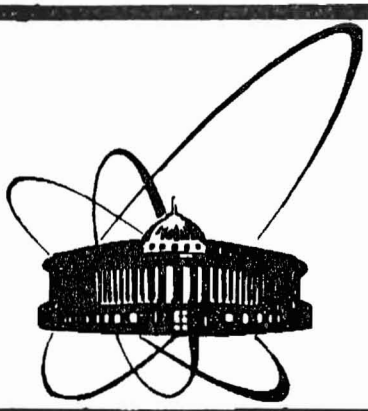


объединенный
институт
ядерных
исследований
дубна

E11-87-101

P.G.Akishin, E.P.Zhidkov, V.D.Kravtsov

INVESTIGATION OF THE INTEGRAL MAGNETIC FIELD EQUATIONS

Submitted to "Journal of Computational Physics"

1987

Introduction

Consider the integral equation formulation for the solution of the magnetostatic problems. Let $\bar{B}(\bar{x})$ is the magnetic induction at the point $\bar{x}$, $\bar{H}(\bar{x})$ is the magnetic field intensity, $\bar{M}(\bar{x})$ - magnetization of the iron, $\mu(|\bar{B}(\bar{x})|)$ is the permeability, $\bar{H}^S(\bar{x})$ is the intensity of the magnetic field due to the currents, calculated by Bio-Savarre law. Let G is the region, occupied by the iron. The integral equation formulation of the magnetostatic problem in three dimensions is [1]:

$$\bar{H}(\bar{a}) = \bar{H}^S(\bar{a}) + \frac{\nabla_{\bar{a}}}{4\pi}\left[\int_G \left(\bar{M}(\bar{x}), \nabla_{\bar{x}} \frac{1}{|\bar{x}-\bar{a}|}\right) dV_{\bar{x}}\right]. \qquad (1)$$

The field vectors $\bar{H}$,$\bar{M}$ and $\bar{B}$ are related by the following equations

$$\bar{H}(\bar{x}) = \frac{\bar{B}(\bar{x})}{\mu_0 \cdot \mu(|\bar{B}(\bar{x})|)}, \qquad (2)$$

$$\bar{M}(\bar{x}) = \frac{\bar{B}(\bar{x})}{\mu_0} - \bar{H}(\bar{x}), \qquad (3)$$

where μ_0 is the permeability of the vacuum.

In two dimensions the equation (1) is reduced to the following form:

$$\bar{H}(\bar{a}) = \bar{H}^S(\bar{a}) - \frac{\nabla_{\bar{a}}}{2\pi}\int_G (\bar{M}(\bar{x}), \nabla_{\bar{a}} \ln|\bar{x}-\bar{a}|) dS_{\bar{x}}. \qquad (4)$$

For the axisymmetrical configurations equation (1) becomes

$$\bar{H}(\bar{a}) = \bar{H}^S(\bar{a}) - \frac{\nabla_{\bar{a}}}{4\pi}\int_G (\bar{M}(\bar{x}), \bar{F}(\bar{x},\bar{a}))\, r_{\bar{x}}\, dr_{\bar{x}}\, dz_{\bar{x}}, \qquad (5)$$

where the components of the vector $\bar{F} = (F_1, F_2)^T$ in (5) are:

$$F_1 = \int_c^{2\pi} \left[\frac{\partial}{\partial r_{\bar{x}}} \left(r_{\bar{x}}^2 + r_{\bar{a}}^2 - 2 r_{\bar{x}} r_{\bar{a}} \cos\varphi + (z_{\bar{x}} - z_{\bar{a}})^2 \right)^{-1/2} \right] d\varphi ,$$

$$F_2 = \int_c^{2\pi} \left[\frac{\partial}{\partial z_{\bar{x}}} \left(r_{\bar{x}}^2 + r_{\bar{a}}^2 - 2 r_{\bar{x}} r_{\bar{a}} \cos\varphi + (z_{\bar{x}} - z_{\bar{a}})^2 \right)^{-1/2} \right] d\varphi .$$

Later on we shall assume that if we consider the intensity $H = H(B)$ as a function of B, the next inequality is valid

$$0 < \alpha \le \mu_0 \frac{\partial H}{\partial B} \le 1 , \qquad (6)$$

where α is a constant different for the various types of the iron and does not depend on H and B. As a rule this inequality is valid. The magnetization $\bar{M}$ has finite module

$$|\bar{M}| \le M_{max} , \qquad (7)$$

where M_{max} corresponds to completely saturated medium.

Suggesting $B_2 \geqslant B_1$ we obtain

$$\mu_0 |M(B_2) - M(B_1)| = \left| \int_{B_1}^{B_2} \left(1 - \mu_0 \frac{\partial H}{\partial B}(x)\right) dx \right| \le (1-\alpha) |B_2 - B_1| . \qquad (8)$$

Hence, any B_1 and B_2 satisfy the inequality

$$\mu_0 |M(B_2) - M(B_1)| \le q |B_2 - B_1| , \qquad (9)$$

where $q < 1$.

The following lemma from [2] is valid:

Lemma 1

Any $\bar{B}_1$ and $\bar{B}_2$ satisfy the inequality

$$\mu_0 |\bar{M}(\bar{B}_2) - \bar{M}(\bar{B}_1)| \le q |\bar{B}_2 - \bar{B}_1| , \qquad (10)$$

where $0 \le q < 1$.

