \mathbb{A}^T_{\varphi\varphi_1} & \mathbb{A}_{\varphi_1\varphi_1} \end{bmatrix} \begin{bmatrix} \varphi \\ \varphi_1 \end{bmatrix} = \begin{bmatrix} \psi \\ \eta \end{bmatrix}$$

where $\mathbb{A}, \varphi, \psi$ are given in (1.11), $\mathbb{A}_{\varphi\varphi_1}$, $\mathbb{A}_{\varphi_1\varphi_1}$ are constructed of P_{ij} (1.4) in accordance with block representation of unknowns.

4. NUMERICAL EXPERIMENTS

In this section we shall present some results of numerical experiments which illustrate the convergence properties of the preconditioning algorithms using **P1L2**, **P1C2**, **P1C3** as preconditioners when used in conjunction with CG method.

In the below examples the domain Ω where initial problem (1.1) is defined, is unit square (cube) partitioned into identical square (cube) subdomains Ω_{ij} (Ω_{ijr}) with sides $a_1^i=a_2^j=a_3^r=a<1$ for all i,j,r; in each subdomain there is uniform mesh with grid size $h_1^i=h_2^j=h_3^r=h=a/N$ for all i,j,r. The integer n is defined to be the number of iterations required to reduce the $\mathbb{A}$-norm of initial error $E_0=\varphi-\varphi_0$ by a factor of $\varepsilon=0.00001$ for two-dimensional problem and of $\varepsilon=0.0001$ for three-dimensional problem, i.e

$$(\mathbb{A}(\varphi_n-\varphi),\varphi_n-\varphi)^{1/2}\le \varepsilon(\mathbb{A}(\varphi_0-\varphi),\varphi_0-\varphi)^{1/2}.$$

ρ_0 is to be the observed reduction defined by

$$\rho_0=\left[\frac{(\mathbb{A}E_n,E_n)}{(\mathbb{A}E_0,E_0)}\right]^{1/(2n)}.$$

Table 2 presents results which illustrate convergence behavior of PCG method with **P1L2**, **P1C2** as preconditioners, in the dependence on discretisation parameter N and on the jumps of elliptic operator coefficients μ_{ij} across the boundaries of subdomains. Problem (1.2) has been considered in domain Ω partitioned into 25 subdomains Ω_{ij}, $i,j=1\div 5$.

Table 2

$\frac{h}{a}=\frac{1}{N}$	log N	P1L2						P1C2					
		Δ		μ1		μ2		Δ		μ1		μ2	
		ρ_0	n	ρ_0	n	ρ_0	n	ρ_0	n	ρ_0	n	ρ_0	n
1/4	2	0.05	4	0.09	5	0.06	4	0.18	7	0.27	9	0.21	8
1/8	3	0.09	5	0.13	6	0.09	5	0.23	8	0.34	11	0.25	9
1/16	4	0.14	6	0.19	7	0.13	6	0.27	9	0.40	13	0.37	11
1/32	5	0.20	7	0.23	8	0.21	7	0.31	10	0 45	15	0.39	12
1/64	6	0.23	8	0.26	9	0.23	8	0.34	11	0.49	17	0.42	13

Column marked "Δ" presents results for the case when Laplace equation in each subdomain ($\mu_{ij}=1$, $i,j=1\div5$) is defined. Columns marked "μ1" and "μ2" present results for the cases when elliptic operator coefficients have jumps across boundaries of subdomains. Figure 4 gives the values of μ_{ij} in each subdomain for "μ1" and "μ2" cases.

$\mu 1$

75	500	10	0.3	800
0.01	10^5	1	20	1100
0.1	700	10^4	920	80
100	200	0.1	1000	10
1	1000	100	0.05	1

$\mu 2$

1	1	1	1	1
1	12000	5000	10000	1
1	10000	1	3000	1
1	5000	700	12000	1
1	1000	1	1	1

Figure 4

Table 3 presents results illustrating convergence behavior of PCG method for three-dimensional problem with preconditioner **P1C3**. The problem has been considered in cube partitioned into 27 subdomains Ω_{ijr}, $i,j,r=1\div 3$.

Table 3

$\frac{h}{a}=\frac{1}{N}$	log N	**P1C3**				**DD2**	
		Δ		$\mu 3$		$\mu 3$	
		ρ_0	n	ρ_0	n	ρ_0	n
1/2	1	0.13	5	0.19	7	0.39	11
1/4	2	0.20	7	0.28	9	0.55	16
1/8	3	0.25	8	0.38	12	0.64	21

Figure 5 gives the values of μ_{ij} in each subcube for "$\mu 3$" case.

3	1	10
1	0.1	10
1000	1	10

$0<z<1/3$

8	3	1
889	22	0.3
47	10	0.88

$1/3<z<2/3$

883	3	33
9	8.8	2
101	3	55

$2/3<z<1$

Figure 5

The example "$\mu 3$" with discontinues coefficients in three-dimensional case have been taken from /5/, and for comparison, in column marked **DD2** in table 3 we present the results from this work /5,pp.15-16/ which illustrate the convergence behavior of PCG method with **DD2** as preconditioner /5/.

The examples presented above give an idea of how the convergence of iterative methods for the solution of (1.11) changes in the dependence on discretization of the problem when partition is fixed, i.e. when the number of subdomains in each direction is fixed. Now let us illustrate the convergence properties in the dependence on the number of subdomains when discretization is fixed: the initial decomposition of the domain Ω in two-dimensional case is chosen as partitioning into 4 subdomains Ω_{ij}, $i,j=1 \div m$, $m=2$ (in three-dimensional case - into 8 subdomains Ω_{ijr}, $i,j,r=1 \div m$, $m=2$), when mesh size in each subdomain is $h=1/32$. Now let us increase the value of m by a factor of two (the whole number of subdomains is m^2 (m^3)) without changing h. In doing so, the number N of grid points in one direction in each subdomain decrease by a factor of two. Table 4 presents results which illustrate convergence behavior of PCG with **P1L2**, **P1C2**, **P1C3** as preconditioners in the dependence of m when Laplace equation in each subdomain is defined.

Table 4

m	N	n		
		P1L2	**P1C2**	**P1C3**
2	16	5	7	8
4	8	4	6	6
8	4	3	5	6
16	2	1	4	–

Data of table 4 are in full accordance with theoretical conclusions that the convergence of iterative processes with preconditioners introduced above is defined by the number N of grid points in one direction in **subdomains** and is independent of the number of subdomains into which initial domain is decomposed.

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