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SUPERSYMMETRIC SOLITONS WITH FRACTIONAL QUANTUM NUMBERS

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1. INTRODUCTION

Recently, much progress has been made toward understanding the role of the fractional quantum numbers on solitons /1/.

There has been found that the interaction of a quantum soliton with charged fermion can lead to degeneracy and fractional charge of the soliton ground state.

Independently it was shown that a natural way of incorporating the fermionic sector in solitonic theories is via a supersymmetric extension /2/.

The degenerate fractionally charged ground states of the soliton arise when the spectrum of the physical fermions exhibits a zero-frequency mode, which was firstly obtained in the second paper of ref. /2/.

However, the class of models in which fermion zero modes can appear is much greater than the class of the supersymmetric models.

In this paper we study fractional quantum fermion charge only from supersymmetric point of view.

2. SUPERSOLITONS

The basic lagrangian density for supersymmetric solitons in (1+1) dimensional field theories has the following form:

$$L = \frac{1}{2}\{\partial_\mu \phi\, \partial^\mu \phi - V^2(\phi) + \bar{\psi}(i\gamma\cdot\partial - V'(\phi))\psi\}, \qquad (2.1)$$

which is invariant under the supersymmetry transformations

$$\delta\phi = \bar{\epsilon}\psi, \quad \delta\psi = (-i\gamma\cdot\partial\phi - V(\phi))\epsilon. \qquad (2.2)$$

The conserved supercurrent has the form:

$$\partial_\mu[\gamma\cdot\partial\phi + iV(\phi)\gamma^\mu\psi] = 0. \qquad (2.3)$$

From (2.3) it is possible to obtain supercharge:

$$Q = \int dx[(\gamma\cdot\partial\phi + iV(\phi))\gamma^0\psi]$$

and to show that the superalgebra has the form:

$$Q_+^2 = P_0 + P_1 = P_+, \quad Q_-^2 = P_0 - P_1 = P_-,$$

$$\{Q_+, Q_-\} = 2 \int_{\phi(-\infty)}^{\phi(+\infty)} d\phi\, V(\phi) = T, \tag{2.4}$$

where $Q_\pm = 1/2(1 \pm i\gamma^5)Q$.

We used the gamma matrix representation:

$$\gamma^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}; \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \quad ''\gamma^5'' = i\gamma^0\gamma^1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

It can be directly seen that the Fermi part of the lagrangian (2.1) is also invariant under the internal phase transformation:

$$\psi \to e^{i\alpha}\psi, \quad \bar{\psi} \to e^{-i\alpha}\bar{\psi}, \quad \alpha \text{ real}, \tag{2.5}$$

which is generated by the conserved charge:

$$Q_F = \frac{1}{2} \int dx [\psi^+, \psi]. \tag{2.6}$$

This conserved charge is called fermion number and this number is carried by generators $Q_\pm$. It is clear that the only way to introduce such a quantum number in supersymmetry is by means of the transformations for supercharges: $Q \to e^{-\alpha\gamma^5}Q$ and for chiral components we get: $Q \to 1/2(1 \pm i\gamma^5)e^{-\alpha\gamma^5}Q = e^{\pm i\alpha}Q_\pm$. Now we shall use as examples of supersymmetric solitons two theories:

a) super sine-Gordon theory:

$$L = \frac{1}{2}\partial_\mu \beta \partial^\mu \phi - \frac{\alpha_0}{\beta}(1 - \cos\beta\phi) + \frac{1}{2}\bar{\psi}(i\gamma\cdot\partial - \sqrt{\alpha_0}\cos\frac{\beta}{2}\phi)\psi, \tag{2.7}$$

which is obtained from (2.1) for sine-Gordon potential

$$U(\phi) = \frac{1}{2}V^2(\phi) = \frac{\alpha_0}{\beta}(1 - \cos\beta\phi);$$

b) super $''\phi^4''$ theory:

$$L = \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - \frac{\Lambda^2}{2\mu^2}(\mu^2 - \phi^2)^2 + \frac{1}{2}\bar{\psi}(i\gamma\cdot\partial - 2\frac{\Lambda}{\mu}\phi)\psi, \tag{2.8}$$

which is obtained from (2.1) for $''\phi^4''$ potential

$$U(\phi) = \frac{1}{2}V^2(\phi) = \frac{\Lambda^2}{2\mu^2}(\mu^2 - \phi^2)^2.$$

At the classical level the solution of equation of motion, which is obtained from lagrangian (2.1) for the Bose sector, satisfies:

$$\frac{d}{dx}\phi_{S(\bar{S})} = \pm V(\phi_{S(\bar{S})}), \quad \frac{d}{dt}\phi_{S(\bar{S})} = 0 \tag{2.9}$$

where the sign (+) is for soliton (s); and (−), for antisoliton ($\bar{s}$).

