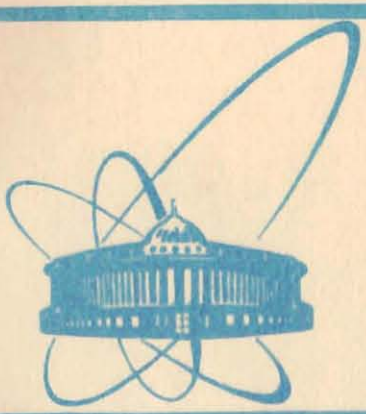

сообщения
объединенного
института
ядерных
исследований
дубна

E11-84-869

V.G.Makhankov, A.B.Shvachka, Z.Sedlaček*

NUMERICAL SOLUTION OF THE KORTEWEG-DE VRIES EQUATION BY PSEUDOSPECTRAL METHOD

* Institute of Plasma Physics,
Czechoslovak Academy of Sciences,
Prague, Czechoslovakia

1984

1. INTRODUCTION

In this paper we describe a computer program for solving the Korteweg-de Vries equation by means of the pseudospectral method [1]. We Fourier transform the KdV equation with respect to the space coordinate to obtain an infinite set of ordinary differential equations for the time evolution of Fourier harmonics of the solution which is truncated and solved by any appropriate numerical method. We have used the RKF45 Runge-Kutta fourth-fifth order subroutine with variable step [2] and ODE, DE/STEP, INTRP predictor-corrector subroutine with variable step and order [3]. The non-linear term in the KdV equation is calculated in the spirit of pseudospectral methods by Fourier synthesizing the solution in the original x-representation from its discrete Fourier harmonics, performing the non-linear operation and returning back to the representation of Fourier harmonics. The Fast Fourier algorithm FOUR67 [4] is used to perform the discrete Fourier synthesis and analysis. Contrary to similar approaches [5,6] we use for the time integration the most precise numerical algorithm at present available, according to recent rigorous tests [7], to eliminate the influence of the time discretization on the numerical stability and precision of the method.

So far we have tested only one soliton solutions of the KdV equation. Very good performance characteristics of our code were found. The algorithm is stable numerically, and the form of the soliton is reproduced with very high precision even with relatively few harmonics used. For a moderately narrow soliton (in comparison with the spatial period of the solution) eight Fourier harmonics were sufficient, in full agreement with the infinite order precision property of spectral methods in general. For a solution of the KdV equation which has derivatives of infinite order, the Fourier harmonics amplitudes decrease faster than any inverse power of the number of harmonics and, correspondingly, also the truncation error diminishes in the same manner. On the other hand, the error of finite-differences numerical schemes is known to decrease only as some inverse power of the number of mesh points.

In future we plan to do further experiments with multisoliton solutions of the KdV equation and to utilize the pseudospectral method for the solution of other non-linear evolution equations including two- and three-dimensional equations of current interest.

2. NUMERICAL METHOD

The KdV equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \qquad (2.1)$$

is Fourier transformated by means of the transformation

$$u(x) = \sum_{n=-\infty}^{\infty} u_n e^{2\pi j n x}, \qquad (2.2)$$

$$u_n = \int_0^1 u(x) e^{-2\pi j n x}\, dx. \qquad (2.3)$$

We thus obtain

$$\dot{u}_n - 6 \int_0^1 \Big(\sum_{m=-\infty}^{\infty} u_m e^{2\pi j m x} \sum_{m=-\infty}^{\infty} 2\pi j m u_m e^{2\pi j m x} \Big) e^{-2\pi j n x} dx - j(2\pi n)^3 u_n = 0, \quad n = 0, \pm 1, \pm 2, \ldots . \qquad (2.4)$$

The infinite sums are now truncated at some $n = N_{max}$ so that a finite set of ordinary differential equations is obtained. Besides, the continuous Fourier transform (2.2), (2.3) is replaced by its discrete analog, the discrete Fourier transform:

$$u(x_i) = \sum_{n=-N_{max}+1}^{N_{max}} U_n e^{2\pi j n \frac{i}{2N_{max}}} \qquad (2.5)$$

$$U_n = \frac{1}{N_{max}} \sum_{i=0}^{2N_{max}-1} u(x_i) e^{-2\pi j n \frac{i}{2N_{max}}}. \qquad (2.6)$$

In our code, this transformation is realized by the Fast Fourier Transform algorithm /4/ which works with 2^q, $q = 3,4,5,\ldots$ harmonics and the function is defined on a mesh with $2^q + 1$ points. In this version of the FFT algorithm the analyzed function U(x) is assumed real so that $U_{-n} = U_n^*$ and, consequently, only U_n with $n = 0,1,2,\ldots, N_{max}$ are calculated.

