объединенный
институт
ядерных
исследований
дубна

E11-85-164

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DETERMINATION OF LIE-BACKLUND SYMMETRIES OF DIFFERENTIAL EQUATIONS USING FORMAC

Submitted to
"Computer Physics Communications"

1985

1. Introduction and Theoretical Background

In recent 10-15 years much attention has been paid to the nonlinear two-dimensional models having soliton-like solutions such as, for example, Korteweg-de Vries equation, sine-Gordon equation, nonlinear Schrödinger equation, etc. Such equations are often completely integrable. They have the representation in the form of commutator of the two linear operators (Lax representation) and solved by the inverse scattering method.

It would be interesting to consider the multidimensional equations with soliton-like solutions especially physically important cases with two and three spatial dimension. Unfortunately, the multidimensional generalizations of the inverse scattering method of full value are nonexistent. On the other hand equations which are integrated by the inverse scattering method have high hidden symmetry. They are invariant with respect to the Lie-Bäcklund transformation groups [1]. N-soliton solutions turn out to be invariant with respect to such groups. Thus search and investigation of the interesting multidimensional models may be carried out supposing that they are invariant with respect to nontrivial Lie-Bäcklund groups. The successful application in recent years of the Lie-Bäcklund groups to investigate the nonlinear parabolic equations (see, for example, [2,3]) should be also mentioned. Thus the determination of Lie-Bäcklund symmetries of differential equations is appeared to be an important problem in the mathematical physics and applied mathematics.

Lie-Bäcklund group is defined as the tangent transformation of the infinite order, that is, the coordinates of the Lie-algebra depend on unlimited number of derivatives [1]. Lie-algebra vector called Lie-Bäcklund operator has the form:

$$X = \xi_i \frac{\partial}{\partial x^i} + \eta_i^a \frac{\partial}{\partial u^a} + \sum_{s \geq 1} \eta^a_{i_1 \dots i_s} \frac{\partial}{\partial u^a_{i_1 \dots i_s}}, \tag{1}$$

where x^i $(i = 1, \dots, n)$ - independent, u^a $(a = 1, \dots, m)$ - dependent variables, $u^a_{i_1 \dots i_s}$ - jet bundle coordinates corresponding to the partial derivatives of u^a with respect to $x^{i_1}, \dots x^{i_s}$. Further we shall call these coordinates breafly "derivatives". Functions ξ_i , $\eta^a, \dots, \eta^a_{i_1 \dots i_s}, \dots$ depend on variables

$x^i, u^\alpha, \ldots, u^\alpha_{i_1 \ldots i_k}, \ldots$ and are connected to each other by usual prolongation formulas

$$\eta^\alpha_{i_1 \ldots i_{s+1}} = D(\eta^\alpha_{i_1 \ldots i_s}) - u^\alpha_{j i_1 \ldots i_s} D(\xi^j),$$

where $D = (D_1, \ldots, D_n)$ is the operator of total differentiation

$$D_i = \frac{\partial}{\partial x^i} + \sum_{s \geq 0} u^\alpha_{i i_1 \ldots i_s} \frac{\partial}{\partial u^\alpha_{i_1 \ldots i_s}}, \quad \eta^\alpha_0 \equiv \eta^\alpha.$$

Lie-Bäcklund operators of the form $X_* = \xi^i_* D_i$, where ξ^i_* are the arbitrary functions of the variables $x^i, u^\alpha, \ldots, u^\alpha_{i_1 \ldots i_k}$, first, form the ideal in the Lie-algebra of the all Lie-Bäcklund operators and secondly, the transformations created by them, leave the arbitrary differential manifold invariant. By term differential manifold one calls the system of equations under consideration with all the differential consequences:

$$[F] : F = 0, \ldots, \underset{\nu}{F} = 0, \ldots, \quad \nu = 1, 2, \ldots$$

The system of the equations $F = 0$ is called invariant with respect to the Lie-Bäcklund group, if the manifold $[F]$ is invariant. There exists the theorem [1] confirming that the differential manifold $[F]$ is invariant with respect to the Lie-Bäcklund transformation group, if and only if $(XP)_{[F]} = 0$. In other words, it is enough to apply operator X only to the initial equations but when transferring to the manifold the differential consequences should be considered.