Consider the discretization investigated in [1]. According to it we must divide G into subregions G_i

$$G = \bigcup_{i=1}^{N} G_i ,$$

$$G_i \cap G_j = 0 , \text{ where } i \neq j.$$

Define $\bar{a}_i$ as a centre of mass of G_i

$$\bar{a}_i = \frac{\int_{G_i} \bar{x}\, dV_{\bar{x}}}{\int_{G_i} dV_{\bar{x}}} , \quad i = 1,2,\ldots,N.$$

We assume that $\bar{M}(\bar{x})$ is a constant in each G_i and equal to $\bar{M}_i$. Then the discretization of (1) is described in the following way

$$\bar{H}_i = \bar{H}^s(\bar{a}_i) + \frac{\nabla_{\bar{a}}}{4\pi} \left[\sum_{j=1}^{N} \int_{G_j} \left(\bar{M}_j, \nabla_{\bar{a}} \frac{1}{|\bar{x} - \bar{a}|} \right) dV_{\bar{x}} \right] \Bigg|_{\bar{a} = \bar{a}_i} , \qquad (11)$$

$$i = 1,2,\ldots,N.$$

In two dimensions the discretization of equation (4) is given by the expression

$$\bar{H}_i = \bar{H}^s(\bar{a}_i) - \frac{\nabla_{\bar{a}}}{2\pi} \left[\sum_{j=1}^{N} \int_{G_j} \left(\bar{M}_j, \nabla_{\bar{a}} \ln|\bar{x} - \bar{a}| \right) dS_{\bar{x}} \right] \Bigg|_{\bar{a} = \bar{a}_i} , \qquad (12)$$

$$i = 1,2,\ldots N.$$

In axisymmetrical case the next approximation for $\bar{M}$ is used in [3] for the discretization of the equation (3). In each subregion $\bar{M}$ (M_1, M_2) is approximated by

$$M_1 = \frac{r}{r_i} M_r^i ,$$

$$M_2 = M_z^i ,$$

where r_i is the r-component of $\bar{a}$ in cylindrical coordinates, and M_r^i, M_z^i are constants in G_i. This approximation is used in order to keep a continuity of $\bar{M}$ at r=0. Using the method of collocations we obtain the following discretization of the equation (5)

$$\bar{H}_i = \bar{H}^s(\bar{a}_i) - \frac{\nabla_{\bar{a}}}{4\pi} \sum_{j=1}^{N} \int_{G_j} \left(M_r^j \frac{r_{\bar{x}}}{r_j} F_1(\bar{x},\bar{a}) + M_z^j F_2(\bar{x},\bar{a}) \right) r_{\bar{x}}\, dr_{\bar{x}}\, dz_{\bar{x}} \Bigg|_{\bar{a} = \bar{a}_i} , \qquad (13)$$

$$i = 1,2,\ldots,N.$$

In this paper the problems, connected with calculation of matrix coefficients of the discretized systems (11) - (13) and the solution of the nonlinear discretized equations, are investigated.

1. Properties of Discretized Operators

Let

$$\hat{\bar{B}} = (\bar{B}_1, \bar{B}_2, \dots, \bar{B}_N)^T,$$
$$\hat{\bar{M}}(\hat{\bar{B}}) = (\bar{M}(\bar{B}_1), \bar{M}(\bar{B}_2), \dots, \bar{M}(\bar{B}_N))^T,$$
$$\hat{\bar{H}}^S = (\bar{H}^S(\bar{a}_1), \bar{H}^S(\bar{a}_2), \dots, \bar{H}^S(\bar{a}_N))^T.$$

Briefly the systems (11) - (13) are described in the following way

$$\frac{\hat{\bar{B}}}{\mu_0} = \hat{\bar{H}}^S + ([A] + [E])\,\hat{\bar{M}}(\hat{\bar{B}}), \qquad (1.1)$$

where $[A]$ is the block, consisting of the N^2 matrices $[A_{ij}]$:

$$[A] = \begin{pmatrix} [A_{11}] & \dots & [A_{1N}] \\ \vdots & & \\ [A_{N1}] & \dots & [A_{NN}] \end{pmatrix}$$

and $[E]$ is the identical matrix.

For the equations (11) $[A_{ij}]$ is the matrix of 3x3 dimensionality:

$$[A_{ij}]\,\bar{M} = \frac{\nabla_{\bar{a}}}{4\pi}\left[\int_{G_j} \left(\bar{M}, \nabla_{\bar{a}} \frac{1}{|\bar{x}-\bar{a}_i|}\right) dV_{\bar{x}}\right]\Big|_{\bar{a}=\bar{a}_i}. \qquad (1.2)$$

For the system (12) $[A_{ij}]$ is the matrix of 2x2 dimensionality:

$$[A_{ij}]\,\bar{M} = -\frac{\nabla_{\bar{a}}}{2\pi}\left[\int_{G_j} \left(\bar{M}, \nabla_{\bar{a}} \ln|\bar{x}-\bar{a}_i|\right) dS_{\bar{x}}\right]\Big|_{\bar{a}=\bar{a}_i}. \qquad (1.3)$$

And, at last, for the system (13) A_{ij} is the matrix of 2x2 dimensionality

$$[A_{ij}]\,\bar{M} = -\frac{\nabla_{\bar{a}}}{4\pi}\left[\int_{G_j} \left(\bar{M}, \bar{U}(\bar{x},\bar{a})\right) \gamma_{\bar{x}}\, d\gamma_{\bar{x}}\, dz_{\bar{x}}\right]\Big|_{\bar{a}=\bar{a}_i}, \qquad (1.4)$$

where $\bar{U}(\bar{x},\bar{a}) = \left(F_1(\bar{x},\bar{a}) \cdot \frac{\gamma_{\bar{x}}}{\gamma_j},\ F_2(\vec{x},\bar{a})\right)^T$.

Assuming that (7) is valid we obtain the following theorem [4].

