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# INVESTIGATION OF NONLINEAR WATER WAVES USING COMPUTER ALGEBRA SYSTEM REDUCE-2

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## 1. INTRODUCTION

Investigation of the hydrodynamical equations is often related to tedious analytical manipulations. Equations, which describe propagation of nonlinear dispersive waves [1], in particular, is the problem of this type. One of interesting aspects of this problem was connected with investigation of nonlinear water gravity waves [2]. In many cases of application, perturbation methods are very useful to solve the problem. The powerful instrument of exploration problem of this type are Computer Algebra Systems (CAS) [3], in particular, universal CAS-system REDUCE-2 [4]. As an example of application of CAS-system REDUCE-2 for investigation of the hydrodynamical equations by perturbation method one can remark the important paper [5]. In the case of ideal (nonviscous) fluid, the equation of motion, kinematical and dynamical boundary conditions have the following form [6]:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad -H \leq z \leq Z(x,t) \qquad (1)$$

$$\left.\begin{aligned} &\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left[ \left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2 \right] + gZ = 0 \qquad (2) \\ &\frac{\partial \Phi}{\partial z} = \frac{\partial Z}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial Z}{\partial t} \qquad (3) \end{aligned}\right\} \quad z = Z(x,t)$$

$$\frac{\partial \Phi}{\partial z} = 0 \qquad z = -H, \qquad (4)$$

where $\Phi$ - velocity potential, $Z$ - wave profile, $H$ - water depth.

## 2. METHOD

One of the methods to solve (1)-(4) problem, when water depth is much higher than characteristic wavelength, was proposed by Stokes [7]. Basic idea consists in expansion of wave characteristics about the mean wave depth $\overline{Z(x,t)} = 0$. Convergence of expansions of this type in the case of stationary wave motion was proved in the classical paper of Levi-Civita /8/. Let us now take the solution as an expansion in an asymptotic series in small parameter $\epsilon$ (wave stepness) /6/:

$$\Phi = \Phi^{(0)} + \epsilon\Phi^{(1)} + \epsilon^2\Phi^{(2)} + \epsilon^3\Phi^{(3)} + \dots , \tag{5}$$

$$Z = Z^{(0)} + \epsilon Z^{(1)} + \epsilon^2 Z^{(2)} + \epsilon^3 Z^{(3)} + \dots . \tag{6}$$

We obtain the explicit form of the terms of series (5), (6) by using CAS-system REDUCE-2. Substituting asymptotic series (5), (6), after constructing Taylor expansion of functions $\Phi^{(i)}(x, z, t)$ near $z = 0$ in (1)-(4), we obtain the set of equations at different orders of small parameter $\epsilon$. At the first order we find well-known solution /6/, where the wave profile and velocity potential have the following form:

$$Z^{(1)} = A\cos(kx - \omega_0 t), \tag{7}$$

$$\Phi^{(1)} = A\frac{\omega_0}{k\,\mathrm{sh}(kH)}\mathrm{ch}\,k(z+H)\sin(kx - \omega_0 t), \tag{8}$$

where $\omega_0$ - wave frequency, $k$ - wave number, which are related by linear dispersion relation $\omega_0^2 = gk\,\mathrm{th}(kH)$; $A$ - wave amplitude, $g$ - acceleration due to gravity.

In the second order boundary conditions have the following form:

$$gZ^{(2)} + \frac{\partial\Phi^{(2)}}{\partial t} = -Z^{(1)}\left(\frac{\partial^2\Phi^{(1)}}{\partial t\partial z}\right)_0 - \frac{1}{2}\left[\left(\frac{\partial\Phi^{(1)}}{\partial x}\right)^2 + \left(\frac{\partial\Phi^{(1)}}{\partial z}\right)^2\right], \tag{9}$$

$$\frac{\partial Z^{(2)}}{\partial t} - \frac{\partial\Phi^{(2)}}{\partial z} = Z^{(1)}\left(\frac{\partial^2\Phi^{(1)}}{\partial z^2}\right)_0 - \frac{\partial Z^{(1)}}{\partial x}\left(\frac{\partial\Phi^{(1)}}{\partial x}\right)_0, \quad (\ )_0 = (\ )_{z=0}. \tag{10}$$

Solutions of (9), (10) are

$$Z^{(2)} = \frac{1}{2}A^2\frac{k}{\mathrm{th}(kH)}\left(1 + \frac{3}{2}\frac{1}{\mathrm{sh}^2(kH)}\right)\cos(2kx - 2\omega t), \tag{11}$$

$$\Phi^{(2)} = \frac{3}{8}A^2\frac{\omega_0}{k\,\mathrm{sh}^4(kH)}\mathrm{ch}\,2k(z+H)\sin(2kx - 2\omega t). \tag{12}$$



At the third order in small parameter boundary conditions con- contain seventeen terms

$$gZ^{(3)} + \frac{\partial\Phi^{(3)}}{\partial t} = -Z^{(1)}\left(\frac{\partial^2\Phi^{(2)}}{\partial z\partial t}\right)_0 - Z^{(2)}\left(\frac{\partial^2\Phi^{(1)}}{\partial z\partial t}\right)_0 - \frac{(Z^{(1)})^2}{2}\left(\frac{\partial^3\Phi^{(1)}}{\partial z^2\partial t}\right)_0 - \left(\frac{\partial\Phi^{(1)}}{\partial x}\right)_0\left(\frac{\partial^2\Phi^{(1)}}{\partial x\partial z}\right)_0 Z^{(1)} - \left(\frac{\partial\Phi^{(1)}}{\partial x}\right)_0\left(\frac{\partial\Phi^{(2)}}{\partial x}\right)_0 - \left(\frac{\partial\Phi^{(1)}}{\partial z}\right)_0\left(\frac{\partial^2\Phi^{(1)}}{\partial z^2}\right)Z^{(1)} - \left(\frac{\partial\Phi^{(1)}}{\partial z}\right)_0\left(\frac{\partial\Phi^{(2)}}{\partial z}\right)_0, \tag{13}$$