As the Lie-Bäcklund operators of the form $X_* = \xi^i_* D_i$ don't contribute to the invariance condition, the factor-algebra of the complete Lie-algebra with respect to ideal formed by the operators X_* may be considered without loss of the generality. Choosing instead of the operators X of the form (1) equivalent to them in the factor-algebra operators with the vanished ξ^i we obtain the operators of the form

$$X = \eta^\alpha \frac{\partial}{\partial u^\alpha} + \ldots, \tag{2}$$

which are called "canonical operators". Transition to the canonical operators essentially simplifies the calculations, as it is enough now to consider m functions instead of $n + m$, what is especially important when the computer is used. Moreover, prolongation formulas take a simple form:

$$\eta^\alpha_{i_1 \ldots i_s} = D_{i_1} \ldots D_{i_s} (\eta^\alpha).$$

With the use of canonical operators the invariance condition (the determining equation) takes the following form

$$\left(\eta^\alpha \frac{\partial F}{\partial u^\alpha} + D_i(\eta^\alpha) \frac{\partial F}{\partial u^\alpha_i} + \ldots\right)_{[F]} = 0. \tag{3}$$

The solutions of the determining equation, depending on the variables $\underset{k}{u}$ (correspond to the k-th derivatives) but independing of the $\underset{k+1}{u}, \ldots$ called the k-th order solutions. If to follow that, when transferring to the differential manifold $[F]$, derivatives of the first order were not expressed by derivatives of more high order, then the first order solutions would correspond to the contact transformations. Point transformations correspond to first order solutions, linear with respect to the derivatives, i.e., to the solutions of the form $\eta^\alpha(x, u) - \xi^i(x, u) u^\alpha_i$, where ξ^i, η^α – are usual coordinates of the Lie-algebra of the point transformations. Note that the point transformation group obtained from the solutions of the Lie-Bäcklund determining equations may be wider than the classical ones. It may occur in case, if some equations of the system under consideration have the order less than maximum order of the system, since when transferring to the manifold the differential consequences of these equations will be used. The classical definition of the invariance does not take into account the differential consequences.

Presently, the Lie-Bäcklund algebras of the evolution equations with one spatial variable, i.e., the equations of the form $u_t = F(x, u, u_x, u_{xx}, \ldots), x \in R^1$ are the most studied. This is connected with the special form of the time differentiation operator inclusion into the equation. The problem comes to the investigation of the differential algebra with one independent variable. This algebra is studied completely enough.

The main way to obtain the solutions of the determining equations is in the following. Let the determining equation has the form: $L\eta = 0$, where L is the linear differential operator. The operator M, satisfying the relation $[L, M] = 0$ called recurrence operator is searched. It is clear, that if $\eta^{(1)}$ is a solution of the determining equation, then $\eta^{(2)} = M\eta^{(1)}$ is also the solution. That is why it is possible to create the new solutions from the known ones (e.g., point and contact) using the recurrence operator. The effective general methods to construct the recurrence operators do not exist. In practice, they are often searched by comparing between each other the low order solutions obtained by direct calculation (see for example [4]). When transferring to the multidimensional problems the complexity of the calculations rapidly increases. Probably the real possibility to deal with multidimensional equations is to applicate the computer algebra.

The program written in the language REDUCE-2 for determining the point and contact symmetries has been proposed in [5]. When computing the Lie-Bäcklund symmetries the effectiveness of the computer algebra system has acquired a special importance. That is why the program proposed here is written in PL/1 - FORMAC.

2. Description of the Program

The space coordinates $Z_k = (x, u, \underset{1}{u}, \ldots, \underset{k}{u})$ are put in a lexicographic order as in [5]. To represent those coordinates the PL/1 symbolic array named ZK is used. To represent the functions η from the formula (3) and their derivatives also put in lexicographic order the symbolic PL/1 array named Z# is used. The determining equations of the Lie-Bäcklund algebra contain the derivatives of the functions η with respect to the derivatives of the functions u. To reduce the complexity of the expressions at the output we accepted the restriction which is in the use of only one-letter symbols for the independent and dependent variables. Let us demonstrate by example the correspondence between the mathematical designations and those used in the program. Let there be independent variables, x, y and dependent ones, u, v. The vector field (2) takes the form

$$X = \overset{*}{U}\frac{\partial}{\partial U} + \overset{*}{V}\frac{\partial}{\partial V} + D_x(\overset{*}{U})\frac{\partial}{\partial (U_x)} + D_y(\overset{*}{U})\frac{\partial}{\partial (U_y)} + D_x(\overset{*}{V})\frac{\partial}{\partial (V_x)} + D_y(\overset{*}{V})\frac{\partial}{\partial (V_y)} + \ldots$$