Theorem 1

The discretized system (1.1) is solvable.

Proof

Consider the operator

$$F(\hat{\bar{B}}) = \left(\hat{\bar{H}}^S - [A]\,\hat{\bar{M}}(\hat{\bar{B}})\right)\mu_0,$$

then F is continuous and maps R^{3N} into R^{3N} in three dimensional and axisymmetrical cases; and R^{2N}, into R^{2N} in two dimensions.

Consider the ball $U = \left\{\hat{\bar{B}} : \frac{1}{\mu_0}\|\hat{\bar{B}}\| \le \|\hat{\bar{H}}^S\| + \|[A]\| \cdot C_0\right\}$,

where $C_0 = \sup \|\hat{\bar{M}}\|$. From the condition (7) we get: $C_0 < \infty$. The operator $F(\hat{\bar{B}})$ transforms the convex U-set into itself

$$F(U) \subset U.$$

The U-set is compact, hence, the Schauder's principle [5] is applicable. The operator $F(\hat{\bar{B}})$ has an immovable point and the theorem 1 is proved.

Consider the continuous operator from the right side of (1)

$$A(\bar{M}) = \frac{\nabla_{\bar{a}}}{4\pi}\left[\int_{G} \left(\bar{M}(\bar{x}), \nabla_{\bar{a}} \frac{1}{|\bar{x}-\bar{a}|}\right) dV_{\bar{x}}\right]. \qquad (1.5)$$

The theorem from [6] takes place:

Theorem 2

The operator A is:

(i) bounded with $\|A\|_G = 1$;

(ii) self-adjoint;

(iii) negative semidefinite:

$$(A(\bar{M}),\bar{M})_G \le 0 , \qquad (1.6)$$

with $$\sup_{\|\bar{M}\|_G=1} (A(\bar{M}),\bar{M})_G = 0 . \qquad (1.7)$$

The validity of this theorem for integral operators from (4) and (5) can be shown as in [6].

In this paper we propound the hypothesis that discretized operators $[A]$ from (1.1) have similar properties namely: the spectrum of $[A]$ is real and belongs to interval $[-1,0]$. The numerical experiments confirm this assumption.

If the stated hypothesis is true we can prove the theorem:

Theorem 3

The system (1.1) has a unique solution.

Proof

Let the system (1.1) has two several solutions $\hat{B}_1$ and $\hat{B}_2$. From (1.1) it follows that:

$$\hat{B}_1 - \hat{B}_2 = \mu_0 ([A]+[E])(\hat{M}(\hat{B}_1) - \hat{M}(\hat{B}_2)).$$

From this expression, taking into account the assumption mentioned before and the result of the Lemma 1, it follows that

$$\|\hat{B}_1 - \hat{B}_2\| \le \mu_0 \|\hat{M}(\hat{B}_1) - \hat{M}(\hat{B}_2)\| \le q \|\hat{B}_1 - \hat{B}_2\| .$$

From the condition $q < 1$ we obtain $\hat{B}_1 = \hat{B}_2$. The theorem is proved.

The problems, connected with the uniqueness of solution for the other discretizations of continuous equations, are considered in [2].

Let us verify the condition (1.7) for the discretized operator $[A]$ from (1.1) in two-dimensional problem.

Let the region G is divided into triangles $\{G_i\}$ in the following way: if $i \ne j$ the triangles G_i and G_j may have a single common point, or a single common side, or they do not cross each other. Let $[\tilde{A}]$ is the discretized operator, which is defined by the expression

$$[\tilde{A}]\hat{M} = -\sum_{i=1}^{N} \frac{\nabla_{\bar{a}}}{2\pi} \int_{G_i} (\bar{M}_i, \nabla_{\bar{a}} \ln|\bar{x}-\bar{a}|)\, dS_{\bar{x}} , \qquad (1.8)$$

where $\bar{a}$ is an arbitrary point ($\bar{a} \in \partial G_i$; $i = 1,2,\dots,N$).

Let $\{\bar{P}_j\}$ is the set of all vertices of the triangles G_i, $\{\tilde{P}_k\}$ is the set of the internal vertices and $\{\hat{\tilde{P}}_i\}$ is the set of the boundary vertices. One defines some function $\varphi(\bar{x})$ which is continuous on G, linear on each triangle G_i and constant on the boundary of G. One may consider the following vector function $\bar{M}(\bar{x})$

$$\bar{M}(\bar{x}) = \frac{\partial\varphi}{\partial y}\bar{e}_1 - \frac{\partial\varphi}{\partial x}\bar{e}_2 = \mathrm{rot}(\bar{e}_3 \varphi(\bar{x})), \qquad (1.9)$$

where $\bar{e}_1$ and $\bar{e}_2$ form a system of orthonormal vectors; x and y are the corresponding coordinates of the vector $\bar{x}$, and $\bar{e}_3 = [\bar{e}_1 \times \bar{e}_2]$.

Obviously, $\bar{M}(\bar{x})$ is a constant on each G_k. According to [7], the following theorem holds:

Theorem 4

Operator $[\tilde{A}]$ from (1.8) identically equals to zero on vector $\bar{M}(\bar{x})$ from (1.9): $[\tilde{A}]\bar{M}(\bar{x}) \equiv 0$.

