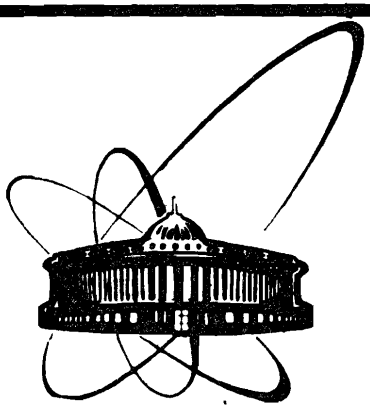

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BOUNDARY INTEGRAL EQUATIONS FOR HARMONIC FUNCTIONS ON SPECIAL SURFACES

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The development of contemporary accelerators stipulates growing requirements to the accuracy of computations of electric and magnetic field parameters. Numerical experiments on computers modelling the behaviour of magnetic fields are significantly faster and cheaper than a comparative prolonged and laborious procedure of physical experiments. The method of boundary integral equations /BIE/ or, in general, the method of boundary elements is being used in solving theoretical and practical problems of experimental physics quite often [1 - 6].

Generally speaking, nonsymmetric dense matrices are obtained when the boundary integral equations are discretized. Besides, there is available only some information on the spectrum location for these matrices. When dealing with problems in practice, the dimension of the system of linear algebraic equations may reach up to 10^3 - 10^4. Centering our interest on the iterative methods of solution of linear systems is therefore obvious.

We survey here some effective iterative methods for solving the integral equations of the potential theory on surfaces of special form. Using these methods we show how to construct economical algorithms emploing both the differential equations and the boundary integral equations in them. We analyse two- and three-dimensional problems of magnetostatics that arise when the stationary Maxwell equation is being solved. The integral equations of magnetostatics,

written down for a domain with constant magnetic permeability, are used in particular to describe exactly the boundary conditions at infinity. To optimize the numerical algorithms that solve the boundary value problems of magnetostatics in unbounded two- or three-dimensional domains one can use the combined methods [7-10]. That, in fact, implies the use of the domain decomposition method with one of the subdomains unbounded.

THE PROPERTIES OF THE BOUNDARY INTEGRAL OPERATORS

We consider boundary integral equations obtained by a direct application of Green's formulae for harmonic functions in the space R^n. Let a closed Lyapunov surface Γ divide the space R^3 into two simply connected domains, Ω_i - the interior one and Ω_e - the exterior one, so that $\Omega_i \cup \Gamma \cup \Omega_e = R^3$ and Ω_e is unbounded.

Let us consider a problem (I) of determining a function $u(M)$, harmonic in one of the domains Ω_i or Ω_e, given the values of u on the boundary Γ /Dirichlet problem/ or the values of its normal derivative on Γ /Neumann problem/. In the case of the exterior problem /Ω_e/, we set $u(\infty) = 0$. We denote by $\partial/\partial n$ the derivative with respect to the inner /in respect to Ω_i/ normal to the boundary Γ and set

$$v(S) = \frac{\partial}{\partial n} u(S), \quad S \in \Gamma.$$

Then there holds Green's formula

$$(\beta E + \alpha K) u(M) - \alpha L v(M) = 0, \quad M \in R^3. \qquad (1.1)$$

The parameter $\alpha = -1$ corresponds to the interior problem; $\alpha = 1$, to the exterior problem. The value of β is defined by the formula

$$\beta = \begin{cases} 1+\alpha, & M \in \Omega_e, \\ 1, & M \in \Gamma, \\ 1-\alpha, & M \in \Omega_i. \end{cases}$$

E is the identity operator and the integral operators K and L are defined by the relations

$$Ku = \int_\Gamma K(M,P)u(P)d\sigma_P = \frac{1}{2\pi}\int_\Gamma \frac{\cos(r_{PM}, n_P)}{|r_{PM}|^2} u(P)d\sigma_P,$$

$$Lv = \int_\Gamma L(M,P)v(P)d\sigma_P = \frac{1}{2\pi}\int_\Gamma |r_{MP}|^{-1} v(P)d\sigma_P,$$

where $|r_{MP}|$ is the euclidean norm of the vector r_{MP}, connecting the points M,P, and n_P is the inner normal vector. Given the function $v(M)$, $M \in \Gamma$ we obtain /from (1.1)/ the boundary integral equation for $u(M)$, $M \in \Gamma$.