For our examples they are given by:

$$\phi_{S(\bar{S})} = \frac{4}{\beta}\,\mathrm{tang}^{-1}\exp(\pm\sqrt{\alpha_0}x), \tag{2.10a}$$

$$\phi_{K(\bar{K})} = \pm\mu\,\tanh\Lambda x. \tag{2.10b}$$

With ϕ_S (or ϕ_K) being the soliton solution the equation of motion for the Fermi sector satisfies:

$$(i\gamma\cdot\partial - V'(\phi_S))\psi = 0 \tag{2.11}$$

and possesses a stable normalizable solution ψ_S localized at the position of the soliton ϕ_S (we shall confine ourselves to studying the case of the soliton, because the same results would be obtained for antisolitons).

For the supersymmetric sine-Gordon theory the solution ψ_S satisfies static Dirac equation:

$$(i\gamma_1\partial_1 - \sqrt{\alpha_0}\cos\frac{\beta}{2}\phi_S)\psi_S = 0 \tag{2.12a}$$

and ψ_S has the form:

$$\psi_S = \frac{N}{(\cosh\sqrt{\alpha_0}x)^2}\begin{pmatrix}1\\1\end{pmatrix}.$$

where N is finite normalization constant.

For the super $''\phi^4''$ theory the solution of the Dirac eq. (2.11) with ϕ_K from (2.10b)

$$(i\gamma_1\partial_1 - 2\frac{\Lambda}{\mu}\phi_K)\psi_K = 0 \tag{2.12b}$$

is given by: $\psi_K = N\exp(-2\frac{\Lambda}{\mu}\int_0^x dx'\tanh\Lambda x')\begin{pmatrix}1\\1\end{pmatrix}$.

The supersymmetric Lagrangian (2.8) and supersymmetric solitons ϕ_K, ψ_K in relations (2.10b) and (2.12b) were proposed by R.Jackiw and C.Rebbi in condense matter theory, without knowledge of supersymmetry /1,3/.

The static solutions ψ_s, ψ_K are zero-eigenvalue solutions. It is clear from the following:

We shall write the stationary solution to eq. (2.11) as usual $\psi(x,t) = e^{-iE_k t}\psi_k(x)$, then from (2.11) follows:

$$(i\gamma_1 \partial_1 - V'(\phi_s))\psi_k(x) = -E_k \gamma_0 \psi_k(x) \qquad (2.13)$$

and eqs. (2.12a,b) are obtained from (2.13) for $\psi_0 \equiv \psi_s$ and $E_0 = 0$.

Eq. (2.13) also gives positive frequency modes $\psi_k^{(+)}$ (for $E_k > 0$) and negative frequency modes $\psi_k^{(-)}$ (for $E_k < 0$).

These eigenmodes satisfy the orthogonality conditions:

$$\int dx\, \psi_s^+(x)\, \psi_s(x) = 1, \qquad (2.14a)$$

$$\int dx\, \psi_s^+(x)\, \psi_k^{(\pm)}(x) = 0, \qquad (2.14b)$$

$$\int dx\, \psi_k^{+(j)}(x)\, \psi_k^{(j')}(x) = \delta_{jj'}\, \delta_{kk'}, \quad j,j' = +,-; \qquad (2.14c)$$

and the completeness relation

$$\psi_s^+(x)\, \psi_s(x') + \sum_k (\psi_k^{+(+)}(x)\, \psi_k^{(+)}(x') + \psi_k^{+(-)}(x)\, \psi_k^{(-)}(x')) = \delta(x - x'). \qquad (2.14d)$$

The charge conjugation matrix $C = -\gamma^0$ transforms eigensolutions as follows: $\psi_s^C = C\bar{\psi}_s^T = \psi_s$ and $\psi_k^{(+)C} = \psi_k^{(-)}$.

Therefore the zero-energy solution ϕ_s is selfconjugate and positive eigenvalue is paired with negative one under charge conjugation.