Proof

Let $\bar{M}_k$ is the value of $\bar{M}(\bar{x})$ on G_k. From (1.8) it follows that

$$[\tilde{A}]\bar{M} = \frac{1}{2\pi} \sum_{\kappa=1}^{N} \oint_{\partial G_\kappa} [(\bar{M}_\kappa, \bar{n}(\bar{x}))\nabla_{\bar{a}} \ln|\bar{x}-\bar{a}|]\, d\ell_{\bar{x}} , \qquad (1.10)$$

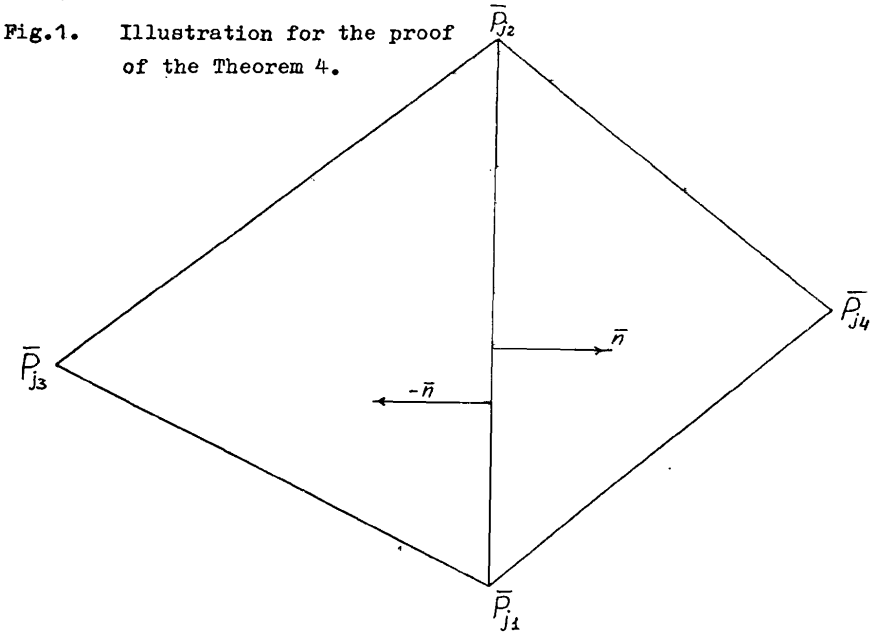
where $\bar{n}(\bar{x})$ is the external normal vector for G_k.

Let L_i is the internal side of the triangles $\{G_k\}$ with the vertices $\bar{P}_{j1}$, $\bar{P}_{j2}$, which belongs to triangles $G_{i1}(\bar{P}_{j1}, \bar{P}_{j2}, \bar{P}_{j3})$ and $G_{i2}(\bar{P}_{j2}, \bar{P}_{j1}, \bar{P}_{j4})$ (fig.1)

Let $\bar{n}$ is the external normal vector for G_{i1} on L_i, then $-\bar{n}$ is the external normal vector for G_{i2} on L_i. Integral on L_i from (1.10) equals

$$\frac{1}{2\pi} \int_{L_i} (\bar{n}, (\bar{M}_{i_2} - \bar{M}_{i_1}))\nabla_{\bar{x}} \ln|\bar{x}-\bar{a}|\, d\ell_{\bar{x}} . \qquad (1.11)$$

Fig.1. Illustration for the proof of the Theorem 4.



Then we shall prove that $(\bar{n}, (\bar{M}_{i2} - \bar{M}_{i1})) = 0$.

Introduce the Cartesian coordinates connecting with the side L_i. Let axis OX is directed along L_i, and axis OY is directed along $\bar{n}$. From the continuity of $\varphi(\bar{x})$ on G and the linearity on each G_k it follows that $\varphi(\bar{x})$ is described on G_{i1} by the expression

$$\varphi(\bar{x}) = a_0 + x \cdot a_x + y \cdot a_{y1}. \qquad (1.12)$$

Then $\varphi(\bar{x})$ is written on G_{i2} thus

$$\varphi(\bar{x}) = a_0 + x \cdot a_x + y \cdot a_{y2}. \qquad (1.13)$$

From (1.12), (1.13) it follows that

$$\bar{M}_{i1} - \bar{M}_{i2} = [\bar{e}_3 \times \bar{n} \cdot (a_{y1} - a_{y2})].$$

And, obviously, $(\bar{n}, (\bar{M}_{i2} - \bar{M}_{i1})) = 0$.

It is easy to show from the constancy of $\varphi(\bar{x})$ on the boundary of G that the external normal vector $\bar{n}$ for the external side L_k is orthogonal to corresponding vector $\bar{M}_{ik}$.

The theorem 4 is proved.

Obviously, the vector $\hat{M} = (\bar{M}_1, \bar{M}_2, \ldots, \bar{M}_N)^T$ is the eigenvector of the operator $[A]$ with the eigenvalue, which is equal to zero.

The system of vectors $\{\hat{e}_k\}$ is constructed in the following way. Let $\varphi_k(\bar{x})$ is the continuous on G and linear on each G_k function satisfying to the following conditions

$$\varphi_i(\tilde{P}_j) = \delta_{ij}, \qquad (1.14)$$
$$\varphi_i(\tilde{\tilde{P}}_k) = 0.$$

We define $\bar{e}_k^i$ from (1.9) by substitution of the $\{\varphi_k(\bar{x})\}$ for $\varphi(\bar{x})$ and supposing that $\bar{x} \in G_i$ $(i = 1,2,\ldots,N)$.

Then $$\hat{e}_k = (\bar{e}_k^1, \bar{e}_k^2, \ldots, \bar{e}_k^N)^T, \qquad (1.15)$$

$$k = 1,2,\ldots,L,$$

where L is the number of the internal apexis of the grid.

We note that $\{\varphi_k(\bar{x})\}$ satisfy the conditions of the theorem 4, hence, $\{\hat{e}_k\}$ are the eigenvectors of the operator $[A]$ with the zero eigenvalue.