Then the derivative $\frac{\partial^2 U}{\partial x \partial y}$ takes the form UXY both at input and at output, and the derivative of the vector field coordinate $\overset{*}{V}$ having the mathematical designation

$$\frac{\partial^3 \overset{*}{V}}{\partial x \partial U \partial (V_{xx})}$$

will take the form V#.(X,U,VXX) at output.

The program executes sequently the following operations:

1) Reading and printing of the input data.
2) Computation of the dimensions of the used spaces and the creation of the working array ZK for the economical representation of the row $(x, u, \underset{1}{u}, \underset{2}{u}, \ldots)$.

3) Computation of the differential consequences of the needed order and exclusion of the dependences out of them. The following remarks should be made here. We consider the differential consequences up to the certain fixed order k. If the system of equations under consideration is not in involution then it may occur that the uncalculated differential consequences of the higher than k order create the relation of the

order $\leq k$ as their algebraic consequences. The inclusion of this new relations may lead to the symmetry group increase. The reduction of the initial system into involution is not carried out in the program. Thus, if there is a suspicion that the input system isn't in involution, then in order to obtain the maximum invariance algebra with guarantee the system must be preliminary reduced into involution. Note, that there are some computer programs for reducing into involution the systems of differential equations, realizing the Cartan's exterior forms method [6] as well as the Riquier-Janet-Thomas method [7].

4) Creation of the working array Z# for the economical representation of the row from the vector field coordinate derivatives. Computation of the vector field prolongations. Computation of the invariance condition (3). When computing this condition the transition to the manifold is executed. In this stage it may occur that some equation of the input system is not explicitly solved with respect to any derivatives. Such, for example, is the equation from [8]:

$$\Box u + \lambda_1 \sin(\Box u) + \lambda_2 \sin\frac{\partial u}{\partial t} = 0,$$

where $\Box = \frac{\partial^2}{\partial t^2} - \Delta$ - d'Alembert operator. In the similar cases when transferring to the manifold the program used only the equations which were solved and the following message will be printed:

** TRANSITION TO MANIFOLD IS NOT COMPLETE.

Of course, in such cases the invariance algebra may turn out to be not maximum.

After transition to the manifold the used differential consequences are printed in the form solved with respect to some derivatives (i.e., pair derivatives and the corresponding right parts are printed). Then the differential consequences are deleted from the computer memory.

5) Separation of the determining equations. Unlike [5] here the separation of the determining equation is carried out not only with respect to different powers of the "free" derivatives but with respect to arbitrary different independent functions of such derivatives as well. The caution is necessary here, as the program does not take into account that the functions may be dependent with particular values of parameters. If such values occur, then the particular form of the equations should be proceeded by the program separately. Below the explanatory example will be considered. When the separation of the determining equations is executed, zeros are deleted at the same time, i.e.,when the one-term determining equations arise they substituted at once into all remaining expressions.

6) Exclusion of the linear dependences from the system of determining equations by reducing to the Hermite normal form using Gauss-Jordan method.

7) Replacing the working symbols of the arrays ZK and Z# by output symbols and printing of the determining equations. The determining equations are printed in the form solved with respect to some derivatives.

3. How to Use the Program. Example

The user must input the following information:

1) The order of the determining equations in the form of PL/1 integer constant.

2) The name of the equation or the system of equations in the form of PL/1 character string of the length not more than 78 characters.

3) Symbols used to represent independent and dependent variables in the form PL/1 character string. At first the independent variables, then the symbol " (double quote) and finally the dependent variables are followed.

4) The equations are introduces by pairs - the left parts of the equations in the form of PL/1 character string and the order of the equations in the form of PL/1 integer constant. The designations of the derivatives must be in agreement with the order specified in the symbol string of the independent variables.