$$\frac{\partial Z^{(3)}}{\partial t} - \frac{\partial\Phi^{(3)}}{\partial z} = Z^{(1)}\left(\frac{\partial^2\Phi^{(2)}}{\partial z^2}\right)_0 + Z^{(2)}\left(\frac{\partial^2\Phi^{(1)}}{\partial z^2}\right)_0 + \frac{(Z^{(1)})^2}{2}\left(\frac{\partial^3\Phi^{(1)}}{\partial z^3}\right)_0 - Z^{(1)}\frac{\partial Z^{(1)}}{\partial x}\left(\frac{\partial^2\Phi^{(1)}}{\partial z\partial x}\right)_0 - \frac{\partial Z^{(1)}}{\partial x}\left(\frac{\partial\Phi^{(2)}}{\partial x}\right)_0 - \frac{\partial Z^{(2)}}{\partial x}\left(\frac{\partial\Phi^{(1)}}{\partial x}\right)_0 . \tag{14}$$

To avoid secular terms we expand wave frequency $\omega$ in the following series:

$$\omega^2 = \omega_0^2(k) + A^2\omega^3(k), \tag{15}$$

i.e. in the third order we obtain well-known Stokes' nonlinear dispersion relation /6/:

$$\omega^2 = gk\,\mathrm{th}(kH)\left\{1 + \left(\frac{9\,\mathrm{th}^4(kH) - 10\,\mathrm{th}^2(kH) + 9}{8\,\mathrm{th}^4(kH)}\right)k^2A^2\right\}. \tag{16}$$

In deep water limit $(kH \to \infty)$ /6/ we have:

$$\omega^2 = gk(1 + k^2A^2). \tag{17}$$

Explicit expression of wave characteristics in case of arbitrary depth are cumbersome. That is the reason to write wave profile in case of deep water limit /9/:

$$Z^{(3)} = \frac{3}{8}k^2A^3\cos(3kx - 3\omega t). \tag{18}$$

The higher order expansions are obtained by analogical method.

## 3. BRIEF DESCRIPTION OF COMPUTING ALGORITHM

The method described in previous section is realized by using CAS-system REDUCE-2 /4/ (version 1979). The programme contains four modules. Every one may work independently.

In the first module we generate series (5), (6) and construct Taylor's expansion of functions $\Phi^{(i)}(x,z,t)$ in (5) near $z=0$. Further we substitute this expansions in boundary equations (2), (3). As a consequence we generate the perturbation scheme, i.e. general form of equation for coefficients of expansions (5), (6) down to the fourth order in small parameter $\epsilon$. The results of the programme, in particular, are equations (9), (10) in the second order and equations (13), (14) in the third order.

Further, using separate block of Fourie analysis, we compute right-hand side of equations (9), (10) with the help of explicit expressions of wave characteristics (7), (8).

Let us take the solution of (9), (10) in the form /6/

$$Z^{(2)} = \mu_2 A^2 \cos(2kx - 2\omega t),$$

$$\Phi^{(2)} = \nu_2 A^2 \,\mathrm{ch}\, 2k(z+H)\sin(2kx - 2\omega t),$$

where $\mu_2, \nu_2$ - unknown coefficients. After computing these coefficients we find the wave characteristics in second order (11), (12). In module 2 we compute time derivative of right-hand side of equation (13), after that using equation (14) we obtain second order equation for function $\Phi^{(3)}(x,z,t)|_{z=0}$ in the third order of small parameter. Further, we find relation, using $\Phi^{(3)}(x,z,t)$ in the same function as in first order. This relation is the so-called nonlinear Stokes' dispersion relation (16), in the case of deep water limit-(17).

In module 3, using Fourie analysis block we computer right-hand sides of equations (13), (14). Further, we construct solutions of these equations, i.e. wave characteristics in the third order using the following functional form:

$$Z^{(3)} = \mu_3' A^3 \cos(kx - \omega t) + \mu_3'' A^3 \cos(3kx - 3\omega t),$$

$$\Phi^{(3)} = \nu_3' A^3 \,\mathrm{ch}\, k(z+H)\sin(kx - \omega t) + \nu_3'' A^3 \,\mathrm{ch}\, 3k(z+H)\sin(3kx - 3\omega t),$$

where $\mu_3', \mu_3'', \nu_3', \nu_3''$ are unknown parameters, which we compute.

The module 4 works in the same manner as the third one. The output results of the module are wave characteristics in the fourth order.

The computing was held in JINR, using ES-1060 computer and the programme run takes about ten minutes CP time.



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Гердт В.П., Костов Н.А., Швачка А.Б. E11-83-750
Исследование нелинейных волн на воде с использованием системы аналитических вычислений REDUCE-2

Исследуется распространение нелинейных гравитационных волн на воде. Система гидродинамических уравнений решается методом теории возмущений. Разработан алгоритм и создана программа на языке системы аналитических вычислений REDUCE-2, которая позволяет вычислять волновые характеристики вплоть до четвертого порядка.

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Gerdt V.P., Kostov N.A., Shvachka A.B. E11-83-750
Investigation of Nonlinear Water Waves Using Computer Algebra System REDUCE-2

In present work we study nonlinear water gravity waves. Using standard perturbation scheme, realized by Computer Algebra System REDUCE-2, we solve the system of hydrodynamical equations. The result of the programme is wave characteristics down to the fourth order in small parameter.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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