The Lemma from [7] holds

Lemma 2

The system of vectors $\{e_k\}$ is linearly independent.

Proof

We admit that such number of values $\{c_i\}$ exists:

$$\sum_{i=1}^{L} c_i \hat{e}_i = \hat{0}.$$

Or

$$\sum_{i=1}^{L} c_i \frac{\partial}{\partial x}(\varphi_i(\bar{x})) = 0,$$

and

$$\sum_{i=1}^{L} c_i \frac{\partial}{\partial y}(\varphi_i(\bar{x})) = 0.$$

Hence

$$\sum_{i=1}^{L} c_i \varphi_i(\bar{x}) = \text{const}.$$

From $\varphi_i(\bar{x}) = 0$ on ∂G it follows that

$$\sum_{i=1}^{L} c_i \varphi_i(\bar{x}) = 0.$$

Supposing $x = \bar{P}_j$, we obtain

$$c_i = 0, \quad i = 1,2,\ldots,L.$$

The Lemma is proved.

The numerical experiments show that the discretized operator $[A]$ has not the other eigenvectors with the zero eigenvalue. The problem of existence of such vectors for the operator $[A]$ in three dimensions is considered in [8].

2. Calculations of Matrix Coefficients for Discretized Equations

In order to write the discretized system (1.1) it is necessary to calculate the following integrals

$$J_k = \nabla_{\bar{a}} \int_{G_k} (\bar{M}, \nabla_{\bar{a}} \ln|\bar{x}-\bar{a}|) d S_{\bar{x}} , \qquad (2.1)$$

where $\bar{M} \equiv \text{const}$.

In this paper the optimized procedure of the calculation of integrals (2.1) is proposed [9].

The integral J_k is reduced to the boundary integral

$$J_k = \oint_{\partial G_k} (\bar{M}, \bar{n}(\bar{x})) \nabla_{\bar{x}} \ln|\bar{x}-\bar{a}| \, d\ell_{\bar{x}} , \qquad (2.2)$$

Let L_1, L_2, L_3 are the sides of the triangle G_k; $\bar{n}_1$, $\bar{n}_2$, $\bar{n}_3$ are the external normal vectors for L_1, L_2, L_3 correspondingly; $\bar{\tau}_1$, $\bar{\tau}_2$, $\bar{\tau}_3$ are vectors which are equal to 1 modulo and determine the positive direction of the round of the triangle G_k. $((\bar{n}_i, \bar{\tau}_i) = 0, i = 1,2,3)$.

Then $J_k = \sum_{i=1}^{3} (\bar{M}, \bar{n}_i) \int_{L_i} \nabla_{\bar{x}} \ln|\bar{x}-\bar{a}| \, d\ell_{\bar{x}}$.

So the expression $\nabla_{\bar{x}} \ln|\bar{x}-\bar{a}|$ can be written as

$$\nabla_{\bar{x}} \ln|\bar{x}-\bar{a}| = \bar{n}_i \frac{\partial}{\partial n_{\bar{x}}} (\ln|\bar{x}-\bar{a}|) + \bar{\tau}_i \frac{\partial}{\partial \ell_{\bar{x}}} (\ln|\bar{x}-\bar{a}|) . \qquad (2.3)$$

The integral $\int_{L_i} \frac{\partial}{\partial \ell_{\bar{x}}} \ln|\bar{x}-\bar{a}| \, d\ell_{\bar{x}}$ is equal up to a sign

$$\int_{L_i} \frac{\partial}{\partial \ell_{\bar{x}}} \ln|\bar{x}-\bar{a}| \, d\ell_{\bar{x}} = \ln|\bar{P}_2-\bar{a}| - \ln|\bar{P}_1-\bar{a}| ,$$

where $\bar{P}_1$ and $\bar{P}_2$ are the vertices of the side L_i.

Analogously, the integral $\int_{L_i} \frac{\partial}{\partial n_{\bar{x}}} \ln|\bar{x}-\bar{a}| \, d\ell_{\bar{x}}$ is equal up to a sign to the angle, under which one can see the side L_i from the point $\bar{a}$.

It follows that J_k is one linear combination of $\varphi_1, \varphi_2, \varphi_3$ and R_1, R_2, R_3, where φ_i is the angle, under which the side L_i can be seen from the point $\bar{a}_i$, and $R_i = \ln|\bar{P}_i-\bar{a}|$, where $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$ are the vertices of the triangle G_k.

Obviously, the triangles G_k may have common sides and vertices. Let $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_N$ are the vertices of all triangles $\{G_k\}$, and $L_1, L_2, \ldots, L_k$ are the sides of all triangles $\{G_k\}$ (we suppose that all sides and all vertices are various).

Then in order to write the coefficients from (1.1) it is necessary to calculate the MxN values:

$$P_{ij} = \ln|\bar{P}_i - \bar{a}_j| ,$$

$i = 1,2,\ldots,M,$

$j = 1,2,\ldots,N,$

and k x N angles φ_{mk}, under which the side L_m is seen from the point $\bar{a}_k$. Similar procedure, which completely remove the repetition in the calculation of the matrix coefficients, was realized for the complex of programmes for the calculation of the axisymmetrical magnetic fields as well [3].

Analogous algorithm for the three-dimensional case is described in [6].

3. The Method of the Solution of Discretized Equation Systems

For the solution of the discretized equation system (1.1) the following iterative process is proposed in [2]:

$$\frac{\hat{B}_{k+1}}{\mu_e} = \hat{H}^s + ([A] + [E])\hat{M}(\hat{B}_k), \qquad (3.1)$$

$$k = 0,1,2,\ldots .$$

The process will be stopped when the relative deviation

$$R_k = \frac{\|\hat{B}_{k+1} - \hat{B}_k\|}{\|\hat{B}_k\|}$$

becomes less than the previously defined value ε.