The fermion number operator Q_F given by relation (2.6) transforms under charge conjugation as

$$Q_F^C = -Q_F. \qquad (2.15)$$

The quantization of the Fermi field is achieved by the investigation of the structure of the Fock space and by the expansion in eigenmodes:

$$\psi = a\psi_s + \sum_k (b_k \psi_k^{(+)} + d_k \psi_k^{(-)}); \qquad (2.16)$$

the anticommutation relation is valid:

$$\{a, a^+\} = 1, \quad \{b_k, b_{k'}^+\} = \{d_k, d_{k'}^+\} = \delta_{kk'} \qquad (2.17)$$

and other anticommutation relations vanish. The operators $b_k^+(b_k)$ create (annihilate) one-particle states for nonzero energy fermions and $d_k^+(d_k)$ for antifermions in the presence of the soliton.

However, the further operator a when operating on the solution state produces another state of the same energy; hence in the presence of the soliton, the ground states are degenerate in energy. To distinguish between them, we may label them as $|\pm, s\rangle$ and the following relations are valid:

$$a|+,s\rangle = |-,s\rangle, \quad a^+|-,s\rangle = |+,s\rangle,$$
$$a|-,s\rangle = 0, \quad a^+|+,s\rangle = 0. \qquad (2.18)$$

Substituting the expansion (2.16) into (2.6) and using (2.17) one finds

$$Q_F = \frac{1}{2}\int dx(\psi^+\psi - \psi\psi^+) = a^+a - \frac{1}{2} + \sum_k (b_k^+ b_k - d_k^+ d_k). \qquad (2.19)$$

Thus it is usual that Q_F changes sign under charge conjugation, because $a^C = a^+$, $b_k^C = d_k$.

Consequently it follows from relations (2.18) and (2.14): $Q_F|\pm,s\rangle = \pm 1/2|\pm,s\rangle$. Since operator a is associated with the fermion zero mode, which is a zero energy wave, ψ_s is localized at the position of the soliton ($\bar{\psi}_s\psi_s = 0$ and $E(\phi_s,\psi_s) = E(\phi_s)$).

The lowest energy states $|\pm,s\rangle$ represent solitons with fermion numbers $\pm 1/2$, i.e., each of the two fermion soliton states carries 1/2 unit charge. In such a way the fermionic soliton affects a transition between the lowest energy states: $\langle -,s|\psi|+,s\rangle \neq 0$.

3. THE VANISHING MASS QUANTUM CORRECTIONS

From the superalgebra (2.4) we obtain:

$$P_+ + P_- = T + (Q_+ - Q_-)^2,$$

$$P_+ + P_- = -T + (Q_+ + Q_-)^2, \quad \text{so} \quad P_+ + P_- \geq |T|$$

and for a single soliton at rest $P_+ = P_- = M$ and so we get:

$$M \geq \frac{1}{2}|T|. \qquad (3.2)$$

From (3.1) the inequality (3.2) is saturated for these states $|s\rangle$ such that

$$(Q_+ \pm Q_-)|s\rangle = 0. \qquad (3.3)$$

It is clearly satisfied classically, because after using (2.9) we obtain: $Q_- - Q_+ = \int dx(\partial_1\phi - V)(\psi_+ - \psi_-) = 0$, and also $Q_+ + Q_- = 0$.

Explicit calculation shows that classically the solitons satisfy

$$M = \frac{1}{2}|T|. \qquad (3.4)$$

If $Q_\pm$ are interpreted as operators, (3.3) can be stated:

$$(\hat{Q}_{-} - \hat{Q}_{+})|s\rangle = 0. \qquad (3.5)$$

Then for a soliton at rest

$$2\langle s|P_0|s\rangle = \langle s|Q_+^2 + Q_-^2|s\rangle = \langle s|(Q_- - Q_+)^2|s\rangle + \langle s|T|s\rangle. \qquad (3.6)$$

But from (3.4) T is a topological invariant and its absolute value is equal to $2M_{cl.}$, then from (3.6) we get [4]: $M = M_{cl.}$

It means that in theory of supersolitons, which respects supersymmetry, there are no quantum corrections to the mass of the soliton, when $(Q_- - Q_+)^2|s\rangle = (\bar{Q})^2|s\rangle = 0$.

It can be explicitly shown using the results in ref.[8], where it is shown that $Q \sim a$ and $\bar{Q} \sim a^+$.

If we use relations (2.18) we obtain for ground states following expression: $\bar{Q}^2|\pm, s\rangle = 0$.

In this way (3.6) is valid in quantum case without assumption of the relation (3.5).