Considering the two-dimensional case /the problem (II)/ the boundary integral equation for a harmonic function u may be derived from (1.1) if the kernels of the operators K and L are defined by relations

$$K(M,P) = \frac{1}{\pi}\frac{\partial}{\partial n_P} \ln r^{-1}(P,M) \equiv \frac{1}{2\pi}\frac{\cos(r_{PM}, n_P)}{|r_{MP}|},$$

$$L(M,P) = \frac{1}{2\pi}\ln r^{-1}(M,P), \quad M, P \in R^2.$$

Axially symmetric harmonic functions u in the space R^3 with cylindrical coordinates satisfy the equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{\partial^2 u}{\partial z^2} = 0, \quad \rho \geqslant 0. \qquad (1.2)$$

If we consider these functions, i.e. the axially symmetric harmonic

functions /the problem (III)/, then the boundary integral equation is obtained from (1.1) for the operators K and L with the kernels

$$K(M,P) = \rho \frac{\partial}{\partial n_P} G(M,P) \ , \quad L(M,P) = \rho G(M,P) \ ,$$

$$G(M,P) = \frac{1}{\pi} \int_0^{\pi} R^{-1}(M,P,\varphi)\, d\varphi \ , \quad M = (\rho, z) \ , \quad P = (\rho_0, z_0) \ ,$$

$$R(M,P,\varphi) = [(z-z_0)^2 + \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\varphi]^{1/2} .$$

The boundary Γ in this case is the envelope of the original axially symmetric surface.

The Neumann problem for each of the above-mentioned problems (I), (II) or (III) leads to a problem of solving an integral equation of the second kind

$$(E + \alpha K)\, u(M) = \Psi \ , \quad \Psi = \alpha L \nu(M) \ , \ M \in \Gamma, \qquad (1.3)$$

and the Dirichlet problem reduces to a problem of solving an equation of the first kind

$$L\nu(M) = \Phi \ , \quad \Phi = (\alpha E + K)\, u(M), \ M \in \Gamma. \qquad (1.4)$$

We will not dwell on the subject of solvability of (1.4), we just mention that these questions are studied in the papers [11,12] and that a stable algorithm in R^2 is suggested in [13]. The Dirichlet problem can obviously be transformed into the equation like (1.3), if its solution is sought in the form of the double layer potential:

$$u(M) = \int_{\Gamma} \frac{\partial}{\partial n_P} r^{-1}(M,P)\, \mu(P)\, d\sigma_P \ , \quad M, P \in R^3,$$

where the unknown potential density $\mu(P)$ satisfies the equation

$$(E - \alpha K)\, \mu = \frac{1}{\gamma(M)}\, u(M) \ , \quad M \in \Gamma,$$

and $\gamma(M)$ is a given function.

Let study the equation (1.3) in more detail. The operators

$K, L : L_2(\Gamma) \longrightarrow C(\Gamma)$ are bounded and $K : C(\Gamma) \longrightarrow C(\Gamma)$ is completely continuous. There holds the following

Theorem 1 [11]. Given the problem (I) or (III) the operator L is positive definite in $L_2(\Gamma)$. Given the problem (II) the condition $L > 0$ holds in the subspace $(\nu, 1) = 0$.

Let us denote the density of Robin potential by $g_0(M)$ /function g_0 is the density of Robin potential when $K^* g_0 = g_0$, $(g_0, g_0) = 1$, K^* is the adjoint operator to K/. We give the connection between the conditions of solvability of the equation (1.3) based on the Fredholm's theorems and the conditions imposed on the given function $\nu = \partial u / \partial n$ [11,12].

Theorem 2. Given $\alpha = 1$ the equation (1.3) has a unique solution, and if $(\nu, 1) = 0$, then $(u, g_0) = 0$ which is equivalent to $u(\infty) = 0$. Now suppose $\alpha = -1$. Then for any $\nu \in L_2(\Gamma)$ such that $(\nu, 1) = 0$ the equation (1.3) has a unique solution $u \in C(\Gamma)$ such that $(u, 1) = 0$ or $(u, g_0) = 0$.

