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REDUCE-2 PROGRAM FOR OBTAINING QUASI-PERIODIC SOLUTIONS OF MODIFIED NONLINEAR SCHRÖDINGER EQUATION

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1. INTRODUCTION. BASIC DEFINITIONS

In the present work a Computer Algebra System REDUCE-2 is applied to obtain finite-gap solutions of the modified non-linear Schrödinger equation (MNLSE) in terms of theta functions:

$$iq_t = -q_{xx} \mp i(|q|^2 q)_x . \tag{1}$$

General aspects of the method of finite-gap integration (periodic analog of the inverse spectral problem) are discussed in review articles /1,2/. The periodic problem for MNLSE was studied in /3,4/. Our work is based on the paper /4/. Next we present some basic definitions and necessary formulas. Let us consider the following system of differential equations

$$iq_t = -q_{xx} + i(wq^2)_x, \qquad iw_t = w_{xx} + i(w^2 q)_x . \tag{2}$$

When $w = \pm\bar{q}$ this system reduces to MNLSE.

The Zakharov-Schabat representation for the system (2) is given by /5/:

$$[\partial_x - U, \ \partial_t - V] = 0, \tag{3}$$

where

$$U = \lambda^2 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \lambda \begin{pmatrix} 0 & q \\ w & 0 \end{pmatrix}, \tag{4}$$

$$V = 2\lambda^4 \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + 2\lambda^3 \begin{pmatrix} 0 & q \\ w & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} -iwq & 0 \\ 0 & iwq \end{pmatrix} + \lambda \begin{pmatrix} 0 & wq^2 + iq_x \\ w^2 q - iw_x & 0 \end{pmatrix}. \tag{5}$$

Let us consider the following system of linear differential equations:

$$\partial_x \Psi = U\Psi , \tag{6}$$

$$\partial_t \Psi = V\Psi . \tag{7}$$

Definition 1 (Baker-Akhiezer function)

Let Γ be the hyperelliptic Riemann surface

$$y^2 = \prod_{j=1}^{2g+2} (z - E_j), \quad E_{2g+2} = 0 \tag{8}$$

of genus g. We denote by P^+ and P^- the points on Γ of type $P=(\infty\pm)$, D is a nonspecial divisor.

A Baker-Akhiezer function $\Psi=\Psi(x,t,P)$ is called the solution of the system (6),(7) with the following properties:

1) $\Psi = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{z} \end{pmatrix}\psi$, $\lambda=\sqrt{z}$, is unique on Γ, meromoprhic on $\Gamma/\{\infty\pm\}$ and its pole divisor $D=\sum_{i=1}^{g} P_i$ doesn't depend on x and t.

2) ψ has the following asymptotic behaviour near P

$$\psi_1^+ = u_1(x,t)\{\xi_{10}^+ + \xi_{12}^+ z^{-1} + \xi_{14}^+ z^{-2} + \dots\}\exp\{-i(zx+2z^2t)\},$$

$$\psi_2^- = u_1(x,t)\{1+\xi_{12}^- z^{-1} + \xi_{14}^- z^{-2} + \dots\}\exp\{i(zx+2z^2t)\}, \tag{9}$$

$$\psi_2^+ = u_2(x,t)\{z^{1/2} + \xi_{21}^+ z^{-1/2} + \xi_{23}^+ z^{-3/2} + \dots\}\exp\{i(zx+2z^2t)\},$$

$$\psi_2^- = u_2(x,t)\{\xi_{21}^- z^{-1/2} + \xi_{23}^- z^{-3/2} + \dots\}\exp\{-i(zx+2z^2t)\}.$$

The function is determined up to indefinite multipliers $u_i(x,t)$, $i=1,2$.

Definition 2 (Realization of hyperelliptic Riemann surface)
Let us consider the hyperelliptic curve:

$$y^2 = R(z) = \prod_{j=1}^{2g+2}(z-E_j).$$

The Riemann surface Γ is constructed from the two copies of the complex plane with cuts $((\infty,E_1], [E_{2j},E_{2j+1}], [E_{2g+2},\infty))$. There are two infinities on this surface $\infty+$ and $\infty-$, on the upper and on the lower sheet, respectively. On Γ we take a canonical homology basis a_j, b_j, $j=1,\dots,g$. For a_j we take a closed contour which surrounds clockwise the cut $[E_{2j},E_{2j+1}]$. For b_j we take a closed contour which starts at E_{2j+1}, goes on the upper sheet as far as E_{2g+2}, then it continues on the lower sheet and ends at E_{2j+1} (Figure).