4. BOSONIZATION AND SUPERSOLITONS WITH FRACTIONAL CHARGE

Bosonization is a procedure which was used by S.Coleman [5] for showing equivalence between massive Thirring model and sine-Gordon model. Also the formation of fractionally charged solitons can be visualized by bosonizing the system [6].

We shall study the supersymmetric lagrangian (2.1) by bosonizing the system using the basic rules of the boson representation as follows:

$$\frac{1}{2} i\bar{\psi}\gamma\cdot\partial\psi = \frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma, \quad \bar{\psi}\psi = G\cos\sqrt{4\pi}\sigma, \quad \bar{\psi}\gamma_\mu\psi = \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial^\nu\sigma, \qquad (4.1)$$

where σ is a scalar field.

We shall write the fermionic charge Q_F in terms of σ

$$Q_F = \int_{-\infty}^{+\infty} dx\,\bar{\psi}\gamma_0\psi = \frac{1}{\sqrt{\pi}}(\sigma(+\infty) - \sigma(-\infty)) = \frac{1}{\sqrt{\pi}}\Delta\sigma. \qquad (4.2)$$

The bosonized version of the Lagrangian (2.1) is given by

$$L = \frac{1}{2}\{\partial_\mu\phi\,\partial^\mu\phi + \partial_\mu\sigma\,\partial^\mu\sigma - V^2(\phi) - GV'(\phi)\cos\sqrt{4\pi}\sigma\}. \qquad (4.3)$$

In order to identify the particle excitations of the system one should identify various classical vacua of the system. They consist of the configuration (ϕ, σ) which minimize the potential $W(\phi,\sigma)$, where

$$W(\phi,\sigma) = -\frac{1}{2}(V^2(\phi) + GV'(\phi)\cos\sqrt{4\pi}\sigma). \qquad (4.4)$$

One can see that, in general, the minima of $W(\phi,\sigma)$ are for super "ϕ^4" model:

a) $\sigma = (n + \frac{1}{2})\sqrt{\pi}$ when $\phi = \phi_{min}$,

b) $\sigma = n\sqrt{\pi}$ when $\phi = -\phi_{min}$,

where ϕ_{min} is the solution of $\partial W(\phi,\sigma)/\partial\phi = 0$, $\partial^2 W(\phi,\sigma)/\partial^2\phi > 0$. From this follows $\Delta\sigma = \pm 1/2\sqrt{\pi}$ and from (4.2) we can see that two solitons acquire a fermionic charge $Q_F = \pm 1/2$.

5. A TWO-DIMENSIONAL BAG MODEL WITH FRACTIONAL QUANTUM NUMBERS

One of the applications of supersymmetric solitons in the elementary particle physics is obtaining a supersymmetric bag model by analogy with the SLAC-bag [7].

It can be shown that in the super "ϕ^4" theory given by lagrangian (2.8), that the kink solution ϕ_K (2.9b) traps a quark ψ_K (2.12b), because the trapping of the quark doesn't play role in the kink energy, i.e., $\bar{\psi}_K\psi_K = 0$.

So for classical energy of the kink with a trapped quark we obtain:

$$E(\phi_K, \psi_K) = E(\phi_K) = \int dx\{\frac{1}{2}(\frac{d}{dx}\phi_K)^2 + U(\phi_K)\} = \frac{4}{3}\mu^2\Lambda \qquad (5.1)$$

In this way as in the SLAC-bag the solutions ϕ_K, ψ_K represent a field theoretical bound state with strong binding and may describe hadrons with confined quarks.

A two-dimensional bag model with fractional quantum numbers is obtained by extension of the study on Fermi fields with $SU(n)\times U(1)$ global symmetry.

The lagrangian density has the form:

$$L = \frac{1}{2}\{\partial_\mu\phi\,\partial^\mu\phi - V^2(\phi) + Tr\bar{\psi}(i\gamma\cdot\partial - V'(\phi))\psi\}, \qquad (5.2)$$

where $1/2\,V^2(\phi) = \Lambda^2/(2\mu^2)(\mu^2 - \phi^2)^2$ and ψ_α^b is a two-component spinor field with internal-symmetry index $b = 1,\dots,n$.

With ϕ being the kink solution the equations of motion for the Fermi sector satisfy:

$$(i\gamma\cdot\partial - V'(\phi_K))\psi^b = 0 \qquad (5.3)$$

and again the physical fermion has a zero-frequency solution.