Assuming the validity of the hypothesis about the spectrums of discretized operators we can prove the following theorem:

Theorem 5

The iterative process (3.1) converges from any initial approximation to the solution of the system (1.1) as a geometric progression:

$$\|\hat{B}_{k+1} - \hat{B}_k\| \le q\|\hat{B}_k - \hat{B}_{k-1}\|,$$

where q was defined in (9).

Proof

From (3.1) it follows that

$$\frac{1}{\mu_0}(\hat{B}_{k+1} - \hat{B}_k) = ([A]+[E])\left(\hat{M}(\hat{B}_k) - \hat{M}(\hat{B}_{k-1})\right), \qquad (3.2)$$

or

$$\|\hat{B}_{k+1} - \hat{B}_k\| \le \mu_0\|\hat{M}(\hat{B}_k) - \hat{M}(\hat{B}_{k-1})\| \le q\|\hat{B}_k - \hat{B}_{k-1}\|.$$

The Theorem 5 is proved.

In fact q is majorised by the value $1 - \frac{1}{\mu_{max}}$, where μ_{max} is the maximal value of the permeability. When the level of the magnetic field is low, μ_{max} achieves

the large value ($\sim 10^5$), hence the iterative process (3.1) converges very slowly. This fact has numerical experiment confirmations.

For the recalculation of the field in the arbitrary point it is necessary to know the magnetization $\hat{M}(\hat{B})$, but not the magnetic induction $\hat{B}$. Obviously it is enough to know $\hat{M}(\hat{B})$ up to an arbitrary linear combination of vectors $\hat{e}_k$. We define $\hat{M}^*(\hat{B})$ in the following way

$$\hat{M}^*(\hat{B}) = \hat{M}(\hat{B}) + \sum_{i=1}^{L} c_i \hat{e}_i, \qquad (3.3)$$

where $\{c_i\}$ is chosen from the orthogonality condition of $\hat{M}^*(\hat{B})$ and $\hat{e}_i$ ($i = 1,2,\ldots,L$).

Let us introduce the new criterion of the iterative process finish. The process will come to its end when the relative deviation

$$R_k^* = \frac{\|\hat{M}^*(\hat{B}_{k+1}) - \hat{M}^*(\hat{B}_k)\|}{\|\hat{M}^*(\hat{B}_k)\|}$$

becomes less than the previously defined value ε. In this case the iterative process is majorised by the geometric progression with the exponent $\gamma = (1 - \frac{1}{\mu_{max}})\cdot(1 + \lambda_{max})$:

$$\|\hat{M}^*(\hat{B}_{k+1}) - \hat{M}^*(\hat{B}_k)\| \le \gamma\|\hat{M}^*(\hat{B}_k) - \hat{M}^*(\hat{B}_{k-1})\|,$$

where λ_{max} is the largest eigenvalue of the discretized operator $[A]$, which is not equal to zero ($\lambda_{max} < 0$).

For the realization of this iterative process it is necessary to construct the fast projection procedure of the arbitrary vector $\hat{M}$ to the subspace of images of the operator $[A]$. In other words it is necessary to solve fastly the following system of linear equations

$$(\hat{M}, \hat{e}_i) + \sum_{j=1}^{L} c_j (\hat{e}_j, \hat{e}_i) = 0, \qquad (3.4)$$

$$i = 1,2,\ldots,L.$$

The matrix from (3.4) is the Gramm matrix of the linearly independent vectors $\{\hat{e}_i\}$; it is positively defined, symmetrical and very sparce. For the solution of this system we use the Incomplete Cholesky Decomposition with the Method of the Conjugated Gradients [10]. The initial approximation for the system (3.4) in the Method of the Conjugated Gradients is transferred from the previous step of the iterative process (3.1).

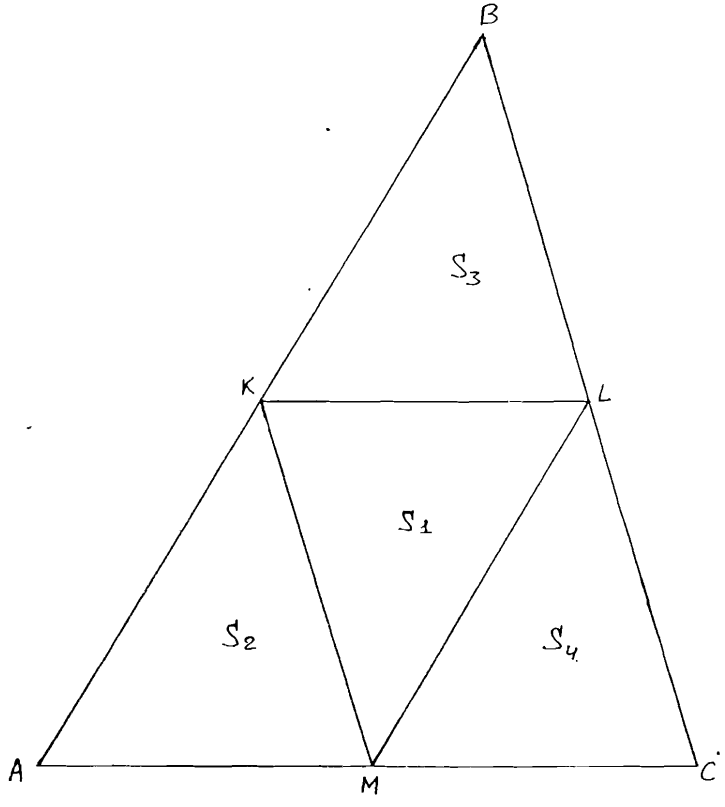


Fig.2. Dividing of a triangle into the four homothetic triangles.