The elements of the input are read by PL/1 operators GET LIST and, hence, must be separated by blanks or commas.

Short example

The first order determining equations of the one-dimensional linear heat equation $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$.

Input:

```
   1
'HEAT EQUATION'
'XT"U'
'UT - UXX'   2
```

Output:

```
INPUT DATA
SYMMETRY ORDER  1
HEAT EQUATION
VARIABLES XT"U
EQUATIONS
UT - UXX = Ø  2
DIMENSIONS OF THE SPACES
N = 2  M = 1  NE = 1  NZK = 8 NDPZK = 5 NMXZK = 12
```



```
EQUATIONS OF THE MANIFOLD
UXX
DFF(1) = UT
-----------
UXXX
DFF(2) = UXT
------------
UXXT
DFF(3) = UTT
------------
DETERMINING EQUATIONS OF THE ORDER 1
U# .(UT,UT)
O(3) = Ø
---------
U# .(UX,UT)
O(2) = -U #.(X.UT)/UT - UX U #.(U.UT)/UT
----------------------------------------
U# .(UX,UX)
O(1) = U# .(T)/UT2- U# .(X,X)/UT2 -
-----------------------------------
UX2 U# .(U.U)/UT2 - 2UX U# .(U.UX)/UT -
---------------------------------------
2 UX U# .(X,U)/UT2 - 2U# .(X,UX)/UT
-----------------------------------
```

Here N is the number of independent variables; M, the number of dependent variables; NE, the number of equations; NZK,NDPZK, NMXZK are the dimensions of the different subspaces of the space Z_k; DFF(i) is the right part of the i-th equation of the differential manifold; O(i) is the right part of the i-th determining equation.

4. Some Results of the Program Application

Let us demonstrate some examples of the program usage to solve the problems of mathematical physics. In [8] the problems for investigation of the symmetry properties of some equations of mathematical physics were stated. In particular, it was proposed to find the invariance algebras of the following nonlinear wave equations.

$$\Box u + \lambda u + \lambda_1 \frac{\partial u}{\partial x_\mu} \frac{\partial u}{\partial x^\mu} = 0, \qquad \text{(a)}$$

$$\Box u + \lambda_1 \frac{\partial u}{\partial t} + \lambda_2 \left(\frac{\partial u}{\partial t}\right)^k = 0, \qquad \text{(b)}$$

$$\Box u + \lambda_1 u + \lambda_2 (1 - \lambda_3 u^2) \frac{\partial u}{\partial t} = 0, \qquad \text{(c)}$$

$$\Box u + \lambda_1 u + \lambda_2 \left(\frac{\partial u}{\partial t}\right)^3 = 0, \qquad \text{(d)}$$

$$\frac{\partial^2 u}{\partial t^2} + \lambda_1 \Delta(u^k) + \lambda_2 (\Delta u)^k = 0. \qquad \text{(e)}$$

Here

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad \Box = \frac{\partial^2}{\partial t^2} - \Delta,$$

$$\frac{\partial u}{\partial x_\mu} \frac{\partial u}{\partial x^\mu} \equiv \left(\frac{\partial u}{\partial t}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 - \left(\frac{\partial u}{\partial z}\right)^2,$$

λ, λ_1, λ_2, λ_3, k are the arbitrary parameters.

Let us present the invariance algebras of these equations, obtained from the first order solutions of the determining equations. The first order symmetries for all these equations turned out to be equivalent to the point ones.

Equations (a)

In case $\lambda \neq 0$ we have the following set of the infinitesimal generators:

$$e_1 = \frac{\partial}{\partial t}, \quad e_2 = \frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial y}, \quad e_4 = \frac{\partial}{\partial z},$$

$$e_5 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad e_6 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad e_7 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},$$

$$e_8 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}, \quad e_9 = y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}, \quad e_{10} = z\frac{\partial}{\partial t} + t\frac{\partial}{\partial z},$$

i.e., when $\lambda \neq 0$, the invariance algebra of the equation (a) is the 10-dimensional Lie algebra of the Poincare group.