To get rid of the aliasing error in calculating the non-linear term, one must use a mesh with double this number of points. In the synthesis operation, the missing harmonics are then simply replaced by zeros, and in the analysis operation the superficial harmonics are dropped. The set of ordinary differential equations which are actually solved by the computer can

then be written as follows:

$$\dot{U}_n = -6 \frac{1}{N_{max}} \sum_{i=0}^{2N_{max}-1} \sum_{m=-N_{max}+1}^{N_{max}} U_m e^{2\pi j m \frac{i}{2N_{max}}} \times \times \sum_{m=-N_{max}+1}^{N_{max}} 2\pi j m e^{2\pi j m \frac{i}{2N_{max}}} - j(2\pi n)^3 U_n = 0, \qquad (2.7)$$

for the aliased case, and

$$\dot{U}_n = -6 \frac{1}{2N_{max}} \sum_{i=0}^{4N_{max}-1} \sum_{m=-N_{max}+1}^{N_{max}} U_m e^{2\pi j m \frac{i}{4N_{max}}} \times \times \sum_{m=-N_{max}+1}^{N_{max}} 2\pi j U_m e^{2\pi j m \frac{i}{4N_{max}}} - j(2\pi n)^3 U_n = 0, \quad n = 0, 1, 2, \ldots, 2^q \qquad (2.8)$$

for the dealiased case.

3. PROGRAM DESCRIPTION

The whole code is written in FORTRAN and has a modular structure so as to make easy to modify and easy to use. In fact, with only minor changes, the present code can be used to solve other non-linear evolution equations as well.

The MAIN program serves only the purpose of allocating the memory space for all the arrays used, read and write the input data and call the main solution subroutine KdV. On finishing the run, the control is returned from KdV to the MAIN program and a new run is started by reading the corresponding new set of input data. If none is found, the program is stopped.

Subroutine KdV first calculates some auxiliary variables which are not changed during one calculation run, calls the initialization subroutine SETF67 for the FFT subroutine FOUR67 /4/, calculates the initial data by means of the FUNCTION subroutine UIC, Fourier transforms them and finally calls iteratively the set of subroutines for solving the set of ordinary differential equations RKF45, RKFS, FEHl /2/. On completion of a time-step, the time integration subroutine returns the value of the monitoring variable IFLAG. According to this value the time-integration is continued or an appropriate measure is taken first, e.g., the relative error tolerance RELERR is changed and then the integration is continued until the prescribed finishing time is reached or untill subroutine RKF45 finds it impossible to continue.

In between, in prescribed time-intervals, the subroutine OUTP is called which activates the system CPU-time measuring procedure, calculates some output variables and calls the printing subroutine PRINT to print the synthesized solution, the Fourier harmonics of the solution and their absolute values, together with the current value of time, the value of IFLAG (the monitor variable of RKF45), the number of time-step performed from the last call of OUTP, CPU-time elapsed from the last call of OUTP and total CPU-time elapsed from the start of the first run.

The right-hand sides of the set of ordinary differential equations (2.7),(2.8) are calculated by the subroutine FCT utilizing the set of FFT subroutines FOUR67, ZERO, MEG, REVNEG, TEOLD, KFOLD. The subroutines DYNAM1, DYNAM2 serve as an interface between the complex type variables that are used in calculating the right-hand sides and the subroutine FOUR67 which in fact performs a real discrete Fourier analysis and synthesis. (The analyzed or synthesized function is assumed real, just as the Fourier coefficients of the sine and cosine series).

Within the program, the arrays are communicated either by dummy arguments or by labeled COMMON blocks such that each block contains only one array. In either case, in subprograms the length of each array is specified to be unity. The real length of all the arrays is to be specified only in the MAIN program and no changes in the subroutines are required.

4. THE USE OF THE CODE

We assume that the code will be used in the batch processing environment so that the user's deck will contain the MAIN program and the FUNCTION subroutine UIC together with the necessary input data cards. All other subroutines may be stored on a disc or a tape.