The convergence of the alternating iterative process is based on the property

$$\gamma_m \sigma(K) = \emptyset \ , \quad \sigma(K) \subset (-1, 1> , \qquad (1.5)$$

$\sigma(K)$ denotes the spectrum of the operator K [14]. The number $\lambda = 1$ is a simple eigenvalue with the eigenfunction $u = 1$, and the function $g_0(M)$ fulfils the relations $(g_0, 1) \neq 0$, $Lg_0 = \text{const}$. And besides, for both operators K and K^*, none of the principal vectors corresponds to the eigenvalue $\lambda = 1$ [11].

The convergence of the iterational processes to the solution of the equation (1.3) on a convex domain is guaranteed by the property of kernels $K(M,P)$ to be nonnegative, $K(M,P) \geq 0$. In this case the matrices approximating the integral operators have nonnegative elements, too.

Let us consider a general approach to the construction of iterative procedures that solve the equation (1.3) without the preliminary discretization. This approach is directly applicable to discrete analogues of (1.3) in the case of convex domains. We suppose that the primary space $X = C(\Gamma)$ may be decomposed into a direct sum of two subspaces X_1 and X_2, and that there is a projector P mapping X onto X_1 : $PX = X_1$. Then to find a solution we use the iterative process [15]

$$u_{k+1} = (E-AP)^{-1}(A(E-P)u_k + \Psi), \quad A = -\alpha K. \qquad (1.6)$$

Here it is suitable to use as the subspace X_1 the space on which is the operator E - AP easily invertible. As a subset of the boundary set that defines the projector P we may elect e.g. segments of the boundary obtained by translation or rotation of its part, sphere, surface of a cylinder or a parallelepiped with a square as its base /in R^3/, perimeter of a circle, boundary of a square /in R^2/, or some other special surfaces. Examples of investigations of the convergence of iterations (1.6) are given in [16].

Now we restrict the equation (1.3), according to the theorem 2, to the invariant /with respect to K/ subspace $E_1 = \{u | (u, g_0) = 0\}$. Since from $(g_0, 1) \neq 0$ follows that X can be decomposed into a direct sum $X = E_1 + R$, we have for the spectral radius $\sigma(K^-)$ /operator K^- is the restriction of K onto E_1/ the inequality $\sigma(K^-) = q < 1$. In an equivalent norm $\|\cdot\|_*$ from [15] we then have $\|K^-\|_* \leq q + \varepsilon$ for arbitrary small $\varepsilon > 0$. This of course implies that for $PX = E_1$ the simple iteration method (1.6) converges at the rate of geometric series. If the domain with boundary Γ is a convex set then $\|\cdot\|_*$ coincides with C-norm and we can set $\varepsilon = 0$.

When we have a convex surface we may establish a sufficient condition for the convergence of iterations (1.6) that is weaker than (1.5). Briefly stated the condition says that the number $\lambda = 1$ is a simple eigenvalue greater than any other, i.e.

$$\max_{\lambda \in \sigma(K),\ \lambda \neq 1} |\lambda| = q < 1. \qquad (1.7)$$

It folows from an integral analogue of Perron's theorem for a completely continuous operator K with positive kernel and a fixed point $Ku = u$, $u = 1$. This property passes on to the matrices that approximate integral operators. That is why the part of the following theorem concerning convex surfaces holds for the /below mentioned/ systems of linear equations that approximate eq. (1.3), too.

Theorem 3 [16]. Let $PX = E_1$ and $q < 1$ is defined by (1.7). Then there exists an equivalent norm on E_1 such that the process (1.6) converges at the rate of a geometric series with the factor $q + \varepsilon$, $\varepsilon > 0$ is arbitrarily small. Let the surface be a convex set. Then for a projector P such that $PX \subset E_1$ process (1.6) converges in C-norm and

$$\|u_k - u\|_C \leq q_1^k \|u_0 - u\|_C, \quad q_1 \leq q.$$

Discretization of BIE (1.3) /on special surfaces/ is performed via piecewise constant interpolation and collocation. Let us remark that when the point M lies on an edge of the surface Γ the parameter α in eq. (1.3) takes on the values not equal to ± 1. Since the points of collocation do not lie on the edges this property of α is not put to use in discretization. Nevertheless, for piecewise linear base functions changes required to compute the matrix elements are easily performed.