When $\lambda = 0$ the invariance algebra includes 15-dimensional conformal algebra, generator of the shift of u and infinite-dimensional subalgebra. The following generators are added to the Poincare algebra:

$$e_{11} = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

$$e_{12} = (t^2 + x^2 + y^2 + z^2)\frac{\partial}{\partial t} + 2xt\frac{\partial}{\partial x} + 2yt\frac{\partial}{\partial y} + 2zt\frac{\partial}{\partial z} - \frac{2t}{\lambda_1}\frac{\partial}{\partial u},$$

$$e_{13} = 2xt\frac{\partial}{\partial t} + (t^2 + x^2 - y^2 - z^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} - \frac{2x}{\lambda_1}\frac{\partial}{\partial u},$$

$$e_{14} = 2yt\frac{\partial}{\partial t} + 2xy\frac{\partial}{\partial x} + (t^2 - x^2 + y^2 - z^2)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} - \frac{2y}{\lambda_1}\frac{\partial}{\partial u},$$

$$e_{15} = 2zt\frac{\partial}{\partial t} + 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (t^2 - x^2 - y^2 - z^2)\frac{\partial}{\partial z} - \frac{2z}{\lambda_1}\frac{\partial}{\partial u},$$

$$e_{16} = \frac{\partial}{\partial u}, \quad e_\infty = \psi(x, y, z, t)\, e^{-\lambda_1 u}\frac{\partial}{\partial u},$$

where the function ψ is the arbitrary solution of the equation $\Box\psi = 0$.

When $\lambda = 0$, the equation (a) turns out to be automorphic, that is, all its solutions may be obtained from the any one solution with the help of group transformations. Solving the Lie equation corresponds to the subalgebra e_∞

$$\frac{du}{d\psi} = e^{-\lambda_1 u}$$

(infinite set of the ordinary differential equations) we obtain the linearizing substitution $u = \ln\psi / \lambda_1$ transforming the equation

$$\Box u + \lambda_1 \frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} = 0$$

into the equation $\Box\psi = 0$. Note, that such a substitution transforms the arbitrary equation of the form

$$\Box u + \phi(u) + \lambda_1 \frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} = 0$$

into the equation $\Box\psi + \lambda_1 \psi\, \phi\left(\frac{\ln\psi}{\lambda_1}\right) = 0.$

Equation (b)

We shall assume, that $k \neq 1$ as the case $k = 1$ is equivalent to the case $\lambda_2 = 0$. We shall not consider also the cases $k = 0$ and $\lambda_2 = 0$, as leading to the linear equations.

Case $k \neq 2$, $\lambda_1 \neq 0$

The symmetry generators are:

$$e_1 = \frac{\partial}{\partial t}, \quad e_2 = \frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial y}, \quad e_4 = \frac{\partial}{\partial z},$$

$$e_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad e_6 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad e_7 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},$$

$$e_\infty = \phi(x, y, z)\frac{\partial}{\partial u},$$

where ϕ is the arbitrary solution of the equation $\Delta\phi = 0$. Generators $e_1, \dots, e_4$ correspond to the shifts of t, x, y, z, generators $e_5 - e_7$ – to the three-dimensional rotations and e_∞ generates the symmetry consisting in possibility of adding the arbitrary harmonic function of the variables x, y, z to u.

Case $k \neq 2$, $\lambda_1 = 0$

Generator

$$e_8 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \frac{k-2}{k-1}u\frac{\partial}{\partial u},$$

which corresponds to the scale transformation, is added to the generators $e_1 - e_7$, e_∞.

Case $k = 2$

Generators $e_1 - e_7$ remain as in previous cases. Generator of the scale transformation takes the form

$$e_8 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - \frac{\lambda_1}{2\lambda_2}t\frac{\partial}{\partial u}.$$

The function ϕ from the operator $e_\infty = \phi(x, y, z)\frac{\partial}{\partial u}$ must satisfy the equation $\Delta\phi + (c_1\lambda_1^2)/(2\lambda_2) = 0$.