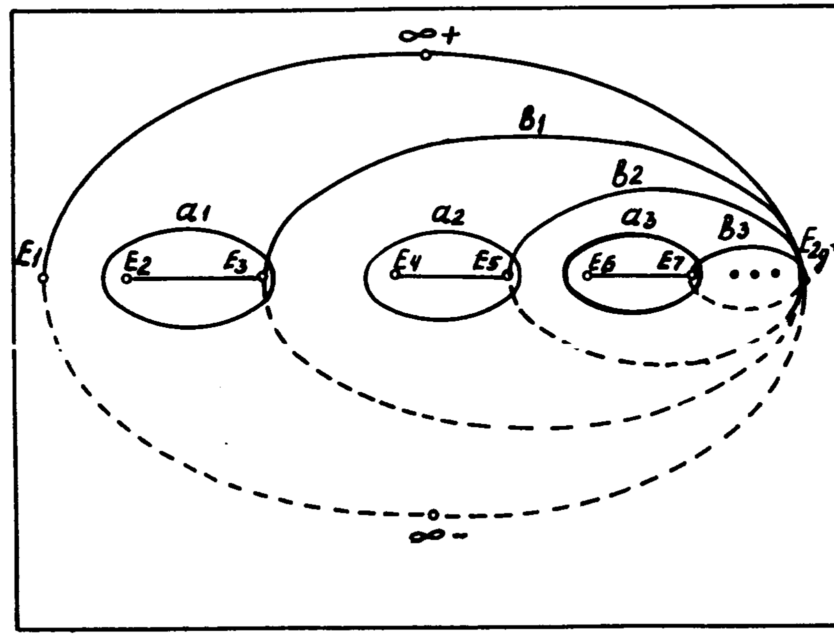
Next we introduce the Abelian differentials of the first kind

$$d\omega_j = \sum_{j=1}^{g}\Gamma_j^k z^{g-k}\frac{dz}{R^{1/2}(z)}, \tag{10}$$

where the matrix of constants Γ_j^k is fixed by the normalization conditions $\int_{a_k} d\omega_j = \delta_{kj}$. From these differentials we define the "period matrix" (Riemann matrix) $B=(B_{kj})$ by $\int_{b_k} d\omega_j = B_{kj}$; and the Abel map A(P), by $A(P)=(\int_0^P d\omega_i,\dots,\int_0^P d\omega_g)$, $P\subset\Gamma$.

Let us also define Abelian differentials of the third kind on Γ, Ω_1, Ω_2 and Ω_3 with zero a-periods, whose singularities are in $\infty\pm$. The explicit form of these differentials is:

$$\begin{aligned}
\Omega_1(P) &= \pm(z - E_0/2 + R_1 z^{-1} + R_2 z^{-2}) + O(z^{-3}), && P\to\infty\pm\\
\Omega_2(P) &= \pm(2z^2 + N_0/2 + F_1 z^{-1} + F_2 z^{-2}) + O(z^{-3}), && P\to\infty\pm\\
\Omega_3(P) &= \ln z/2 + \gamma z^{-1} + \gamma_1 z^{-2} + O(z^{-3}), && P\to\infty+\\
\Omega_3(P) &= -\ln z/2 + \ln\omega_0 + \delta z^{-1} + \delta_1 z^{-2} + O(z^{-3}), && P\to\infty-.
\end{aligned} \tag{11}$$



Definition 3 (Theta function)
The multidimensional theta function of argument v is

$$\theta(v|B) = \sum_{k\subset Z^g}\exp\{\pi i\langle Bk,k\rangle + 2\pi i\langle k,v\rangle\}, \tag{12}$$

where B is the g-dimensional symmetric complex matrix, $\langle\,,\rangle$ denotes the scalar product $\langle k,v\rangle=\sum_{i=1}^{g}k_i v_i$, $\langle Bk,k\rangle=\sum_{i,j=1}^{g}B_{ij}k_i k_j$ and the sum is over the lattice Z^g, i.e., the set of g-dimensional vectors with integer (real) components k_i ($k_i=0,\pm1,\pm2,\dots,$

$i = 1,2,\ldots, g$), $\sum\limits_{k \subset Z^g} = \sum\limits_{k_1=-\infty}^{+\infty}, \ldots, \sum\limits_{k_g=-\infty}^{+\infty}$. The series (12) will be convergent if there exists $C > 0$ such that $\mathrm{Im}<Bk, k> \geq C<k, k>$. If matrix B is the Riemann matrix we define g-dimensional theta function associated with the Riemann surface.

