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AN IMPLEMENTATION OF KOVACIC'S ALGORITHM FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS IN FORMAC

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1. INTRODUCTION

In the recent paper [1] an algorithm for finding a "closed-form" solution of the following differential equations is given

$$y'' + a(x)y' + b(x)y = 0 . \qquad (1)$$

where a and b are rational functions of the independent variable x . The "closed-form" solution means the Liouvillian solution, i.e. one that can be, expressed in terms of algebraic functions, exponentials and indefinite integrals, see (Kovacic [1]), for precise definition. Kovacic's algorithm provides a Liouvillian solution of (1) or reports that no such solution exists. The main result obtained by Kovacic is the following.

Theorem. Equation (1) has a Liouvillian solution if and only if it has a solution of the form

$$y = \exp\{\int(\omega - a/2)\,dx\},$$

where ω is an algebraic function of x of degree 1, 2, 4, 6 or 12. The last means that ω, satisfying the Ricatti equation

$$\omega' + \omega^2 = R(x), \qquad R = a'/2 + a^2/4 - b$$

solves a polynomial equation $G(\omega, x) = 0$, where $G(\omega, x) = \sum_{i=0}^{N} g_i(x)\,\omega^i$, g_i are rational functions of x and $N \in \{1,2,4,6,12\}$.

An algorithm for finding the polynomial G is based on the knowledge of the even order poles of R and consists in constructing and testing a finite number of possible candidates for G. If each candidate is not a desired polynomial then eq.(1) has no Liouvillian solutions.

Kovacic's algorithm has been already implemented in the Computer Algebra Systems MACSYMA [2] and MAPLE [3]. Our implementation is based on the Computer Algebra System FORMAC, see [4]. In the second section of this paper we give the complete algorithm description in such a way that an interested reader can immediately start to implement it using a suitable

Computer Algebra System. In the third (final) section some implementation aspects and the computational experience in the FORMAC are discussed.

2. ALGORITHM DESCRIPTION

Notation:

C denotes the complex numbers; **C**(x), the rational functions over **C**; **C**[x], the polynomials over **C**, **Z** the integers, L a finite set. For s, t ∈ **C**[x] gcd(s, t) denotes the greatest common divisor of s and t, deg t the leading degree of t and lc(t) the leading coefficient of t.

Problem:

Given $R \in \mathbf{C}(x)$

Find $G(\omega, x) = \sum_{i=0}^{N} g_i(x)\omega^i$, $g_i \in \mathbf{C}(x)$, $N \in \{1, 2, 4, 6, 12\}$,
such that $\omega' + \omega^2 = R$ and $G(\omega, x) = 0$.

Algorithm:

1. Partitioning of R
   $L' := \emptyset$
   $R := s/t$; [$s, t \in \mathbf{C}[x]$, $\gcd(s, t) = 1$, $lc(t) = 1$]
   $m := \deg s - \deg t$;
   Computer the square-free factorization of t :
   $t := t_1 \cdot t_2^2 \cdot \ldots \cdot t_\ell^\ell$; [$t_\ell \neq 1$].
2. Necessary conditions for N
   if $\forall_{k>0}\ t_{2k+1} = 1$ and ($m/2 \in \mathbf{Z}$ or $m < -2$) then $L := L \cup \{1\}$;
   if $\exists_{k>0}\ t_{2k+1} \neq 1$ or $t_2 \neq 1$ then $L := L \cup \{2\}$;
   if $\forall_{k>2}\ t_k = 1$ and $m \leq -2$ then $L := L \cup \{4, 6, 12\}$;

if $L = \emptyset$ then return 'no solution exists';

3. Constructing of candidates
   $d_0 := \frac{1}{4}(\min(2, -m) - \deg t - 3\deg t_1)$; $\theta_0 := \frac{1}{4}(t'/t + 3t_1'/t_1)$;

   Find the roots c_i of t_2; [$i = 1, 2, \ldots, n_2$]
   for i := 1 to n_2 do begin $d_i := \sqrt{1 + 4\lim_{x \to c_i}(x - c_i)^2 R}$;
   $\theta_i := d_i/(x - c_i)$; end;
   if $m \leq -2$ then begin $n_2 := n_2 + 1$; $\theta_{n_2} := 0$; end;
   if $m < -2$ then $d_{n_2} := 1$ else if $m = -2$ then
   $d_{n_2} := \sqrt{1 + 4 lc(s)/lc(t)}$;