Equation (c)

This equation is the multidimensional generalization of the Van der Pol equation. Let's assume that $\lambda_2 \neq 0$ as otherwise it is the Klein-Gordon equation. The invariance algebra is 10-dimensional and includes:

the shifts of t, x, y, z:

$$e_1 = \frac{\partial}{\partial t}, \quad e_2 = \frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial y}, \quad e_4 = \frac{\partial}{\partial z},$$

three-dimensional rotations:

$$e_5 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad e_6 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad e_7 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},$$

the scale transformation:

$$e_8 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + f(u, t)\frac{\partial}{\partial u},$$

and two additional transformations of u:

$$e_9 = g(u, t)\frac{\partial}{\partial u}, \quad e_{10} = h(u, t)\frac{\partial}{\partial u},$$

where the functions f, g and h are defined by the equations:

$$\overset{*}{u}(u, t) = c_8 f + c_9 g + c_{10} h,$$

$$\overset{*}{u}_{tt} + \lambda_1\overset{*}{u} - \lambda_1 u\overset{*}{u}_u + \overset{*}{u}_t\lambda_2(1 - u^2\lambda_3) + 2\lambda_1 u c_8 = 0,$$

$$\overset{*}{u}_{tu} - \lambda_2\lambda_3 u\overset{*}{u} + \frac{c_8}{2}\lambda_2(1 - u^2\lambda_3) = 0.$$

Here c_8, c_9, c_{10} are the constants, corresponding to the generators e_8, e_9, e_{10} The last two equations come to the relation

$$\overset{*}{u}_{ttt} + \lambda_2(1 - \lambda_3 u^2)\overset{*}{u}_{tt} + \lambda_1\overset{*}{u}_t - \lambda_1\lambda_2\lambda_3 u^2\overset{*}{u} +$$

$$+ c_8\frac{\lambda_1\lambda_2}{2}(1 - \lambda_3 u^2)u = 0,$$

which is the ordinary differential equation with the constant coefficients. Thus the problem comes to the solution of the algebraic equation of the third power:

$$p^3 + p^2\lambda_2(1 - \lambda_3 u^2) + p\lambda_1 - u^2\lambda_1\lambda_2\lambda_3 = 0.$$

The final expressions are compound. For example, even in case $\lambda_1 = 0$ the solution is expressed by the probability integrals.

Equation (d)

The case $\lambda_2 = 0$ leads to Klein-Gordon equation, the case $\lambda_1 = 0$ is reduced to the considered particular case of the equation (b). Thus, we shall assume that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

The symmetry group generators are the following

$$e_1 = \frac{\partial}{\partial t}, \quad e_2 = \frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial y}, \quad e_4 = \frac{\partial}{\partial z},$$

$$e_5 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad e_6 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad e_7 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},$$

$$e_\infty = \phi(x, y, z)\frac{\partial}{\partial u},$$

where ϕ is the arbitrary solution of the Helmholtz equation $\Delta\phi = \lambda_1 \phi$.

Equation (e)

While investigating this equation, the situation mentioned in item 5) of the program work description is occurred. This situation requires attention, that is why let us consider it in more detail.

The invariance condition has a form

$$D_t^2(\overset{*}{u}) + \lambda_1 \{ k(k-1)(k-2)u^{k-3}(\nabla u)^2 \overset{*}{u} +$$

$$+ k(k-1)u^{k-2}(D_x + D_y + D_z)\overset{*}{u} + k(k-1)u^{k-2}\Delta u \overset{*}{u} +$$

$$+ k u^{k-1}(D_x^2 + D_y^2 + D_z^2)\overset{*}{u} \} + \lambda_2 k(\Delta u)^{k-1}(D_x^2 + D_y^2 + D_z^2)\overset{*}{u} = 0.$$

The underlined terms (after transferring to the manifold) contain variables u_{xx}, u_{yy}, u_{zz} in the power 0, 1, 2 and the last term contains this variables in the power $k-1$ and higher. Hence, the cases $k = 1,2,3$ must be executed by the program separately.

Let us consider the general case, i.e., let us assume that $k \neq 1,2,3$. The solution of the determining equations leads to the necessity to distinguish the following cases.