Calculations on the sequence of grids are used for the decreasing the time of run , which is necessary to solve the system of nonlinear equations [11].

Primarily the problem (1.1) is solved on the rough grid by using the iterative process (3.1). Then each triangle G_k is divided into four homothetic triangles (fig.2). We solve the problem (1.1) for this grid, in addition the initial approximation in (3.1) for the fine grid is transferred from the solved problem (1.1) for the rough grid. This procedure essentially decreases the summary expenditure of the time of run because the greatest number of iterations is made on more cheap (from the calculating expense point of view) rough grid. The solution of the problem (1.1) on the fine grid begins with the good initial approximation.

4. Numerical Experiments

This method was used for creating of the complex of programmes for the calculation of the magnetostatic fields. In order to calculate matrix coefficients of the system of discretized equations (1.1) on rough and fine grids we use the following fact (in two dimensions): matrix coefficients on the rough grid are expressed by matrix coefficients of the discretized system on the fine one. Let S is the triangle from the rough grid (fig.2). Then, obviously, the following identity holds

$$\nabla_{\bar{a}} \int_S (\bar{M}, \nabla_{\bar{a}} \ln|\bar{x}-\bar{a}|)\, dS_{\bar{x}} = \sum_{i=1}^{4} \nabla_{\bar{a}} \int_{S_i} (\bar{M}, \nabla_{\bar{a}} \ln|\bar{x}-\bar{a}|)\, dS_{\bar{x}} ,$$

where $\bar{M} \equiv \text{const}$.

The centres of mass of triangles from the fine grid contain the centres of mass from the rough one.

For the testing of the calculation accuracy we calculated the problem, which has an analytic solution: the infi-

nite cylinder with μ = const in the diametrical homogenious magnetic field. The result, when the circle quarter was divided into 240 elements, was in the good agreement with the theoretical calculation (the relative deviation is about 10^{-4}). The formation of the matrix, the solution of the nonlinear system and the recalculation of the field at 20 points required 118 seconds of CDC-6500 CP time.

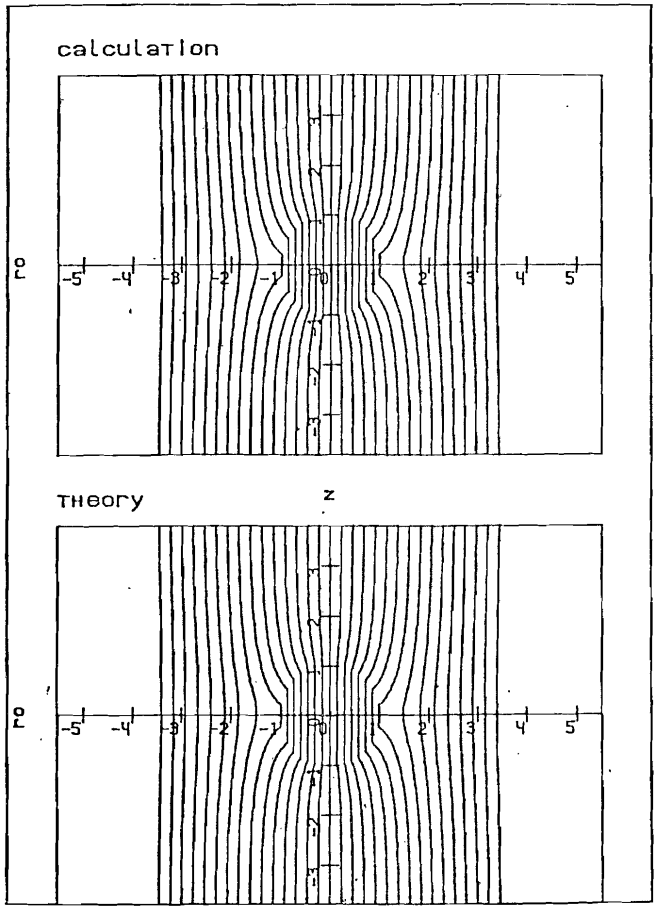


Fig.3. Magnetic field of the paramagnetic ball on the external homogenious field.

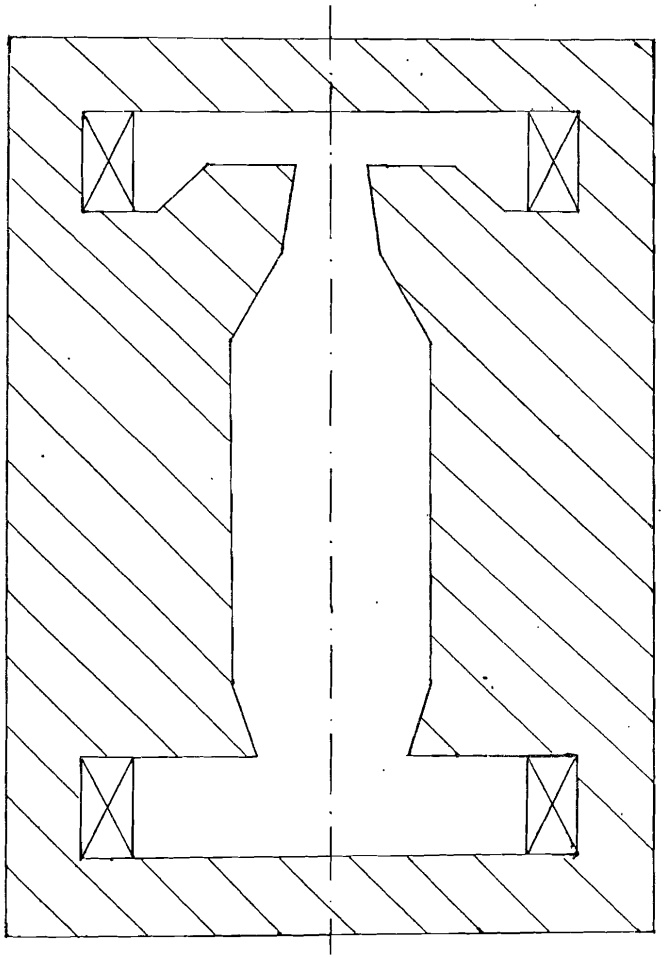


Fig.4. Cross-section of the Synhrophasotron of JINR.