Using Definition 2 we have the following explicit expressions of $\Psi = (\Psi_1, \Psi_2)$ in theta functions

$$\Psi_1 = \frac{\theta(A(P) - g(x,t))\theta(g(0,0) + r)}{\theta(A(P) - g(0,0))\theta(g(x,t) + r)} \exp\{i\Omega_1(P)x + i\Omega_2(P)t - \frac{ixE_0}{2} + \frac{itN_0}{2}\}, \quad (13)$$

$$\Psi_2 = \frac{\theta(A(P) - g(x,t) - r)\,\theta(g(0,0) - r)}{\theta(A(P) - g(0,0))\,\theta(g(x,t))} \exp\{i\Omega_1(P)x + i\Omega_2(P)t + \Omega_3(P) + \frac{ixE_0}{2} - \frac{itN_0}{2}\},$$

where (14)

$$g_j(x,t) = -2i\Gamma_j^1 x - 4i(\Gamma_j^2 + \frac{1}{2}\Gamma_j^1 \sum_{i=1}^{g} E_i)t + g_j(0,0), \quad j = 1,\ldots,g,$$

$$g_j(0,0) = -\sum_{m=1}^{g}\int_0^{P_m} d\omega_j + \frac{j}{2} - \frac{1}{2}\sum_{m=1}^{g} B_{mj}, \quad r = (\int_0^{\infty+} d\omega_1 \ldots \int_0^{\infty+} d\omega_g),$$

$g_j(x,t)$ and $g_j(0,0)$ are g-dimensional vectors. We take the last integral along the left-hand side of the cut of the upper sheet.