In the MAIN program, the user must specify the correct length of all the arrays. The length depends on

1) the number of Fourier harmonics used (LFIN),

2) the number of mesh points used by the FFT subroutine (2^{IQ}),

3) the time-integration subroutine used (RKF45 or ODE, DE/STEP, INTRP).

Thus, the formulas for the array length are as follows:

CUP/LFIN/
CUPIC/LFIN/
ACUP/LFIN/
C1/LFIN/
C2/LFIN/
S1/LFIN/
S2/LFIN/
U/2^{IQ}/
Y/2xLFIN+2/
WORK/3+6 x (2 x LFIN + 2) for RKF45
WORK/100 + 21 x (2 x LFIN + 2)/ for ODE, DE/STEP, INTRP
IWORK/5/
CDIN(LFIN/
CDOUT/LFIN/
F67IN/2^{IQ} + 1/
F67OUT/2^{IQ} + 1/
INDEX/2^{IQ-1} + 1/
SI/2^{IQ-1} + 1/

The FUNCTION subprogram should contain the Fortran deck for calculating the initial condition as a function of x. The value of x is communicated by means of a single dummy argument.

The input data cards should be in the NAMELIST format and should specify the value of the following parameters:

TSTART	the value of time at which the time-integration subroutine starts (normally 0.0)
TFINAL	the value of time at which the time integration subroutine should stop
PRDEL	the time interval between two calls of the subroutine OUTPUT
RELERR	the bound on the relative error
ABSERR	the bound on the absolute error
IFLAG	the starting value of the monitor variable (normally 1)
IQ	2^{IQ}/IQ = 3/ is equal to the number of mesh points
LFIN	the number of Fourier harmonics used (normally 2^{IQ-1} for the aliased case, 2^{IQ-2} for the dealiased case)
A	the parameter occurring in the FUNCTION subprogram UIC defining the height of the soliton (see sec.5, the parameter a in (5.1))
S1,S2	the spectral windows, should be specified as arrays of LFIN length, normally all elements are unity.

These cards should be inserted between the NAMELIST input card & INPUT and &END.

The use of other time-integration subroutines other than RKF45 requires changes only in the KdV subroutine.

5. RESULTS

At present, we have a relatively closed set of results concerning the performance of the code corresponding to the one-soliton initial condition

$$u(x, 0) = -\frac{a^2}{2}\cosh^2\frac{ax}{2}. \qquad (5.1)$$

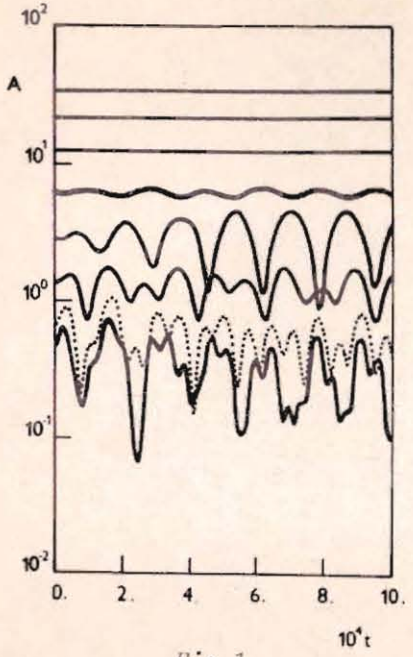

Fig. 1

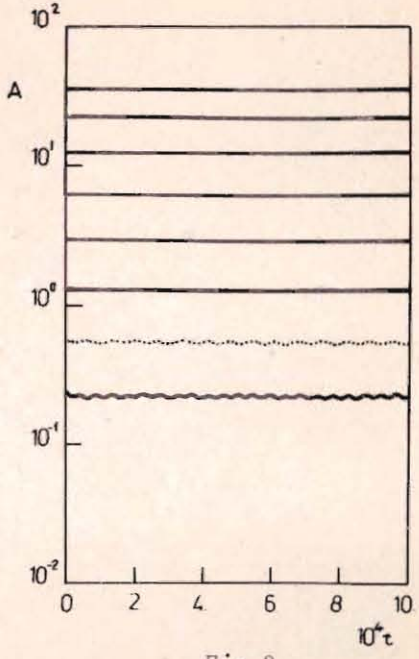

Fig. 2

We have tried various numbers of the Fourier harmonics (LFIN = = 8,16) corresponding to a given soliton amplitude (fixed by the value of a = 20). Further we have examined the influence of various values of RELERR and ABSERR on the global precision of the calculation. By increasing the number of mesh points to twice that necessary for a given number of harmonics, we have been able to completely eliminate the aliasing error [7] which arises in calculation of the quadratic non-linear term in the KdV equation. This enabled us to isolate the effect of aliasing error from other errors. In each of these experiments the sophisticated time-step controlling algorithm of the subroutine RKF45 practically eliminated any possible source of errors due to the time-integration procedure.