Since L is assumed invariant under $\psi \to e^{-i\alpha_a I^a}\psi$ and $\psi \to e^{i\alpha}\psi$, where I^a are classical generators of $SU(n)$; $[I^a, I^b] = i\epsilon^{abc}I^c$, the following conserved charges exist:

$$Q_{F_a} = \frac{(I^a)_{bd}}{2} \int dx\, [\psi^{+b}, \psi^d], \tag{5.4a}$$

$$Q_F = \frac{1}{2} \int dx\, \mathrm{Tr}[\psi^+, \psi]. \tag{5.4b}$$

Following Y. Leblanc and G. Semenoff [8] we obtain the multiplicity of the kink states, which increases with n and all these states always carry nonvanishing quantum numbers.

Denoting the diagonal generators of SU(n), U(1) as F_i, F and using notation $F_i|s\rangle = f_i|s\rangle$, $i = 1,\dots,n-1$, $F|s\rangle = f|s\rangle$, so that $|s\rangle = |f, f_1, \dots, f_i\rangle$, then examples of the fermionic quantum numbers up to $SU(3) \times U(1)$ are following:

U(1):

f
1/2
−1/2 ,

SU(2)×U(1):

f	f_1
1	0
0	−1/2
0	1/2
−1	0 ,

SU(3)×U(1):

f	f_1	f_2
3/2	0	0
1/2	−1/2	$-1/2\sqrt{3}$
1/2	1/2	$-1/(2\sqrt{3})$
−1/2	0	$-1/\sqrt{3}$
1/2	0	$1/\sqrt{3}$
−1/2	−1/2	$1/(2\sqrt{3})$
−1/2	1/2	$1/(2\sqrt{3})$
−3/2	0	0 .

In this way we get a supersymmetric bag model with one kink field and color quark fields with fractional quantum numbers.

One can speculate to investigate fermion internal $SU(n) \times U(1)$ symmetry locally and obtain connection with fractional quantum numbers in supersymmetric bag with physical charges, which can play the role of the central charges in superalgebra [9].

A candidate for this model in (3+1) dimension has the lagrangian:

$$L = -\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} + \frac{1}{2} (D_\mu \phi)_a (D^\mu \phi)_a - \frac{1}{2} V^2(\phi_a \phi_a) + \frac{1}{2} \mathrm{Tr}\bar{\psi}(i\gamma \cdot D - V'(\phi_a \phi_a))\psi,$$

where (5.5)

$$F^{\mu\nu}_a = \partial^\mu A^\nu_a - \partial^\nu A^\mu_a + e C_{abc} A^\mu_b A^\nu_c ,$$

$$(D^\mu \phi)_a = \partial^\mu \phi_a + e C_{abc} A^\mu_b \phi_c ,$$

$$V^2(\phi_a \phi_a) = \frac{\Lambda^2}{\mu^2} (\mu^2 - \phi_a \phi_a)^2, \quad V'(\phi_a \phi_a) = 2\frac{\Lambda}{\mu} I^a \phi_a ,$$

$$D^\mu \psi = (\partial^\mu - ieI^a A^\mu_a)\psi .$$

For SU(2) group (5.5) gives exactly Jackiw-Rebbi extension of the Hooft-Polyakov monopole, where zero energy solution and fractional quantum numbers are found [10].

The utility of the lagrangian (5.5) and connection of the fractional fermion charge with physical charges is in progress.

6. CONCLUSIONS

In this paper deep connection between supersymmetric solitons and theories with fractional quantum charge was shown.

The support for old result, that quantum mass corrections in theories with supersymmetric solitons vanish, was given.

A super-quark-bag model with fractional quantum charge is presented.

There can exist connection between fermion quantum numbers and central charges in superalgebra with SU(n) internal symmetry, the exact quantum mechanical mass spectrum can be determined from the mass formula.

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Грубы Я. E11-83-887

Суперсимметричные солитоны с дробными квантовыми числами

Исследуются суперсимметричные солитоны с дробными зарядами. Получена точная формула для массы солитона. Показана возможность исследования фермионных дробных зарядов в рамках суперсимметричной модели "кваркового мешка".

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Supersymmetric Solitons with Fractional Quantum Numbers

The supersymmetric solitons with fractional charges are studied. Exact mass formula is proved. A way for using fermionic fractional charges in the supersymmetric quark bag model is shown.

The investigation has been performed at the Laboratory of the Computing Techniques and Automation, JINR.

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