if $1 \in L$ then begin
   Find the roots c_i of $t_4 \cdot t_6 \cdot \ldots \cdot t_\ell$; [$i = n_2 + 1, n_2 + 2, \ldots, n$]
   for i := $n_2 + 1$ to n do begin
      $\nu := m_i/2$; [m_i is the order of the pole c_i in R]
      for k := 0 to $\nu - 1$ do $\lambda_{\nu-k} := \frac{d^k}{dx^k}((x - c_i)^\nu \sqrt{R(x)})|_{x=c_i}$;
      $d_i := 2\lambda_1$; $\theta_i := 2\sum_{k=1}^{\nu} \lambda_k/(x - c_i)^k$;
   end;
   if $m > -2$ then begin
      $n := n + 1$;
      $\nu := m/2$;
      for k := 0 to $\nu + 1$ do $\lambda_{\nu-k} := \frac{d^k}{dx^k}(x^\nu \sqrt{R(1/x)})|_{x=0}$;
      $d_n := 2\lambda_{-1}$; $\theta_n := 2\sum_{k=0}^{\nu} \lambda_k x^k$;
   end;
end 'if $1 \in L$';

4. Testing of candidates
   for each $N \in L$ [in increasing order] do begin
      if N = 1 then k := n else k := n_2;
      for j := 0 to k do $s_j := -N/2$;
      for ℓ := 1 to $(N + 1)^k$ do begin
         j := 1;
         while $s_j = N/2$ do begin $s_j := -N/2$; j := j + 1; end;
         $s_j := s_j + 1$;
         $d := N \cdot d_0 - \sum_{i=1}^{k} s_i d_i$;
         if $d \in \mathbf{Z}$ and $d \geq 0$ then begin
         $\theta := N \cdot \theta_0 + \sum_{i=1}^{k} s_i \theta_i$;
         $P := \sum_{i=0}^{d} a_i x^i$; [with undetermined a_i]
         $P_N := P$;
            for i := N step -1 to 0 do $P_{i-1} := -P_i' - \theta P_i -$
               $-(N+i)(i+1) R P_{i+1}$;
            Solve equation '$P_{-1} = 0$' for P
                  [linear algebraic system for a_i]
            if solution $P = \tilde{P}$ is found
               then return $G := \sum_{i=0}^{N} \frac{\omega^i}{(N - i)!} P_i |_{P=\tilde{P}}$;
         end 'if d...';

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    end 'for ℓ';
end 'for each N;
return 'no solution exists';
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3. IMPLEMENTATION IN FORMAC

The above algorithm is implemented in the FORMAC Computer Algebra System. The choice of FORMAC is caused by its high execution velocity and comparatively small memory needed, so that one can run our program in IBM and derivative computers with 512K memory. To implement Kovacic's algorithm we developed a number of routines extending the capabilities of FORMAC, such as polynomial division, polynomial gcd's computation, square-free polynomial factorization, determination of the rational roots of polynomials, partial fraction decomposition of rational functions, solving the linear algebraic systems.

The algorithm requires exact numeric computations to be carried out in a quadratic extension of the initial number field $\mathbf{F}$, which includes the coefficients and the even order poles of R. The current version of our program can be applied to the limited class of eqs.(1) with $\mathbf{F}=\mathbf{Q}$ (rational number field). It means that the program works with the numbers of the form $q_1+q_2\sqrt{q_3}$ which are automatically simplified to the canonical form: $q_1'+q_2'\sqrt{m}$ ($q_i, q_i' \in \mathbf{Q}$, m is a square-free integer). To extend the class of input equations it's sufficient to modify the procedure SIMP implementing the simplification of "numeric expressions".

The program has been tested successfully on examples in [1,5] Moreover we tested 70 equations for which the infinite power series solutions are given in [6] Among them about 30 equations were found to have Liouvillian solutions. For example, a power series solution of the equation

$$y'' + \frac{3-x}{x}y' - \frac{5}{x}y = 0 \quad \text{presented in [6] is } y = \sum_{n=0}^{\infty}(n+3)(n+4)x^n/n!$$

Using the program developed we found a closed-form solution: $y=(x^2+8x+12)e^x$. All tested examples take from 3 to 10 sec. of ES-1061 running time and less than 200K memory.

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Реализация на языке FORMAC алгоритма Ковачича для решения обыкновенных дифференциальных уравнений

Рассмотрена реализация на языке аналитических вычислений FORMAC алгоритма Ковачича для нахождения лиувиллевских решений дифференциальных уравнений вида $y'' + a(x)y' + b(x)y = 0$, где a, b - рациональные функции x. Приведено формальное описание алгоритма, позволяющее легко реализовать его в любой подходящей системе компьютерной алгебры.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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An Implementation of Kovacic's Algorithm for Solving Ordinary Differential Equations in FORMAC

An implementation of Kovacic's algorithm for finding Liouvillian solutions of the differential equations $y'' + a(x)y' + b(x)y = 0$ with rational coefficients a(x) and b(x) in the Computer Algebra System FORMAC is described. The algorithm description is presented in such a way that one can easily implement it in a suitable Computer Algebra System.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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