Let us divide Γ into N parts: $\Gamma = \bigcup_{i=1}^{N} \Gamma_i$, diam $\Gamma_i \leq h > 0$, and represent $u(s)$ in the form $u(s) = \sum_{i=1}^{N} u_i \varphi_i(s)$, where $\varphi_i(s)$ is the characteristic function of Γ_i, $i = \overline{1,N}$.

Suppose the centres of gravity of Γ_i are the collocation points s_i. Then we get a system of linear equations

$$(E + \alpha K_h) U = \Psi_h \ , \quad \Psi_h = L_h V \ , \qquad (1.8)$$

approximating eq. (1.3), where

$$U = (u_1, \ldots, u_N)^T \ , \quad V = (v(s_1), \ldots, v(s_N))^T, \ v(s) = \partial u(s)/\partial n,$$

$$K_h = \{ k_{ij} \} \ , \quad L_h = \{ \ell_{ij} \} \ , \quad i,j = \overline{1,N},$$

and

$$k_{ij} = \int_\Gamma K(s_i, s) \varphi_j(s) ds \ , \quad \ell_{ij} = \int_\Gamma L(s_i, s) \varphi_j(s) ds .$$

ITERATIVE METHODS FOR EQUATION (1.8)

The choice of the methods depends on the properties of the system. One of the factors influencing the behaviour of the system is structure of matrices K_h and L_h. For example, the block structure of matrices K_h, L_h [2], in the case the boundary Γ is a circle or a square in problems (II),(III) and a parallelepiped with square basis [17,18] or a cylinder in problem (I) shows that for special surfaces of the given type it is possible to attain significant economy in computational resources. The results of these investigations enable for instance to reduce essentially dimension of the array where the matrices K_h, L_h are stored, or to use fast algorithms based on the Fourrier transform to multiply these matrices by a vector /here comes into play the Toeplitz block structure of these matrices/. When we consider also the spatial symmetries /asymmetries/ of Γ we may lower the number of unknowns of the system (1.8).

Let A denote a matrix K_h for any of the above-discussed

surfaces and let the dimension of the problem with this matrix be N. Equation (1.8) with its matrix of the system $E + A$ / $\alpha = 1$/ is solved by the splitting method for the nonselfadjoint operator $B = E + A$:

$$B\tau(U^{k+1} - U^k) = 2\tau(BU^k - \Psi), \qquad (2.1)$$

where

$$B\tau = (E + \tau K_1)(E + \tau K_2) \ , \quad B = K_1 + K_2 . \qquad (2.2)$$

Lower and upper block-triangular matrices K_1, K_2 are selected so as to have $K_1 = E + K_3$, $K_3 \geq 0$ and to ensure the invertibility of their diagonal blocks. For already listed special surfaces the matrices K_1, K_2 with mentioned properties can be constructed [2, 18]. Optimal specification of the parameter τ is done in practice with the help of numerical experiment. Confidence in thus defined τ stem from the fact that the spectral radius of the iterational operator is virtually independent of the discretization step and is determined only by a geometry of Γ. It therefore suffices to perform a few computations on a low-dimensional grid. The convergence of the process defined by eq. (2.1) for convex surfaces is established in [2,18].

When we take into account the spatial symmetry of Γ we solve eq. (1.8) by a Seidel method to get

$$B(U^{k+1} - U^k) = -(E + A)U^k + \Psi. \qquad (2.3)$$

The analysis of convergence can be found in [2,6].

We will now give an example of numerical computations performed on EC-1061 using FORTRAN programs intended for solving eq. (1.8) by the above-mentioned methods. We have used the program solving BIE on the surface of a parallelepiped with square as its basis. Given below in Table 1 are the results of computations and

fundamental characteristics of an algorithm for a sequence of three grids that have been constructed on the surface $\Gamma = \partial\Pi$, where Π denotes a cube with an edge a=9. As a test function we have used the harmonic function $u(M) = |M - M_0|^{-1}$, $M_0 \in \overline{\Pi}$. The grid ω_k, $k = \overline{1,3}$ is defined by p_k points on every edge of the basis $p_k = p_0 2^{k-1}$, $p_0=4$, and q_k points on every edge perpendicular to the basis, $q_k = q_0 2^{k-1}$, $q_0=16$. The dimension N of the problem is then $N = N(k) = 2p_k(p_k + 2q_k)$. All computations have been done double-precision. Indexed letters T_K, T_L, T_ψ denote the computation time /sec./ required to compute the matrices K_h, L_h and $\Psi = L_h V$ in that order, and T denotes the time /sec./ needed to solve the system (1.8). Letter G denotes the general case of the iteration method (2.1) and S denotes the special case, where the iteration method (2.3)was constructed taking into consideration three planes of symmetry for Π. The number of iterations of (2.3) till the relative accuracy 10^{-4} for S is not reached and the number of iterations of (2.1) till the absolute accuracy 10^{-4} for G is not reached in both cases on the grid ω_k is denoted by N_k.