Case $\lambda_1 \neq 0$, $\lambda_2 = 0$, $k = 1/5$

The invariance algebra is 12-dimensional and includes the following generators:

$$e_1 = \frac{\partial}{\partial t}, \quad e_2 = \frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial y}, \quad e_4 = \frac{\partial}{\partial z},$$

$$e_5 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \quad e_6 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad e_7 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},$$

$$e_8 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \frac{5}{2}u\frac{\partial}{\partial u},$$

$$e_9 = (x^2 - y^2 - z^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} - 5xu\frac{\partial}{\partial u},$$

$$e_{10} = 2xy\frac{\partial}{\partial x} + (y^2 - x^2 - z^2)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} - 5yu\frac{\partial}{\partial u},$$

$$e_{11} = 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (z^2 - x^2 - y^2)\frac{\partial}{\partial z} - 5zu\frac{\partial}{\partial u},$$

$$e_{12} = t\frac{\partial}{\partial t} - \frac{5}{2}u\frac{\partial}{\partial u}.$$

The generators $e_9 - e_{11}$ correspond to the conformal transformations.

Case $\lambda_1 \neq 0$, $\lambda_2 = 0$, $k = -3$

The invariance algebra is 10-dimensional: $e_1 - e_7$ are as in previous case,

$$e_8 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \quad e_9 = t\frac{\partial}{\partial t} + \frac{u}{2}\frac{\partial}{\partial u},$$

$$e_{10} = t^2\frac{\partial}{\partial t} + ut\frac{\partial}{\partial u}.$$

Case $\lambda_1 \neq 0$, $\lambda_2 = 0$, $k \neq -3, 1/5$

The invariance algebra is 9-dimensional: $e_1 - e_8$ are as in previous case,

$$e_9 = \frac{1-k}{2}t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

Case $\lambda_1 \neq 0$, $\lambda_2 \neq 0$

The invariance algebra is 7-dimensional and includes $e_1 - e_7$ of the previous case.

Case $\lambda_1 = 0$, $\lambda_2 \neq 0$

The invariance algebra is infinite-dimensional. It contains the generators $e_1 - e_7$ from the previous case, scale transformation generator

$$e_8 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u}$$

and infinite-dimensional subalgebra

$$e_\infty = \{\phi(x, y, z) + t\psi(x, y, z)\}\frac{\partial}{\partial u},$$

ϕ and ψ are an arbitrary solutions of the equations $\Delta\phi = \Delta\psi = 0$.

When $k = 2,3$ four additional cases arise:

Case $\lambda_1 \neq 0,\ \lambda_2 \neq 0,\ k = 2$

The invariance algebra is 8-dimensional and includes $e_1 - e_7$ from the previous cases and

$$e_8 = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

Case $\lambda_1 \neq 0,\ \lambda_2 = 0,\ k = 3$

The invariance algebra is 8-dimensional. The generator

$$e_8 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

is added to the generators $e_1 - e_7$.

Case $\lambda_1 = 0,\ \lambda_2 \neq 0,\ k = 2$

Additional generators are:

$$e_8 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u},$$

$$e_9 = t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

$$e_\infty = \{\phi(x,y,z) + t\psi(x,y,z)\}\frac{\partial}{\partial u}.$$

$$\Delta\phi = \Delta\psi = 0.$$

Case $\lambda_1 = 0,\ \lambda_2 \neq 0,\ k = 3$

Additional generators are:

$$e_8 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2u\frac{\partial}{\partial u}.$$

$$e_9 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},$$

$$e_\infty = \{\phi(x,y,z) + t\psi(x,y,z)\}\frac{\partial}{\partial u}, \quad \Delta\phi = \Delta\psi = 0.$$

The case $k = 1$ leads to well-known linear wave equation.

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Received by Publishing Department
on March 5, 1985.

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Федорова Р.Н., Корняк В.В. E11-85-164
Определение симметрий Ли-Беклунда дифференциальных уравнений с помощью системы FORMAC

В работе описана программа на языке PL/1-FORMAC для вычисления симметрий Ли-Беклунда дифференциальных уравнений. Приведены алгебры симметрий некоторых нелинейных волновых уравнений, полученные с помощью этой программы.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1985

Fedorova R.N., Kornyak V.V. E11-85-164
Determination of Lie-Bäcklund Symmetries of Differential Equations Using FORMAC

The program written in PL/1-FORMAC language to calculate Lie-Bäcklund symmetries of differential equations is described. Symmetry algebras of some nonlinear wave equations, obtained with the help of the program, are presented.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1985