2. DESCRIPTION OF THE PROGRAMME

Module 1

We define the matrices U, V and the series (9) at the beginning of the programme. Next we write the equations $\partial_x \Psi - U\Psi = 0$, $\partial_t \Psi - V\Psi = 0$ near the point at infinity $\infty\pm$ and obtain the relations between the terms of the series (9). These relations are coefficients at different orders of the local parameter z. We illustrate some of calculations with the following expressions

```
2*I*K14P(X,T)*U1(X,T) - Q*K23P(X,T)*U2(X,T)
+ DF(K12P(X,T),X)*U1(X,T) + DF(U1(X,T),X)*K12P(X,T),
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i.e.,

$$2i\xi_{14}^+ u_1 - q\xi_{23}^+ u_2 + \partial_x \xi_{12}^+ u_1 + \partial_x u_1 \xi_{12}^+ = 0,$$

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2*I*K12P(X,T)*U1(X,T) - Q*K21P(X,T)*U2(X,T)
+ DF(K1ØP(X,T),X)*U1(X,T) + DF(U1(X,T),X)*K1ØP(X,T),
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i.e., $2i\xi_{12}^+ u_1 - q\xi_{21}^+ u_2 + \partial_x \xi_{10}^+ u_1 + \partial_x u_1 \xi_{10}^+ = 0$, 2*I*K1ØP(X,T)*U1(X,T) - Q*U2(X,T),

i.e., $2i\xi_{10}^+ u_1 - qu_2 = 0$.

Module 2

In this module we obtain expressions for the terms of the series (9) near the points at infinity $\infty\pm$, taking into account the asymptotics [4]:

$$\ln\theta(A(P) - g(x,t)) \underset{P\to\infty+}{=} \ln\theta(g(x,t) - r) + \frac{i}{2}\frac{\partial}{\partial x}\ln\theta(g(x,t) - r)z^{-1} + \frac{1}{8}(-\frac{\partial^2}{\partial x^2} + i\frac{\partial}{\partial t})\ln\theta(g(x,t) - r)z^{-2} + O(z^{-3}),$$

(15)

$$\ln\theta(A(P) - g(x,t)) \underset{P\to\infty-}{=} \ln\theta(g(x,t) + r) - \frac{i}{2}\frac{\partial}{\partial x}\ln\theta(g(x,t) + r)z^{-1} - \frac{1}{8}(\frac{\partial^2}{\partial x^2} + i\frac{\partial}{\partial t})\ln\theta(g(x,t) + r)z^{-2} + O(z^{-3}).$$

After the transition $g(x,t) \to g(x,t) + r$ we obtain asymptotic expressions for $\ln\theta(A(P) - g(x,t) - r)$ near the points at infinity $\infty\pm$. The programme compares the logarithm of Ψ which follows from (9) and the analogical expression, which is derived from (13), (14) after substituting the asymptotics of the Abels differentials (11). We obtain the following result:

$$\xi_{10}^+ = \frac{\theta(g(0,0) + r)\;\theta(g(x,t) - r)}{\theta(g(0,0) - r)\;\theta(g(x,t) + r)} \exp(-iE_0 x + iN_0 t), \quad (16)$$

$$\xi_{12}^+ = \xi_{10}^+(\frac{i}{2}\frac{\partial}{\partial x}\ln\theta(g(x,t) - r) - \frac{i}{2}\frac{\partial}{\partial x}\ln\theta(g(0,0) - r) + ixR_1 + iF_1 t), \quad (17)$$

$$\xi_{14}^+ = \xi_{10}^+[\frac{1}{8}(-\frac{\partial^2}{\partial x^2} + i\frac{\partial}{\partial t})\ln\theta(g(x,t) - r) - \frac{1}{8}(-\frac{\partial^2}{\partial x^2} + i\frac{\partial}{\partial t})\ln\theta(g(0,0) - r) + iR_2 x + iF_2 t + 1/2\,\xi_{12}^{+2}], \quad (18)$$

$$\xi_{12}^- = -\frac{i}{2}\frac{\partial}{\partial x}\ln\theta(g(x,t) + r) + \frac{i}{2}\frac{\partial}{\partial x}\ln\theta(g(0,0) + r) - iR_1 x - iF_1 t, \quad (19)$$

$$\xi_{14}^- = -\frac{1}{8}(\frac{\partial^2}{\partial x^2} + i\frac{\partial}{\partial t})\ln\theta(g(x,t) + r) + \frac{1}{8}(\frac{\partial^2}{\partial x^2} + i\frac{\partial}{\partial t})\ln\theta(g(0,0) + r) - iR_2 x - iF_2 t + \frac{1}{2}\xi_{12}^{-2} \quad (20)$$

$$\xi_{21}^+ = \frac{i}{2}\frac{\partial}{\partial x}\ln\theta(g(x,t)) - \frac{i}{2}\frac{\partial}{\partial x}\ln\theta(g(0,0) - r) + ixR_1 + iF_1 t + \gamma, \quad (21)$$

$$\xi_{23}^+ = \frac{1}{8}(-\frac{\partial^2}{\partial x^2} + i\frac{\partial}{\partial t})\ln\theta(g(x,t)) - \frac{1}{8}(-\frac{\partial^2}{\partial x^2} + i\frac{\partial}{\partial t})\ln\theta(g(0,0) - r) + iR_2 x + iF_2 t + \gamma_1 + \frac{1}{2}\xi_{21}^{+2}, \quad (22)$$

$$\xi_{21}^- = \omega_0 \frac{\theta(g(x,t) + 2r)\;\theta(g(0,0) - r)}{\theta(g(x,t))\;\theta(g(0,0) + r)} \exp(iE_0 x - iN_0 t), \quad (23)$$

$$\xi_{23}^- = \xi_{21}^-(-\frac{i}{2}\frac{\partial}{\partial x}\ln\theta(g(x,t) + 2r) + \frac{i}{2}\frac{\partial}{\partial x}\ln\theta(g(0,0) + r) - iR_1 x - iF_1 t + \delta). \quad (24)$$

Module 3

In this module we derive two basic identities:

$$\xi^{+}_{10}\xi^{-}_{21} = \gamma + E_0/2 + \frac{i}{2}\frac{\partial}{\partial x}\ln\frac{\theta(g(x,t))}{\theta(g(x,t)+r)}, \tag{25}$$

$$2\xi^{+}_{10}\partial_x\xi^{-}_{21} - 2\partial_x\xi^{+}_{10}\xi^{-}_{21} - 4i(\xi^{+}_{10})^2(\xi^{-}_{21})^2 = 8i\gamma_1 - iN_0 + \frac{\partial}{\partial t}\ln\frac{\theta(g(x,t)+r)}{\theta(g(x,t))}. \tag{26}$$

Relations (25), (26) can be verified by applying the relations obtained in module 1:

$$\xi^{+}_{10}\xi^{-}_{21} = \frac{i}{2}\frac{\partial_x\xi^{+}_{10}}{\xi^{+}_{10}} - \frac{\xi^{+}_{12}}{\xi^{+}_{10}} + \xi^{+}_{21}, \tag{27}$$

$$i(2\xi^{+}_{10}\partial_x\xi^{-}_{21} - 2\partial_x\xi^{+}_{10}\xi^{-}_{21} - 4i(\xi^{+}_{10})^2(\xi^{-}_{21})^2) =$$

$$-i\frac{\partial_t\xi^{+}_{10}}{\xi^{+}_{10}} + 4\frac{\xi^{+}_{14}}{\xi^{+}_{10}} - 4\xi^{+}_{23} + 4\xi^{-}_{21}\xi^{+}_{12} - 2i\frac{\partial_x\xi^{+}_{10}}{\xi^{+}_{10}}\xi^{+}_{21}, \tag{28}$$

$$\xi^{+}_{23} - \frac{\xi^{+}_{14}}{\xi^{+}_{10}} - \xi^{-}_{21}\xi^{+}_{12} + \frac{i}{2}\frac{\partial_x\xi^{+}_{12}}{\xi^{+}_{10}} = 0, \tag{29}$$

$$\partial_x\xi^{+}_{21} - 2i\xi^{+}_{10}\xi^{-}_{21}\xi^{+}_{21} + 2i\xi^{-}_{21}\xi^{+}_{12} = 0. \tag{30}$$

The relation (25) is the direct result of substituting (16), (17), (21) into (27). Applying the procedure described in[4] and taking into account (27)-(30) we obtain (26).

Also in module 1 we obtain relations, which are used to determine the normalization functions $u_1(x,t)$, $u_2(x,t)$:

$$\frac{\partial}{\partial x}\ln u_1 = -\frac{\partial}{\partial x}\ln u_2 = 2i\xi^{+}_{10}\xi^{-}_{21}, \tag{31}$$

$$\frac{\partial}{\partial t}\ln u_1 = -\frac{\partial}{\partial t}\ln u_2 = 2\xi^{+}_{10}\partial_x\xi^{-}_{21} - 2\partial_x\xi^{+}_{10}\xi^{-}_{21} - 4i(\xi^{+}_{10})^2(\xi^{-}_{21})^2. \tag{32}$$

After the integration of (31), (32) we have the following result:

$$u_1(x,t) = u_2^{-1}(x,t) = \frac{\theta(g(x,t)+r)}{\theta(g(x,t))}\exp\{iE_0 - 2i\gamma)x - (iN_0 - 8i\gamma_1)t\}. \tag{33}$$

In module 1 we obtain the following basic expressions:

$$w = -2i\frac{u_2}{u_1}\xi^{-}_{21}, \tag{34}$$

$$q = 2i\frac{u_1}{u_2}\xi^{+}_{10}, \tag{35}$$

$$wq = 4\xi^{+}_{10}\xi^{-}_{21}. \tag{36}$$

Making use of (25), (33)-(36) and the explicit form of coefficients ξ^{+}_{10}, ξ^{-}_{21} in terms of theta functions we finally arrive at the exact solution of system (2)[4]:

$$w = -2i\omega_0\frac{\theta(g(x,t))\theta(g(x,t)+2r)\,\theta(g(0,0)-r)}{\theta^2(g(x,t)+r)\,\theta(g(0,0)+r)}\exp\{-i(E_0-4\gamma)x + i(N_0-16\gamma_1)t\}, \tag{37}$$

$$q = 2i\frac{\theta(g(x,t)+r)\,\theta(g(x,t)-r)\,\theta(g(0,0)+r)}{\theta^2(g(x,t))\,\theta(g(0,0)-r)}\exp\{i(E_0-4\gamma)x - i(N_0-16\gamma_1)t\}. \tag{38}$$

The result of calculations is also one interesting basic relation between the theta functions:

$$wq = 2i\frac{\partial}{\partial x}\ln\frac{\theta(g(x,t))}{\theta(g(x,t)+r)} + 4\gamma + 2E_0 = 4\omega_0\frac{\theta(g(x,t)+2r)\,\theta(g(x,t)-r)}{\theta(g(x,t))\,\theta(g(x,t)+r)}. \tag{39}$$

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Костов Н.А., Швачка А.Б. E11-83-767
Программа для нахождения квазипериодических решений модифицированного нелинейного уравнения Шредингера на языке системы REDUCE-2

Описан метод конечнозонного интегрирования на примере периодической задачи для модифицированного нелинейного уравнения Шредингера. Разработан алгоритм и создана программа на языке системы аналитических вычислений REDUCE-2 для получения конечнозонных решений модифицированного нелинейного уравнения Шредингера в тэта-функциях.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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REDUCE-2 Program for Obtaining Quasi-Periodic Solutions of Modified Nonlinear Schrödinger Equation

The method of finite-gap integration is considered. Using the Computer Algebra System REDUCE-2 we obtain solutions of the modified nonlinear Schrödinger equation in terms of theta functions.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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