Table 1

ω_k		T_K	T_L	T_ψ	T	N_k
4x4x16	S	1.65	1.20	0.33	6.14	8
	G	1.65	1.16	4.28	29.0	4
8x8x32	S	12.7	8.59	2.49	33.4	5
	G	12.6	8.63	32.6	160.1	3
16x16x64	S	99.5	65.6	20.4	331.0	4
	G	112.7	65.2	290.1	1312.0	3

The use of Richardson's extrapolation on the given sequence of grids has given an additional improvement of effectivity of the computational process.

The estimate of the computational time T to solve the problem (1.8) on the grid ω_4 of dimension 32x32x128 with the symmetries taken into account is about 48 min.

Note that the dimension of the system (1.8) on the grid ω_3 is $N(3) = 4608$, and the time of solution is cca. 22 min. /for the accuracy $\geq 10^{-4}$/. Estimates /for EC-1061/ show that more than 200 hours would be needed to solve a similar system by the Gauss method.

APPLICATIONS IN MAGNETOSTATICS

Economical algorithms for solving BIE on special surfaces [8, 17,18] can be used in magnetostatics to restrict the domain of investigation for two- or three-dimensional problems involving scalar potential. Efficient application of BIE is possible also when a vector potential is used in two-dimensional problems [7].

Consider the system of Maxwell's steady-state equations for isotropic medium :

$$\operatorname{rot} H = \frac{4\pi}{c} j \,, \quad \lim_{|M| \to \infty} H(M) = 0 \,, \qquad (3.1)$$
$$\operatorname{div} B = 0 \,, \quad B = \mu(|H|) H \,,$$

with the conditions at the boundary between different media :

$$n \cdot (B_2 - B_1) = 0 \,, \quad n \times (H_2 - H_1) = 0. \qquad (3.2)$$

Problem (3.1)-(3.2) is transformed into a problem of finding two scalar functions φ_1 and φ_μ defined by

$$H = H_0 - \operatorname{grad} \varphi_1 \,, \quad M \in \Omega_1 \,, \qquad (3.3)$$
$$H = -\operatorname{grad} \varphi_\mu \,, \quad M \in \Omega_\mu \,,$$

where $\Omega_1 \cup \Gamma_\mu \cup \Omega_\mu = R^3$, $\mu = 1$, $M \in \Omega_1$, and H_0 is defined by the formula

$$H_0(M) = \frac{1}{c} \int_{V'} \frac{\gamma_0 \times (R - R')}{|R - R'|^3} dV'.$$

The functions φ_1, φ_μ satisfy equations

$$\begin{aligned} &div(\mu(|grad\,\varphi_\mu|)grad\,\varphi_\mu) = 0, \quad M \in \Omega_\mu, \\ &\Delta\varphi_1 = 0, \quad \varphi_1(\infty) = 0, \quad M \in \Omega_1, \end{aligned} \tag{3.4}$$

with conditions on the boundary Γ_μ

$$\begin{aligned} &\frac{\partial\varphi_1}{\partial n} - \mu(|grad\,\varphi_\mu|)\frac{\partial\varphi_\mu}{\partial n} + (H_0, n) = 0, \\ &\varphi_1(P) - \varphi_\mu(P) = \int_\gamma H_0 ds, \quad P \in \Gamma_\mu, \gamma \in \Gamma_\mu, \end{aligned} \tag{3.5}$$

where γ is a continuous curve connecting points P_0, $P \in \Gamma_\mu$. To account for the condition $\varphi_1(\infty) = 0$ we use a boundary integral equation (1.3).