The computational results are summarized in Figs.1 and 2. In Fig.1 the time evolution of the absolute values of the Fourier harmonics is plotted for LFIN = 8 with 16 mesh points used (full aliasing included). We see that all the Fourier amplitudes oscillate with various frequencies about their constant correct values. In this sense our code is absolutely stable. The relative amplitude of these oscillations decreases with decreasing harmonics number and for the lowest harmonics is practically negligible. The amplitude of the oscillations does not grow even for many passages of the soliton through the basic interval(0,1).

In Fig.2 the same variables are plotted for 32 mesh points (no aliasing). This picture represents the pure effect of truncation of the infinite set of ordinary differential equations. In comparison with Fig.1 we see that the amplitudes of oscillations diminished substantially and are in fact significant only for the highest two. Thus it is seen that the truncation errors are probably of much less importance as compared with the aliasing errors.

In comparison with RKF45, the performance of the ODE, DE/STEP, INTRP time-integration subroutines was poorer. With the same RELERR, ABSERR requirements, more integration steps were needed, so that there was no overall efficiency gain, although less right-hand sides computations per time-step were performed. Besides, very slow time-growth of harmonic of amplitudes was detected (numerical instability).

REFERENCES

1. Gottlieb D., Orszag S.A. Numerical Analysis of Spectral Methods: Theory and Applications. Society for Industrial and Applied Mathematics. Philadelphia, Pennsylvania, 1977.
2. Forsythe G.E., Malcolm M.A., Moler C.B. Computer Methods for Mathematical Computations. Prentice-Hall, Inc., Englewood Cliffs, 1977.
3. Shampine L.F., Gordon M.K. Computer Solution of Ordinary Differential Equations. The Initial Value Problem. W.H.Freeman and Company. San Francisco, 1975.
4. Christiansen J.P., Hockney R.W. Comp.Phys.Comm., 1971, 2, p.127.
5. Schamel H., Elsässer K. J.Comp.Phys., 1976, 22, p.501.
6. Fornberg B., Whitham G.B. Phil.Trans.Roy.Soc.(London), 1978, A289, p.373.
7. Shampine L.F. et al. SIAM Rev., 1976, 18, p.376.

Received by Publishing Department
on December 29,1984.

В Объединенном институте ядерных исследований начал выходить сборник *"Краткие сообщения ОИЯИ"*. В нем будут помещаться статьи, содержащие оригинальные научные, научно-технические, методические и прикладные результаты, требующие срочной публикации. Будучи частью "Сообщений ОИЯИ", статьи, вошедшие в сборник, имеют, как и другие издания ОИЯИ, статус официальных публикаций.

Сборник "Краткие сообщения ОИЯИ" будет выходить регулярно.

The Joint Institute for Nuclear Research begins publishing a collection of papers entitled *JINR Rapid Communications* which is a section of the JINR Communications and is intended for the accelerated publication of important results on the following subjects:

Physics of elementary particles and atomic nuclei.
Theoretical physics.
Experimental techniques and methods.
Accelerators.
Cryogenics.
Computing mathematics and methods.
Solid state physics. Liquids.
Theory of condensed matter.
Applied researches.

Being a part of the JINR Communications, the articles of new collection like all other publications of the Joint Institute for Nuclear Research have the status of official publications.

JINR Rapid Communications will be issued regularly.

Маханьков В.Г., Швачка А.Б., Седлачек З. E11-84-869

Численное решение уравнения Кортевега-де Вриза методом быстрого преобразования Фурье

Описывается программа для решения уравнения Кортевега-де Вриза методом дискретного преобразования Фурье. В работе представлены некоторые результаты анализа ошибок, связанных со "стробоскопическим" эффектом.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1984

Makhankov V.G., Shvachka A.B., Sedlaček Z. E11-84-869

Numerical Solution of the Korteweg-de Vries Equation by Pseudospectral Method

A computer program for solving the Korteweg-de Vries equation based on discrete Fourier transformation is described and some results concerning the analysis of aliasing errors are presented.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1984