Similarly , in the axisymmetrical case we calculated the field of the paramagnetic ball in the external homogenious magnetic field. It required 60 seconds of CDC-6500 CP time; the relative deviation is about $5 \cdot 10^{-3}$, it is mainly determined by the error of the approximation of the circle by the polygon (fig.3).

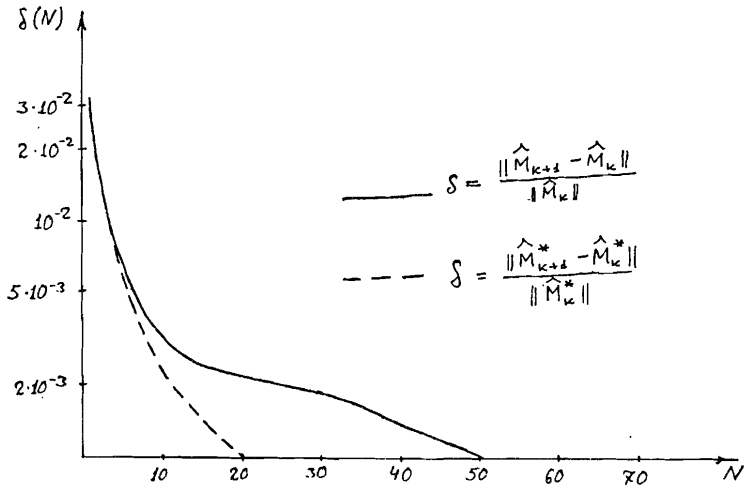


Fig.5. Relative deviations of the iterative process (3.1).

Table

Time expenditure for iterative processes

Number of iterations	CP time for iterative process (3.1)	CP time for modernized iterative process
5	53.041	53.384
10	105.304	108.040
15	157.505	162.280
20	209.725	217.401

In two dimensions we estimated the cancellation of the calculating time by using the optimized algorithm, which is described in part 2. The calculation of the matrix coefficient, when the region of the iron was devided into 224 elements (in addition, 3 reflections took place) required 134 seconds of CDC-6500 CP time, if we used the optimized algorithm; and 409 seconds - if the nonoptimized one was used.

We calculated the magnetostatic field of JINR Synhrophasotron (fig.4) in the two-dimensional approximation. In fig.5 and the table one can see comparative characteristics of the iterative process (3.1) and the modernized process, which was proposed in part 3. This comparison shows that the application of the modernized algorithm essentially decreases the calculating expenditure of the solution in the case of large permeability .

References

1. G.W.Trowbridge et al. GFUN3D User Guide, RL-76-029/A.
2. P.G.Akishin, E.P.Zhidkov. Commun. of JINR, P11-83-427, Dubna, 1983.
3. P.G.Akishin, E.P.Zhidkov, V.D.Kravtsov. Commun. of JINR, P11-85-552, Dubna, 1985.
4. P.G.Akishin, E.P.Zhidkov. Commun. of JINR, P11-81-826, Dubna, 1981.
5. L.V.Kantorovich, G.G.Akilov. "Functional Analysis", "Nauka", Moscow, 1977, (in Russian).
6. M.J.Friedman. SIAM J.Appl.Math. Vol.39, 1980, pp.14-20.

7. P.G.Akishin, E.P.Zhidkov, V.D.Kravtsov. Commun. of JINR, P11-86-718, Dubna, 1986.

8. Akishin P.G.. Commun. of JINR, P11-86-522, Dubna, 1986.

9. P.G.Akishin, E.P.Zhidkov, V.D.Kravtsov. Commun. of JINR, P11-86-534, Dubna, 1986.

10. J.A.Mejerink, H.A.Van der Vorst. Math. of Comput., Vol. 31,137, 1977, pp.148-162.

11. N.S.Bakhvalov, Z. Vycisl. Mat. i Mat. Fiz.,v.6,No5, 1966, pp.861-883.

Received by Publishing Department
on February 18, 1987.

Акишин П.Г., Жидков Е.П., Кравцов В.Д. E11-87-101
Исследование интегральных уравнений магнитостатики

Исследуется интегральная постановка уравнений магнитостатики. Доказаны некоторые свойства дискретизованного оператора. Предложена оптимизированная процедура вычисления коэффициентов матриц. На основе полученных свойств дискретизованного оператора модифицирован итерационный процесс решения системы нелинейных уравнений. Приведены результаты численных расчетов.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1987

Akishin P.G., Zhidkov E.P., Kravtsov V.D. E11-87-101
Investigation of the Integral Magnetic Field Equations

The magnetostatic problem is investigated in terms of the integral equations. Some properties of the discretized operator are proved. The optimized procedure for calculating of the matrix coefficients is proposed. Making use of the properties of the discretized operators we modify the iterative process for the solution of nonlinear equations, which we use before. The results of the numerical experiments are discussed.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1987