Let Π be a cylinder or a parallelepiped with square basis and suppose that Π contains the domain Ω_μ, i.e. $\Pi = \Omega_\mu \cup D$, where $D = \Pi \cap \Omega_1$. Now we consider a boundary value problem, equivalent to (3.4)-(3.5), for which the equation $\Delta\varphi_1 = 0$ is defined only in domain D, and the condition

$$(E - K)\varphi_1 - L\frac{\partial\varphi_1}{\partial n} = 0, \quad M \in \Gamma \tag{3.6}$$

holds at the exterior boundary Γ. We denote

$$u(M) = \begin{cases} \varphi_\mu, & M \in \bar{\Omega}_\mu, \\ \varphi_1, & M \in D = \Pi \cap \Omega_1. \end{cases}$$

Let us suppose that eqs. (3.4),(3.5) define on Π a differential operator Gu. Then, just as in the papers [7,8,17], one of the following iterative processes can be employed to solve the system (3.4)-(3.6).

Algorithm 1. Given an approximation $u^k(\xi)$ we compute $u^{k+1}(\xi)$ successively solving the following equations.

$$\begin{aligned} &Gu = \Psi(M), \quad M \in \Pi, \; u(\xi)|_\Gamma = u^k(\xi), \\ &\nu^{k+1} = \frac{\partial u(\xi)}{\partial n}, \quad z = \mu(|grad\,u(\xi)|)\nu^{k+1}, \\ &(E + K)u^{k+1/2} - Lz = 0, \\ &u^{k+1} = (1 - h)u^k + hu^{k+1/2}, \quad 0 < h < 1. \end{aligned} \tag{3.7}$$

One step of this process corresponds to solving the Dirichlet problem in Π and to solving the integral equation at the boundary for the exterior Neumann problem, with a subsequent relaxation.

Algorithm 2. Given an approximation $u^k(\xi)$, $\xi \in \Gamma$, $(u^k, g_0) = 0$ we compute $u^{k+1}(\xi)$ successively solving the following equations.

$$(E + K)u^k - L\nu^{k+1} = 0, \quad \xi \in \Gamma,$$

$$\begin{cases} Gu = \Psi(M), \; \dfrac{\partial u}{\partial n} = \mu(|grad\,u^k|)^{-1}\nu^{k+1}, \; M \in \Pi, \\ (u, g_0) = 0, \; u^{k+1/2} = u(\xi), \end{cases} \tag{3.8}$$

$$u^{k+1} = (1 - h)u^{k+1/2} + hu^k, \quad 0 < h \leq 1.$$

Here we solve the Neumann boundary value problem with the operator G in Π, provided $(u, g_0) = 0$ /this is equivalent to $(u, 1) = 0$/ and BIE at the boundary Γ for the exterior Dirichlet problem with relaxation.

It is important to notice, that after the system (3.4)-(3.6) has been discretized, the rate of convergence of alternating iterative processes (3.7) and (3.8) practically does not depend on the discretization step, it depends only on the function $\mu(t)$ and on the geometry of domains Ω_M and Π.

The questions of solvability for nonlinear systems (3.4)-(3.6) in Sobolev spaces and verification of convergence of iterative processes (3.7), (3.8) have been studied in [19,10]. Numerical examples illustrating solutions of the combined system (3.4)-(3.6) using BIE on special surfaces may be found in [2,18,19].

Considerable part of results that have been described in here can also be applied to a wider class of problems that appear in mathematical physics.

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Грегуш М., Жидков Е.П., Хоромский Б.Н. E11-87-586

Граничные интегральные уравнения для гармонических функций на специальных поверхностях

Рассмотрены экономичные итерационные процессы решения граничных интегральных уравнений, определенных на двумерных или трехмерных областях специального вида. Построены алгоритмы, основанные на комбинировании метода граничных интегральных уравнений с сеточными методами. Приводятся примеры численных расчетов пространственных задач магнитостатики в неограниченной области.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1987

Gregu$\check{s}$ M., Zhidkov E.P., Khoromsky B.N. E11-87-586

Boundary Integral Equations for Harmonic Functions on Special Surfaces

Economical iterative processes to solve the boundary integral equations defined on two-dimensional and three-dimensional regions of special type are considered. The algorithms, that are obtained by combining the method of boundary integral equations with the grid methods, are presented. The examples of numerical solutions of the magnetostatic problems for unbounded domains in space are given.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1987